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BOUNDARY BEHAVIOUR OF THE BERGMAN INVARIANT AND RELATED QUANTITIES

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Abstract. Using Fefferman's classical result on the boundary singularity of the Bergman kernel, we give an analogous description of the boundary behaviour of various related quantities like the Bergman invariant, the coefficients of the Bergman metric, of the associated Laplace-Beltrami operator, of its curvature tensor, Ricci curvature and scalar curvature. The main point is that even though one would expect a bit stronger singularities than the one for the Bergman kernel, due to the differentiations involved, all these quantities turn out to have — except for a different leading power of the defining function — the same kind of singularity as the solution of the Monge-Ampére equation.

1. Introduction

Let Ω be a (bounded) strictly-pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary, $K(x,y)$ the Bergman kernel function of Ω, and $K(z)=K(z,z)$ its restriction to the diagonal. This kernel gives rise to a Riemannian metric, the Bergman metric, on Ω by the recipe

$$ds^2=\sum_{j,k=1}^n g_{j\bar k}\,dz_j\,d\bar z_k, \qquad \text{where } g_{j\bar k}(z)=\frac{\partial^2\log K(z)}{\partial z_j\partial\bar z_k}. \tag{1}$$

The usual objects associated to this metric are the Riemannian volume element $g(z)\,dz$, where

$$g(z)=\det[g_{j\bar k}(z)] \tag{2}$$

and dz stands for the $2n$-dimensional Lebesgue measure; the Laplace (or Laplace-Beltrami) operator

$$\widetilde\Delta=g^{\bar jk}(z)\,\frac{\partial^2}{\partial\bar z_j\partial z_k}; \tag{3}$$

and the curvature tensor, whose components are given by

$$R_{i\bar jk\bar l}=\frac{\partial^2 g_{i\bar j}}{\partial z_k\partial\bar z_l}-g^{\bar rp}\,\frac{\partial g_{i\bar r}}{\partial z_k}\,\frac{\partial g_{p\bar j}}{\partial\bar z_l}. \tag{4}$$

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Here in the last two formulas, $[g^{\overline{r}p}(z)]$ denotes the inverse matrix to $[g_{i\overline{j}}(z)]$, and we have started using the standard summation convention of summing automatically over any index which appears once in the upper and once in the lower position. The contraction

$$\mathrm{Ric}_{i\bar{l}} = g^{\bar{j}k} R_{i\bar{j}k\bar{l}} = \frac{\partial^2 \log g}{\partial z_i \partial \overline{z}_l} \tag{5}$$

of $R_{i\bar{j}k\bar{l}}$ is the Ricci tensor, and the double contraction

$$R = g^{\bar{l}i}\, \mathrm{Ric}_{i\bar{l}} = g^{\bar{l}i} g^{\bar{j}k} R_{i\bar{j}k\bar{l}} \tag{6}$$

is the scalar curvature of the Bergman metric (1).

The *Bergman canonical invariant* is the function on Ω defined by

$$\beta(z) = \frac{g(z)}{K(z)} = \frac{\det[\partial\overline{\partial} \log K(z)]}{K(z)}. \tag{7}$$

From the transformation formula for the Bergman kernel under a biholomorphic mapping $f : \Omega \to \Omega'$,

$$K_\Omega(z) = K_{\Omega'}(f(z)) \cdot |\det f'(z)|^2, \tag{8}$$

it is easy to see that all the objects (1) – (7) above are biholomorphic invariants. This makes them — and, in particular, their boundary behaviour — of natural interest in complex geometry.

Another noteworthy object of this kind is the solution u to the Monge-Ampére equation

$$u^{n+1} \det[\partial\overline{\partial} \log u] = (-1)^n \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

The first of these equations can be rewritten as

$$J[u] = 1 \text{ on } \Omega,$$

where $J[u]$ is the *Monge-Ampére determinant*

$$J[u] := (-1)^n \det \begin{bmatrix} u & \partial u \\ \overline{\partial} u & \partial\overline{\partial} u \end{bmatrix}. \tag{9}$$

The function u transforms under biholomorphic maps as

$$u_\Omega(z) = u_{\Omega'}(f(z)) \cdot |\det f'(z)|^{-2/(n+1)},$$

from which it follows that $\partial\overline{\partial} \log(1/u)$ again defines a metric on Ω invariant under biholomorphisms, which is in fact the unique complete Kähler-Einstein metric on Ω with Kähler-Einstein constant equal to -1 [CY].

The boundary behaviour of the Bergman kernel $K(z)$ has been known since the 1974 seminal work of Fefferman [Fe]. Namely, if $\rho(z)$ is any positively-signed defining function for Ω, meaning that

$$\rho \in C^\infty(\overline{\Omega}), \quad \rho > 0 \text{ on } \Omega, \quad \rho = 0,\ \partial\rho \neq 0 \text{ on } \partial\Omega,$$

then there exist functions $a, b \in C^\infty(\overline{\Omega})$ such that

$$K(z) = \frac{a(z)}{\rho(z)^{n+1}} + b(z) \log \rho(z) \qquad \forall z \in \Omega. \tag{10}$$

Furthermore, for any $z \in \partial\Omega$,

$$a(z) = \frac{n!}{\pi^n} J[\rho](z) > 0, \tag{11}$$

where $J[\rho]$ is again the Monge-Ampére determinant (9), whose positivity on $\partial\Omega$ follows from the strict-pseudoconvexity of Ω. The leading order term in (10) was obtained even earlier by Hörmander [Hö] and the derivatives up to second order — from which one can read off the leading order boundary asymptotics for the Bergman metric coefficients $g_{\bar{j}k}$ — by Diederich [Di]. These two authors also showed that the Bergman invariant $\beta(z)$ tends to $(n+1)^n \pi^n / n!$ as $z \to \partial\Omega$; for $1 \le n \le 3$, this discovery was later duplicated by Hachaichi [Ha]. The leading order terms for the curvature tensor $R_{i\bar{j}k\bar{l}}$ were given by Klembeck [Kl]. The whole subject, as much as the leading order terms are concerned, was pushed much further by Boas, Straube and Yu [BSY] and Krantz and Yu [KY], who, in addition to treating the Ricci tensor $\mathrm{Ric}_{i\bar{l}}$ and the scalar curvature R, were able to handle also some weakly pseudoconvex domains (the *h-extendible*, or *pseudoregular*, ones). We refer to [KY] for further bibliographic references concerning this subject.

For the Monge-Ampére solution u, a complete description (i.e. not just the leading-order term) of the boundary singularity was given by Lee and Melrose [LM]: there exist functions $\eta_j \in C^\infty(\overline{\Omega})$, $j = 0, 1, 2, \dots$, such that

$$u(z) = \rho(z) \sum_{j=0}^{\infty} \eta_j(z) \left(\rho(z)^{n+1} \log \rho(z) \right)^j. \tag{12}$$

Here the last sum should be understood in the asymptotic sense, i.e. as a resolution of singularities: that is, for any $N = 1, 2, \dots$, the difference

$$u - \rho \sum_{j=0}^{N-1} (\rho^{n+1} \log \rho)^j \eta_j$$

belongs to $C^{(n+1)N}(\overline{\Omega})$ and vanishes on $\partial\Omega$ together with all its partial derivatives of orders $\le (n+1)N$.

The aim of this paper is to give a similar complete description of the boundary singularity of the quantities $g_{j\bar{k}}$, $g^{\bar{j}k}$, $R_{i\bar{j}k\bar{l}}$, $\mathrm{Ric}_{i\bar{l}}$, R and β. With a bit of labour and patience, a brute force computation starting from Fefferman's expansion (10) reveals that all these objects admit expansions of a similar form as the Monge-Ampére solution u, namely, of the form

$$\rho^m \sum_{j=0}^{\infty} (\rho^{n+1-k} \log \rho)^j \eta_j, \qquad \eta_j \in C^\infty(\overline{\Omega}),$$

with suitable integers m and with k equal to the number of differentiations involved, that is, $k = 2$ for $g_{j\bar{k}}$, g, $g^{\bar{j}k}$ and β, and $k = 4$ for $R_{i\bar{j}k\bar{l}}$, $\mathrm{Ric}_{i\bar{l}}$ and R. Our main result is that, in fact, a much better assertion is true: we actually can take in all these cases $k = 0$, i.e. obtain the same expansion as (12), except for a different leading power of ρ.

Theorem 1. *Let $\Omega \subset \mathbf{C}^n$ be strictly pseudoconvex with smooth boundary and ρ a positively-signed defining function for Ω. Then:*

(a) *The Bergman invariant β has an asymptotic expansion at $\partial\Omega$ of the form*

$$\beta = \sum_{j=0}^{\infty} (\rho^{n+1} \log \rho)^j \, \eta_j, \qquad \eta_j \in C^\infty(\overline{\Omega}), \tag{13}$$

with

$$\eta_0|_{\partial\Omega} = \frac{(n+1)^n \pi^n}{n!}.$$

(b) *$\rho^{n+1} g$ has the asymptotic expansion of the form* (13), *with*

$$\eta_0|_{\partial\Omega} = (n+1)^n J[\rho].$$

(c) *$\rho^2 g_{i\bar{j}}$ has the asymptotic expansion of the form* (13), *with*

$$\eta_0|_{\partial\Omega} = (n+1)\rho_i \rho_{\bar{j}}.$$

(d) *$\rho^{-1} g^{\bar{j}k}$ has the asymptotic expansion of the form* (13), *with*

$$\eta_0|_{\partial\Omega} = \frac{-1}{n+1} \frac{1}{\rho} [\log \rho]^{\bar{j}k}.$$

(e) *$\rho^4 R_{i\bar{j}k\bar{l}}$ has the asymptotic expansion of the form* (13), *with*

$$\eta_0|_{\partial\Omega} = 2(n+1)\rho_i \rho_{\bar{j}} \rho_k \rho_{\bar{l}}.$$

(f) *$\rho^2 \operatorname{Ric}_{i\bar{l}}$ has the asymptotic expansion of the form* (13), *with*

$$\eta_0|_{\partial\Omega} = (n+1)\rho_i \rho_{\bar{l}}.$$

(g) *R has the asymptotic expansion of the form* (13), *with*

$$\eta_0|_{\partial\Omega} = n.$$

Here in (c) and (d), we have started using the shorthands

$$\rho_i(z) = \frac{\partial \rho(z)}{\partial z_i}, \quad \rho_{\bar{j}}(z) = \frac{\partial \rho(z)}{\partial \bar{z}_j}, \quad \rho_{i\bar{j}}(z) = \frac{\partial^2 \rho(z)}{\partial z_i \partial \bar{z}_j}, \quad \text{etc.},$$

and similarly for $(\log \rho)_{j\bar{k}}$; and $[\log \rho]^{\bar{j}k}$ stands, in analogy with $g_{k\bar{j}}$ and $g^{\bar{j}k}$, for the inverse matrix of $[(\log \rho)_{k\bar{j}}]$. We will see that the latter vanishes to the first order on $\partial\Omega$, so that $\frac{1}{\rho}[\log \rho]^{\bar{j}k}$ extends smoothly to the boundary. All these asymptotic expansions are understood in the sense of "resolution of singularities" as in (12).

Of course, the assertions about the boundary values of η_0 in (e)–(g) are just the findings of [Kl] and [KY, Corollary 2].

Concerning β and g, we even have the following slightly more general result, which makes it possible to get an analogue of Theorem 1(a) also e.g. for the invariant

$$\beta_{\rm Sz} := \frac{\det[\partial\bar{\partial} K_{\rm Sz}]}{K_{\rm Sz}^{(n+1)/n}}$$

associated in a similar way to the Szegö kernel $K_{\rm Sz}$ of Ω. This kernel is known to have the same asymptotic expansion (10) as K, only with the exponent $n+1$ replaced by n.

Theorem 2. *Let F be any zero-free function on Ω having an asymptotic expansion at $\partial\Omega$ of the form*

$$F = \rho^q \sum_{j=0}^{\infty} (\rho^m \log \rho)^j \eta_j, \qquad \eta_j \in C^\infty(\overline{\Omega}), \tag{14}$$

with $q \in \mathbf{R}$, $q \neq 0$, m a positive integer, and $\eta_0 \neq 0$ on $\partial\Omega$. Then $\det[\partial\overline{\partial} \log F]$ also has an asymptotic expansion at $\partial\Omega$ of the form (14), with the same m, $q = -n-1$ and some $\eta'_j \in C^\infty(\overline{\Omega})$ in the place of η_j, where

$$\eta'_0 = (-q)^n J[\rho] \qquad \text{on } \partial\Omega. \tag{15}$$

Note that the last expression does not depend on F.

We remark that, in principle, it is also possible to define a metric (as well as the associated volume element, Laplace-Beltrami operator, curvature tensor, etc.) starting from the Szegö kernel K_{Sz} instead of the Bergman kernel K; if the surface measure on $\partial\Omega$, used for defining K_{Sz}, is chosen appropriately, K_{Sz} again obeys the transformation rule like (8) (only $|\det f'(z)|^2$ has to be replaced by $|\det f'(z)|^{2n/(n+1)}$, cf. [HK]) and it follows that the resulting metric, as well as all the other objects associated to it, will again be invariant under biholomorphisms. Though this "Szegö invariant" and "Szegö metric" have not received as much attention as their Bergman counterparts, it is not difficult to prove for them a complete analogue of Theorem 1 (the only difference being that $\rho^{n+1} \log \rho$ gets replaced by $\rho^n \log \rho$ throughout). In fact, it is even possible to replace K_{Sz} by any function F satisfying (14).

Theorem 3. *Let F be as in Theorem 2. Denote*

$$g_{i\bar{j}}(z) := \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log F(z).$$

Assume that the matrix $[g_{i\bar{j}}]$ is nonsingular. Let $[g^{\bar{j}k}]$ denote the inverse matrix and let the quantities $R_{i\bar{j}k\bar{l}}$, $\mathrm{Ric}_{i\bar{l}}$ and R be defined by the formulas (4)–(6). Then

(i) *$\rho^2 g_{i\bar{j}}$ has an asymptotic expansion at $\partial\Omega$ of the form*

$$\sum_{k=0}^{\infty} (\rho^m \log \rho)^k \, \eta'_k, \qquad \eta'_k \in C^\infty(\overline{\Omega}), \tag{16}$$

with $\eta'_0|_{\partial\Omega} = (-q)\rho_i \rho_{\bar{j}}$;

(ii) *$\rho^{-1} g^{\bar{j}k}$ has an asymptotic expansion at $\partial\Omega$ of the form (16), with*

$$\eta'_0|_{\partial\Omega} = \frac{1}{q} \frac{1}{\rho} [\log \rho]^{\bar{j}k};$$

(iii) *$\rho^4 R_{i\bar{j}k\bar{l}}$ has an asymptotic expansion at $\partial\Omega$ of the form (16), with*

$$\eta'_0|_{\partial\Omega} = -2q \rho_i \rho_{\bar{j}} \rho_k \rho_{\bar{l}};$$

(iv) $\rho^2 \operatorname{Ric}_{i\bar l}$ *has an asymptotic expansion at* $\partial\Omega$ *of the form* (16), *with*

$$\eta'_0|_{\partial\Omega} = (n+1)\rho_i\rho_{\bar l};$$

(v) R *has an asymptotic expansion at* $\partial\Omega$ *of the form* (16), *with*

$$\eta'_0|_{\partial\Omega} = n.$$

The assumption that $[g_{i\bar j}]$ be nonsingular on Ω can in fact be dropped: indeed, by Theorem 2 the determinant $\det[g_{i\bar j}]$ is always nonzero in some neighbourhood of the boundary, thus $[g_{i\bar j}]$ is invertible there, and (i)–(v), which all concern only the behaviour at the boundary, still remain in force, except in (16) the asymptotic expansion has to be understood in the sense that

$$\rho^2 g_{i\bar j} - \sum_{k=0}^{N-1} (\rho^m \log\rho)^k \, \eta'_k$$

belongs to $C^{mN-1}(\overline{\Omega}\setminus U)$ and vanishes to order mN on $\partial\Omega$, for each $N = 1, 2, \dots$ and any neighbourhood U in Ω of the set $\{z : \det[g_{i\bar j}(z)] = 0\}$; similarly for (ii)–(v).

Note also that, again, the values of η'_0 on $\partial\Omega$ do not depend on the η_j in (14), but only on q. In particular, up to the constant factor of $\frac{-q}{n+1}$ (for (i) and (iii)), $\frac{n+1}{-q}$ (for (ii)), or 1 (for (iv) and (v)), these values are the same for the Bergman metric (1) as for the "Szegö metric" mentioned above (then $q = -n$), or for the Cheng-Yau metric corresponding to $F = 1/u$ (then $q = -1$).

The proof of Theorem 2 and of the parts (a) and (b) of Theorem 1 is given in Section 2; the proof of Theorem 3 and of the remaining parts (c)–(g) of Theorem 1 occupies Section 3.

2. The Bergman invariant

It will be expedient to introduce the notation, for m an integer ≥ 1,

$$\mathcal{A}^m := \{F \in C^\infty(\Omega) : \; F = \sum_{j=0}^{\infty} (\rho^m \log\rho)^j \, \eta_j, \; \eta_j \in C^\infty(\overline{\Omega})\}, \tag{17}$$

for the space of all functions in $C^\infty(\Omega)$ which have the asymptotic expansion of the form (14) with $q = 0$ — the expansion being, as always, understood in the sense of the resolution of singularities. For a given $F \in \mathcal{A}^m$, we will denote the corresponding functions η_j by $[F]_j$ when needed; note that in general these are not uniquely determined by F, only their boundary jet is. It is clear that $\mathcal{A}^m$ is an algebra, whose group of invertible elements consists of those F which do not vanish on Ω and for which $[F]_0$ does not vanish on $\partial\Omega$; we will denote this group by $\mathcal{A}^m_*$. We can also topologize $\mathcal{A}^m$ by declaring that $F_n \to F$ if $F_n - F$ belongs to $C^{k_n}(\overline{\Omega})$ and vanishes to order k_n at $\partial\Omega$, where $k_n \to +\infty$ as $n \to +\infty$; note that, for each fixed k_n, this condition involves only finitely many of the coefficients $[F_n]_j$ and $[F]_j$, $j = 0, 1, 2, \dots$. For any $F \in \mathcal{A}^m_*$, we can choose an η_0 which is nonzero on $\overline{\Omega}$; using the factorization $F = \eta_0 \cdot (1 + \sum_{j=1}^\infty (\rho^m \log\rho)^j \frac{\eta_j}{\eta_0})$ and the fact that the binomial formula $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \dots$ converges in the above topology if $x = \sum_{j=1}^\infty (\rho^n \log\rho)^j \frac{\eta_j}{\eta_0}$, we conclude that F^α also belongs to $\mathcal{A}^m$ for any $\alpha \in \mathbf{Z}$. (In fact, if $F > 0$ on Ω, or if $\overline{\Omega}$ is simply connected, we could even have $\alpha \in \mathbf{C}$.) It is convenient to summarize these observations as a lemma.

Lemma 4. *For any integer $m \geq 1$,*

(a) *$F, G \in \mathcal{A}^m$ and $\lambda \in \mathbf{C}$ imply $F + G$, λF, $FG \in \mathcal{A}^m$;*
(b) *$F \in \mathcal{A}^m_*$ and $\alpha \in \mathbf{Z}$ imply $F^\alpha \in \mathcal{A}^m$; in particular, $1/F \in \mathcal{A}^m$.*

Remark. A more fancy way of looking at $\mathcal{A}^m$ is the following: let $C^\infty_0(\overline{\Omega})$ be the subspace in $C^\infty(\overline{\Omega})$ of all functions that vanish on $\partial\Omega$ together with all their derivatives, and let $\mathcal{I} = C^\infty(\overline{\Omega})/C^\infty_0(\overline{\Omega})$ be the quotient space, i.e. the space of all jets of $C^\infty(\overline{\Omega})$-functions at $\partial\Omega$. Then $\mathcal{A}^m/C^\infty_0(\overline{\Omega})$ is topologically isomorphic to the ring of all formal power series $\mathcal{I}[[L]]$ over $\mathcal{I}$ in the variable

$$L = \rho^m \log \rho. \tag{18}$$

The following easy lemma will prove handy.

Lemma 5. *For any $A_{i\bar{j}} \in \mathcal{A}^m$, $i, j = 1, \dots, n$, and $Q \in \mathcal{A}^m$, the determinant*

$$\det \begin{bmatrix} \rho & \rho_{\bar{j}} \\ \rho_i & A_{i\bar{j}} + Q\dfrac{\rho_i \rho_{\bar{j}}}{\rho} \end{bmatrix}$$

belongs to $\mathcal{A}^m$.

Proof. By elementary matrix manipulations, the determinant equals

$$\det \begin{bmatrix} \rho & \rho_{\bar{j}} \\ \rho_i - Q\rho_i & A_{i\bar{j}} \end{bmatrix}.$$

All the entries in the last determinant belong to $\mathcal{A}^m$, by Lemma 4. One more application of Lemma 4(a) thus yields the conclusion. □

Proof of Theorem 2. Let us write $F = \rho^q v$, with $v \in \mathcal{A}^m_*$. Since F is zero-free by hypothesis, we may assume that $[v]_0$ does not vanish on $\overline{\Omega}$. By the well-known formula

$$\det[\partial\overline{\partial} \log F] = \frac{1}{F^{n+1}} \det \begin{bmatrix} F & F_{\bar{j}} \\ F_i & F_{i\bar{j}} \end{bmatrix},$$

where, as before for ρ, we have adopted the shorthands $F_i = \partial F/\partial z_i$, etc., for the derivatives. By elementary matrix manipulations,

$$\det \begin{bmatrix} F & F_{\bar{j}} \\ F_i & F_{i\bar{j}} \end{bmatrix} = \det \begin{bmatrix} \rho^q v & q\rho^{q-1}\rho_{\bar{j}} v + \rho^q v_{\bar{j}} \\ q\rho^{q-1}\rho_i v + \rho^q v_i & \begin{matrix} q(q-1)\rho^{q-2}\rho_i\rho_{\bar{j}} v + q\rho^{q-1}\rho_{i\bar{j}} v \\ + q\rho^{q-1}\rho_{\bar{j}} v_i + q\rho^{q-1}\rho_i v_{\bar{j}} + \rho^q v_{i\bar{j}} \end{matrix} \end{bmatrix}$$

$$= v^{n+1}\rho^{(q-1)(n+1)} \det \begin{bmatrix} \rho & q\rho_{\bar{j}} + \rho\dfrac{v_{\bar{j}}}{v} \\ q\rho_i + \rho\dfrac{v_i}{v} & q(q-1)\dfrac{\rho_i\rho_{\bar{j}}}{\rho} + q\rho_{i\bar{j}} + q\rho_{\bar{j}}\dfrac{v_i}{v} + q\rho_i\dfrac{v_{\bar{j}}}{v} + q\dfrac{v_{i\bar{j}}}{v} \end{bmatrix}$$

$$= v^{n+1}\rho^{(q-1)(n+1)} \det \begin{bmatrix} \rho & q\rho_{\bar{j}} + \rho\dfrac{v_{\bar{j}}}{v} \\ q\rho_i & q(q-1)\dfrac{\rho_i\rho_{\bar{j}}}{\rho} + q\rho_{i\bar{j}} + q\rho_i\dfrac{v_{\bar{j}}}{v} + \rho\Big(\dfrac{v_{i\bar{j}}}{v} - \dfrac{v_i v_{\bar{j}}}{v^2}\Big) \end{bmatrix}$$

$$= v^{n+1}\rho^{(q-1)(n+1)} \det \begin{bmatrix} \rho & q\rho_{\bar j} \\ q\rho_i & q(q-1)\dfrac{\rho_i\rho_{\bar j}}{\rho} + q\rho_{i\bar j} + \rho\Big(\dfrac{v_{i\bar j}}{v} - \dfrac{v_i v_{\bar j}}{v^2}\Big) \end{bmatrix}$$
$$= v^{n+1}\rho^{(q-1)(n+1)} \det \begin{bmatrix} \rho & q\rho_{\bar j} \\ \rho_i & q\rho_{i\bar j} + \rho\Big(\dfrac{v_{i\bar j}}{v} - \dfrac{v_i v_{\bar j}}{v^2}\Big) \end{bmatrix}$$
$$= q^n v^{n+1}\rho^{(q-1)(n+1)} \det \begin{bmatrix} \rho & \rho_{\bar j} \\ \rho_i & \rho_{i\bar j} + \dfrac{\rho}{q}\Big(\dfrac{v_{i\bar j}}{v} - \dfrac{v_i v_{\bar j}}{v^2}\Big) \end{bmatrix}.$$

Observe that, for L as in (18),

$$L_i = m\rho^{m-1}\rho_i \log\rho + \rho^{m-1}\rho_i = m\,\frac{\rho_i}{\rho}\,L + \rho^{m-1}\rho_i$$

(and similarly for $L_{\bar j}$). Thus, if, say,

$$v = \sum_{k=0}^{\infty} \eta_k\, L^k, \qquad \eta_j \in C^\infty(\overline{\Omega}),$$

then

$$\begin{aligned} v_i &= \sum_k (\eta_k)_i L^k + \eta_k k\, L^{k-1}\Big(m\frac{\rho_i}{\rho}\,L + \rho^{m-1}\rho_i\Big) \\ &= \sum_k [(\eta_k)_i + (k+1)\eta_{k+1}\rho^{m-1}\rho_i]\,L^k + \sum_k m\,\frac{\rho_i}{\rho}\,k\eta_k\,L^k \\ &\equiv D_i v + \frac{\rho_i}{\rho}\,Mv, \end{aligned}$$

and, similarly,

$$v_{\bar j} = D_{\bar j} v + \frac{\rho_{\bar j}}{\rho}\,Mv,$$

where we have introduced the operators

$$\begin{aligned} D_i :\ \sum_k \eta_k L^k &\longmapsto \sum_k \frac{\partial \eta_k}{\partial z_i}\,L^k, \qquad i = 1,\dots,n, \\ D_{\bar j} :\ \sum_k \eta_k L^k &\longmapsto \sum_k \frac{\partial \eta_k}{\partial \bar z_j}\,L^k, \qquad j = 1,\dots,n, \\ M :\ \sum_k \eta_k L^k &\longmapsto \sum_k \big((k+1)\eta_{k+1}\rho^m + mk\eta_k\big)\,L^k. \end{aligned} \tag{19}$$

For later use, we also introduce one more operator

$$M_0 :\ \sum_k \eta_k L^k \longmapsto \sum_k (k+1)\eta_{k+1}\,L^k.$$

Note that $D_i, D_{\bar{j}}, M$ as well as M_0 all map $\mathcal{A}^m$ into itself. Further, we have the obvious relations, for any $f \in C^\infty(\Omega)$,

$$\begin{aligned} D_i(fv) &= f_i v + f\, D_i v, \\ M(fv) &= f\, Mv, \\ D_i M v &= M\, D_i v + \frac{\rho_i}{\rho} \cdot m\rho^m\, M_0 v \end{aligned} \tag{20}$$

(and similarly for $D_{\bar{j}}$). For later use, we also record that

$$Mv = O(\rho^m \log \rho) \tag{21}$$

for any $v \in \mathcal{A}^m$; in particular, $Mv|_{\partial\Omega} = 0$.

Iterating the computation of v_i and $v_{\bar{j}}$, we further get

$$\begin{aligned} v_{i\bar{j}} &= \Big(D_i + \frac{\rho_i}{\rho}\, M\Big)\Big(D_{\bar{j}} v + \frac{\rho_{\bar{j}}}{\rho}\, Mv\Big) \\ &= D_i D_{\bar{j}} v + D_i\Big(\frac{\rho_{\bar{j}}}{\rho} Mv\Big) + \frac{\rho_i}{\rho} M D_{\bar{j}} v + \frac{\rho_i}{\rho} M\Big(\frac{\rho_{\bar{j}}}{\rho} Mv\Big). \end{aligned}$$

Upon using (20), this gives

$$v_{i\bar{j}} = D_i D_{\bar{j}} v + \frac{\rho_{i\bar{j}}\rho - \rho_i \rho_{\bar{j}}}{\rho^2}\, Mv + \frac{\rho_{\bar{j}}}{\rho}\Big(M D_i v + \frac{\rho_i}{\rho}\, m\rho^m M_0 v\Big) + \frac{\rho_i}{\rho}\, M D_{\bar{j}} v + \frac{\rho_i \rho_{\bar{j}}}{\rho^2}\, M^2 v.$$

On the other hand, from the formulas for v_i and $v_{\bar{j}}$,

$$v_i v_{\bar{j}} = D_i v \cdot D_{\bar{j}} v + \frac{\rho_i}{\rho}\, D_{\bar{j}} v \cdot Mv + \frac{\rho_{\bar{j}}}{\rho}\, D_i v \cdot Mv + \frac{\rho_i \rho_{\bar{j}}}{\rho^2} (Mv)^2.$$

Combining the last two formulas, we see that

$$\rho\Big(\frac{v_{i\bar{j}}}{v} - \frac{v_i v_{\bar{j}}}{v^2}\Big) = B_{i\bar{j}} + \frac{\rho_i \rho_{\bar{j}}}{\rho} B, \tag{22}$$

where

$$\begin{aligned} B_{i\bar{j}} = \frac{\rho D_i D_{\bar{j}} v}{v} &+ \frac{\rho_{i\bar{j}} Mv + \rho_{\bar{j}} M D_i v + \rho_i M D_{\bar{j}} v}{v} \\ &- \frac{\rho D_i v \cdot D_{\bar{j}} v + \rho_{\bar{j}} D_i v \cdot Mv + \rho_i D_{\bar{j}} v \cdot Mv}{v^2} \end{aligned} \tag{23}$$

and

$$B = \frac{m\rho^m M_0 v + M^2 v - Mv}{v} - \frac{(Mv)^2}{v^2} \tag{24}$$

both belong to $\mathcal{A}^m$ owing to Lemma 4(b). For later use, we note that, by (21),

$$B_{i\bar{j}} = O(\rho \log \rho), \quad B = O(\rho^m \log \rho) \qquad \text{as } \rho \to 0. \tag{25}$$

Applying Lemma 5, with $A_{i\bar j} = \frac{1}{q} B_{i\bar j}$ and $Q = \frac{1}{q} B$, we thus get

$$\det \begin{bmatrix} \rho & \rho_{\bar j} \\ \rho_i & \rho_{i\bar j} + \frac{\rho}{q}\Big(\frac{v_{i\bar j}}{v} - \frac{v_i v_{\bar j}}{v^2}\Big) \end{bmatrix} \in \mathcal{A}^m. \tag{26}$$

Consequently, since $F^{n+1} = \rho^{n+1} v^{n+1}$,

$$\begin{aligned} \det[\partial\bar\partial \log F] &= \frac{1}{F^{n+1}} \det \begin{bmatrix} F & F_{\bar j} \\ F_i & F_{i\bar j} \end{bmatrix} \\ &= \frac{q^n}{\rho^{n+1}} \det \begin{bmatrix} \rho & \rho_{\bar j} \\ \rho_i & \rho_{i\bar j} + \frac{\rho}{q}\Big(\frac{v_{i\bar j}}{v} - \frac{v_i v_{\bar j}}{v^2}\Big) \end{bmatrix} \\ &\in \rho^{-n-1}\mathcal{A}^m, \end{aligned}$$

and we are done.

It remains to compute the leading term. However, when $\rho = 0$, the determinant (26) becomes simply

$$\det \begin{bmatrix} \rho & \rho_{\bar j} \\ \rho_i & \rho_{i\bar j} \end{bmatrix} = (-1)^n J[\rho],$$

and (15) follows immediately. This completes the proof. □

Proof of parts (a) *and* (b) *of Theorem 1.* Applying Theorem 2 to $F = K$, which belongs, by (10), to $\rho^q \mathcal{A}^m_*$ for $m = n+1$ and $q = -n-1$, we obtain

$$g = \det[\partial\bar\partial \log K] \in \rho^{-n-1}\mathcal{A}^{n+1}$$

with $[\rho^{n+1} g]_0|_{\partial\Omega} = (n+1)^n J[\rho]|_{\partial\Omega}$; this settles (b). Furthermore, appealing to (10) once more,

$$\beta = \frac{g}{K} = \frac{\rho^{n+1} g}{\rho^{n+1} K}$$

belongs to $\mathcal{A}^{n+1}$ by Lemma 4(a), and for its leading term we have on $\partial\Omega$

$$[\beta]_0 = \frac{[\rho^{n+1} g]_0}{[\rho^{n+1} K]_0} = \frac{(n+1)^n J[\rho]}{n! J[\rho]/\pi^n} = \frac{(n+1)^n \pi^n}{n!}$$

by (11). This proves (a) (and recovers the familiar result of [Hö] and [Di]). □

3. The curvatures

Proof of parts (i), (ii) *and* (iv) *of Theorem 3.* Let us again write $F = \rho^q v$, where $v \in \mathcal{A}^m_*$. Then we have

$$g_{i\bar j} = (\log F)_{i\bar j} = q\,(\log \rho)_{i\bar j} + (\log v)_{i\bar j}. \tag{27}$$

From (22), it is evident that

$$\rho^2 (\log v)_{i\bar j} = \rho^2 \frac{v_{i\bar j} v - v_i v_{\bar j}}{v^2} = \rho B_{i\bar j} + \rho_i \rho_{\bar j} B \in \mathcal{A}^m. \tag{28}$$

On the other hand,

$$\rho^2(\log\rho)_{i\bar{j}} = \rho_{i\bar{j}}\rho - \rho_i\rho_{\bar{j}} \in C^\infty(\overline{\Omega}). \tag{29}$$

Thus indeed

$$\rho^2 g_{i\bar{j}} \in \mathcal{A}^m.$$

As for the leading term, clearly $\rho^2(\log v)_{i\bar{j}} \to 0$ as $\rho \to 0$, by (28) and (25); thus by (27) and (29)

$$\lim_{\rho\to 0} \rho^2 g_{i\bar{j}} = q \lim_{\rho\to 0} \rho^2(\log\rho)_{i\bar{j}} = (-q)\rho_i\rho_{\bar{j}},$$

or $[\rho^2 g_{i\bar{j}}]_0 = (-q)\rho_i\rho_{\bar{j}}$ on $\partial\Omega$. This establishes (i).

For (iv), note that

$$\mathrm{Ric}_{i\bar{l}} = g^{\bar{j}k}R_{i\bar{j}k\bar{l}} = \frac{\partial^2}{\partial z_i \partial \overline{z}_l}\log g$$

is obtained from $\log g$ in the same way as

$$g_{i\bar{l}} = \frac{\partial^2}{\partial z_i \partial \overline{z}_l}\log F$$

is obtained from $\log F$. Since, by Theorem 2, g has the form $\rho^{-n-1}w$ with $w \in \mathcal{A}^m_*$, the proof of (iv) is the same as for (i).

Concerning $[g^{\bar{j}k}] = [g_{k\bar{j}}]^{-1}$, recall that, quite generally, for any invertible matrix M

$$M^{-1} = \frac{\mathrm{Ad}\, M}{\det M},$$

where the (j,k)-entry of $\mathrm{Ad}\, M$, the adjoint matrix of M, is by definition $(-1)^{j+k}$ times the determinant of the submatrix obtained upon deleting from M the j-th row and k-th column. Apply this to $M = [g_{k\bar{j}}]$. The determinant $\det M = g$ is then precisely what we computed in the preceding section to be of the form

$$g = \rho^{-n-1}G, \qquad \text{with } G \in \mathcal{A}^m,$$

where

$$[G]_0 = (-q)^n J[\rho] \quad \text{on } \partial\Omega.$$

However, an easy check reveals that the arguments from the preceding section apply, without any changes, also to the subdeterminants mentioned a few lines above; the only difference being that the size of the submatrix is then smaller by 1. Thus the (j,k)-entry of $\mathrm{Ad}\, M$ has the form

$$[\mathrm{Ad}\, M]_{jk} = \rho^{-n}G_{jk}, \quad \text{with } G_{jk} \in \mathcal{A}^m,$$

where

$$[G_{jk}]_0 = (-q)^{n-1}J_{jk}[\rho] \quad \text{on } \partial\Omega,$$

$J_{jk}[\rho]$ being $(-1)^{j+k+n-1}$ times the determinant of the submatrix obtained upon deleting from $\begin{bmatrix} \rho & \rho_{\bar{j}} \\ \rho_i & \rho_{i\bar{j}} \end{bmatrix}$ the $(j+1)$-st row and the $(k+1)$-st column. In other words, by standard matrix manipulations,

$$(-1)^{n-1} J_{jk}[\rho] = \rho^n \operatorname{Ad}[\partial\overline{\partial} \log \rho]_{jk}$$

(just as $(-1)^n J[\rho] = \rho^{n+1} \det[\partial\overline{\partial} \log \rho]$). Consequently,

$$g^{\bar{j}k} = \frac{[\operatorname{Ad} M]_{jk}}{g} = \frac{\rho^{-n} G_{jk}}{\rho^{-n-1} G} = \rho \, \frac{G_{jk}}{G} \in \rho \mathcal{A}^m$$

by Lemma 4; and the leading term on the boundary equals

$$[\rho^{-1} g^{\bar{j}k}]_0 = \frac{[G_{jk}]_0}{[G]_0} = \frac{(-q)^{n-1} J_{jk}[\rho]}{(-q)^n J[\rho]} = \frac{1}{-q} \, \frac{\rho^n (-1)^{n-1} \operatorname{Ad}[\partial\overline{\partial} \log \rho]_{jk}}{\rho^{n+1} (-1)^n \det[\partial\overline{\partial} \log \rho]}$$
$$= \frac{1}{q} \, \frac{1}{\rho} \, ([\partial\overline{\partial} \log \rho]^{-1})_{jk} = \frac{1}{q} \, \frac{1}{\rho} \, [\log \rho]^{\bar{j}k},$$

as claimed. Since $J_{jk}[\rho]$ and $J[\rho]$ are smooth on $\overline{\Omega}$ and $J[\rho]$ does not vanish on $\partial\Omega$, it is further evident from the second equality in the chain that $\frac{1}{\rho}[\log \rho]^{\bar{j}k} = -J_{jk}[\rho]/J[\rho]$ extends smoothly to $\partial\Omega$. This finishes the proof of part (ii). □

Note, in particular, that (ii) implies that the coefficients $g^{\bar{j}k}$ of the Laplace-Beltrami operator (3) always vanish at the boundary to (at least) first order.

A small point which we have glossed over in the end of the last proof is that $J[\rho]$, though always positive on $\partial\Omega$ and, hence, by continuity, also in some neighbourhood of $\partial\Omega$ in $\overline{\Omega}$, may vanish at some point in the interior of Ω, meaning that the matrix $[\partial\overline{\partial} \log \rho]$ is not invertible there and thus $[\log \rho]^{\bar{j}k}$ does not make sense. From the point of view of the last proof this is immaterial, since there we were only interested in the behaviour near the boundary; however, the defect can in fact be removed completely if desired by noting that the classes $\mathcal{A}^m$ are independent of the choice of defining function. (Indeed, if ρ' is another positively-signed defining function for Ω, then $\rho = \rho' e^h$ for some $h \in C^\infty(\overline{\Omega})$, whence

$$\rho^m \log \rho = \rho'^m e^{mh} (\log \rho' + h) \in \rho'^m \log \rho' \cdot C^\infty(\overline{\Omega}) + \rho'^m C^\infty(\overline{\Omega})$$

so $\mathcal{A}^m_\rho = \mathcal{A}^m_{\rho'}$.) Thus we can from the very beginning choose a defining function ρ such that $J[\rho] > 0$ on all of $\overline{\Omega}$ (which is always possible in view of the strict pseudoconvexity).

For convenience, we will assume from now on that such a ρ has been chosen.

In order to prove the two remaining parts of Theorem 3, we state some lemmas and a slight sharpening of the part (ii) that we have already proved.

Lemma 6. *For any $k = 1, \dots, n$,*

$$[\log \rho]^{\bar{j}k} \rho_{\bar{j}} \in \rho^2 C^\infty(\overline{\Omega}),$$

and similarly for $[\log \rho]^{\bar{j}k} \rho_k$.

Proof. By the definition of $[\log \rho]^{\bar{j}k}$,

$$[\log \rho]^{\bar{j}k} \Big(\frac{\rho_{m\bar{j}} \rho - \rho_m \rho_{\bar{j}}}{\rho^2} \Big) = [\log \rho]^{\bar{j}k} [\log \rho]_{m\bar{j}} = \delta^k_m.$$

Thus

$$[\log\rho]^{\bar{j}k}\rho_{\bar{j}}\rho_m = \rho[\log\rho]^{\bar{j}k}\rho_{m\bar{j}} - \rho^2\delta^k_m \in \rho^2 C^\infty(\overline{\Omega}),$$

since $\frac{1}{\rho}[\log\rho]^{\bar{j}k} \in C^\infty(\overline{\Omega})$ by the end of the proof of part (ii) of Theorem 3. Multiplying by $\rho_{\overline{m}} = \overline{\rho}_m$, summing over m, and recalling that $\|\partial\rho\|^2 > 0$ on $\partial\Omega$ since ρ is a defining function, the claim follows. □

In the same manner, since, by (ii), $\rho^{-1}g^{\bar{j}k} = \frac{1}{q}\rho^{-1}[\log\rho]^{\bar{j}k} + \rho C^\infty(\overline{\Omega}) + \rho^m \log\rho \cdot \mathcal{A}^m$, it follows that

$$[\log\rho]_{l\bar{j}}g^{\bar{j}k} = \tfrac{1}{q}\delta^k_l + C^\infty(\overline{\Omega}) + \rho^{m-1}\log\rho\cdot\mathcal{A}^m.$$

The next proposition shows that in fact the last two summands can be improved by a factor of ρ.

Proposition 7. *There exist functions $H^k_l \in \mathcal{A}^m$, $k, l = 1, \dots, n$, such that*

$$g^{\bar{j}k} = [\log\rho]^{\bar{j}l}H^k_l.$$

Furthermore, $[H^k_l]_0 = \frac{1}{q}\delta^k_l$ on $\partial\Omega$.

Proof. In principle this should be possible to extract from the proof of part (ii) of Theorem 3, but it seems better to proceed directly. Let us temporarily denote by $\mathbf{G}$, $\mathbf{G}^{-1}$, $\mathbf{Q}$, $\mathbf{Q}^{-1}$ and $\mathbf{V}$ the matrices $[g_{i\bar{j}}] = [\partial\overline{\partial}\log F]$, $[g^{\bar{j}k}] = [\partial\overline{\partial}\log F]^{-1}$, $[\partial\overline{\partial}\log\rho]$, $[\partial\overline{\partial}\log\rho]^{-1}$ and $[\partial\overline{\partial}\log v]$, respectively; here again $v = \rho^{-q}F$. Then by (27)

$$\mathbf{G} = q\mathbf{Q} + \mathbf{V} = (qI + \mathbf{V}\mathbf{Q}^{-1})\mathbf{Q}.$$

Consequently,

$$\mathbf{G}^{-1} = \mathbf{Q}^{-1}(qI + \mathbf{V}\mathbf{Q}^{-1})^{-1} \equiv \mathbf{Q}^{-1}\mathbf{H},$$

the invertibility of the matrix in the parentheses being a consequence of the invertibility of $\mathbf{G}$ and $\mathbf{Q}$. Now by (22),

$$\mathbf{V}_{i\bar{j}} = \frac{B_{i\bar{j}}}{\rho} + \frac{\rho_i\rho_{\bar{j}}}{\rho^2}B,$$

with $B_{i\bar{j}}, B \in \mathcal{A}^m$ given by (23) and (24). Using the fact that $\mathbf{Q}^{-1} \in \rho C^\infty(\overline{\Omega})$ and Lemma 6, we thus get

$$\begin{aligned}[\mathbf{V}\mathbf{Q}^{-1}]^k_i = \mathbf{V}_{i\bar{j}}[\log\rho]^{\bar{j}k} &= B_{i\bar{j}}\,\frac{[\log\rho]^{\bar{j}k}}{\rho} + \frac{[\log\rho]^{\bar{j}k}\rho_i\rho_{\bar{j}}}{\rho^2}\,B \\ &= B_{i\bar{j}}\,C^\infty(\overline{\Omega}) + \frac{\rho^2 C^\infty(\overline{\Omega})}{\rho^2}\,B\end{aligned}$$

which belongs to $\mathcal{A}^m$ by Lemma 4.

At the same time, by (25), both $B_{i\bar{j}}$ and B are $O(\rho\log\rho)$ as $\rho \to 0$; thus also $\mathbf{V}\mathbf{Q}^{-1}$ vanishes at $\partial\Omega$. Thus near the boundary, we can compute the inverse $\mathbf{H} = (qI + \mathbf{V}\mathbf{Q}^{-1})^{-1} = \frac{1}{q}(I + \frac{1}{q}\mathbf{V}\mathbf{Q}^{-1})^{-1}$ by the Neumann series

$$\mathbf{H} = \frac{1}{q}\Big(I + \sum_{j=1}^\infty \Big(\frac{\mathbf{V}\mathbf{Q}^{-1}}{-q}\Big)^j\Big).$$

As $\mathbf{V}\mathbf{Q}^{-1} \in \mathcal{A}^m$, it thus follows by Lemma 4 that $\mathbf{H} \in \mathcal{A}^m$, and $\mathbf{H}|_{\partial\Omega} = \frac{1}{q}I$. Taking H^k_l to be the (l, k)-entry of $\mathbf{H}$, we get the result. □

Lemma 8. *The quantity*

$$\frac{1}{\rho^2}[\log\rho]^{\bar{j}k}\rho_{\bar{j}}\rho_k \tag{30}$$

tends to -1 *at the boundary.*

Proof. First of all, let us see what happens if we replace ρ by another defining function, say $\rho' = \rho e^h$ with some $h \in C^\infty(\overline{\Omega})$. Clearly

$$\frac{\rho'_{\bar{j}}\rho'_k}{\rho'^2} = \Big(\frac{\rho_{\bar{j}}}{\rho} + h_{\bar{j}}\Big)\Big(\frac{\rho_k}{\rho} + h_k\Big). \tag{31}$$

On the other hand, keeping the notation $\mathbf{Q}$ from the proof of the preceding proposition,

$$\begin{aligned}[\log\rho']^{\bar{j}k} &= [\partial\overline{\partial}\log\rho + \partial\overline{\partial}h]^{-1} \\ &= (\mathbf{Q} + \partial\overline{\partial}h)^{-1} \\ &= (I + \mathbf{Q}^{-1}\cdot\partial\overline{\partial}h)^{-1}\mathbf{Q}^{-1}.\end{aligned}$$

Since $\mathbf{Q}^{-1} = O(\rho)$, expanding the last inverse again by the Neumann series shows that

$$[\log\rho']^{\bar{j}k} = (\delta^{\bar{j}}_{\bar{l}} + W^{\bar{j}}_{\bar{l}})\,[\log\rho]^{\bar{l}k}$$

where $W^{\bar{j}}_{\bar{l}} = O(\rho)$. Combining this with (31) yields

$$[\log\rho']^{\bar{j}k}\frac{\rho'_{\bar{j}}\rho'_k}{\rho'^2} = (\delta^{\bar{j}}_{\bar{l}} + W^{\bar{j}}_{\bar{l}})\,[\log\rho]^{\bar{l}k}\Big(\frac{\rho_{\bar{j}}\rho_k}{\rho^2} + \frac{\rho_{\bar{j}}h_k}{\rho} + \frac{\rho_k h_{\bar{j}}}{\rho} + h_k h_{\bar{j}}\Big).$$

By Lemma 6, the contribution from the third term in the last brackets is $O(\rho)$, as is that from the last term there since $[\log\rho]^{\bar{l}k} = O(\rho)$. In the contribution from the second term,

$$(\delta^{\bar{j}}_{\bar{l}} + W^{\bar{j}}_{\bar{l}})\,[\log\rho]^{\bar{l}k}\frac{\rho_{\bar{j}}h_k}{\rho},$$

the part coming from $W^{\bar{j}}_{\bar{l}}$ is $O(\rho)$ since both W and $[\partial\overline{\partial}\log\rho]^{-1}$ are $O(\rho)$, while the part coming from $\delta^{\bar{j}}_{\bar{l}}$ is also $O(\rho)$ by Lemma 6 again. Finally, in the contribution from the first term the part coming from $W^{\bar{j}}_{\bar{l}}$ is again $O(\rho)$ by $W = O(\rho)$ and Lemma 6. Thus

$$[\log\rho']^{\bar{j}k}\frac{\rho'_{\bar{j}}\rho'_k}{\rho'^2} = [\log\rho]^{\bar{j}k}\frac{\rho_{\bar{j}}\rho_k}{\rho^2} + O(\rho).$$

Consequently, the boundary limit of (30) does not depend on the choice of the defining function.

We may therefore assume that $-\rho$ is strictly plurisubharmonic on $\overline{\Omega}$; that is, that the Hessian matrix

$$\mathbf{O} := [\partial\overline{\partial}\rho] = [\rho_{\bar{j}k}]$$

is negative definite and, hence, in particular, invertible on $\overline{\Omega}$. Denote, for brevity,

$$X = \partial\rho$$

(viewed as a vector in $\mathbf{C}^n$). Thus

$$\mathbf{Q} = [\partial\overline{\partial}\log\rho] = \frac{1}{\rho}\mathbf{O} - \frac{1}{\rho^2}\langle\,\cdot\,, X\rangle X.$$

We claim that

$$\mathbf{Q}^{-1} = \rho\mathbf{O}^{-1} + \frac{\rho}{c}\langle\,\cdot\,, \mathbf{O}^{-1}X\rangle\mathbf{O}^{-1}X, \tag{32}$$

where

$$c := \rho - \langle\mathbf{O}^{-1}X, X\rangle \tag{33}$$

does not vanish on Ω. Indeed,

$$\begin{aligned}&\Big(\rho\mathbf{O} - \langle\,\cdot\,, X\rangle X\Big)\Big(c\mathbf{O}^{-1} + \langle\,\cdot\,, \mathbf{O}^{-1}X\rangle\mathbf{O}^{-1}X\Big) = \\ &\quad = \rho cI + \rho\langle\,\cdot\,, \mathbf{O}^{-1}X\rangle X - c\langle\,\cdot\,, \mathbf{O}^{-1}X\rangle X - \langle\,\cdot\,, \mathbf{O}^{-1}X\rangle\langle\mathbf{O}^{-1}X, X\rangle X = \rho cI,\end{aligned}$$

since $\rho - c - \langle\mathbf{O}^{-1}X, X\rangle = 0$. Thus

$$\rho c\mathbf{O}^{-1} + \rho\langle\,\cdot\,, \mathbf{O}^{-1}X\rangle\mathbf{O}^{-1}X = \rho^2 c(\rho^2\mathbf{Q})^{-1} = c\mathbf{Q}^{-1}, \tag{34}$$

whence

$$\begin{aligned}c\langle\mathbf{Q}^{-1}X, X\rangle &= \rho c\langle\mathbf{O}^{-1}X, X\rangle + \rho|\langle\mathbf{O}^{-1}X, X\rangle|^2 \\ &= \rho c(\rho - c) + \rho(\rho - c)^2 = \rho^2(\rho - c) = \rho^2\langle\mathbf{O}^{-1}X, X\rangle.\end{aligned}$$

Since $\mathbf{O}$ is negative definite on $\overline{\Omega}$, it follows that, indeed, $c \neq 0$ on Ω, and (34) gives (32). (If $X = 0$, then $c = \rho > 0$ by (33). Note also that $\mathbf{Q}$ is invertible, since $\det\mathbf{Q} = \rho^{-n-1}(-1)^n J[\rho] \neq 0$ throughout Ω; cf. the comments before Lemma 6.)

Now from (32) we have

$$\begin{aligned}\frac{1}{\rho^2}[\log\rho]^{\overline{j}k}\rho_{\overline{j}}\rho_k &= \frac{1}{\rho^2}\langle\mathbf{Q}^{-1}X, X\rangle \\ &= \frac{1}{\rho^2}\Big(\rho\langle\mathbf{O}^{-1}X, X\rangle + \frac{\rho}{c}|\langle\mathbf{O}^{-1}X, X\rangle|^2\Big) \\ &= \frac{1}{\rho^2}\Big(\rho(\rho - c) + \frac{\rho}{c}(\rho - c)^2\Big) \\ &= \frac{1}{\rho^2}\rho(\rho - c)\Big(1 + \frac{\rho - c}{c}\Big) \\ &= \frac{\rho^2(\rho - c)}{\rho^2 c} = \frac{\rho - c}{c} \\ &= \frac{\langle\mathbf{O}^{-1}X, X\rangle}{\rho - \langle\mathbf{O}^{-1}X, X\rangle}.\end{aligned}$$

Since $\mathbf{O}^{-1}$ is negative definite on $\partial\Omega$ (by virtue of our choice of ρ) and X is nonzero there (since ρ is a defining function), we have $\langle \mathbf{O}^{-1}X, X\rangle \neq 0$ on $\partial\Omega$, so the last expression tends to -1 as $\rho \to 0$. $\square$

Proof of parts (iii) *and* (v) *of Theorem 3.* Recall that

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}k\bar{l}} - g^{\bar{p}r} g_{ik\bar{p}} g_{\bar{j}\bar{l}r}, \tag{35}$$

where as expected we write $g_{i\bar{j}k\bar{l}} = \partial^4(\log F)/\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{z}_l$, and similarly for $g_{ik\bar{p}}$ and $g_{\bar{j}\bar{l}r}$. By (27),

$$g_{i\bar{j}k\bar{l}} = q\,(\log\rho)_{i\bar{j}k\bar{l}} + (\log v)_{i\bar{j}k\bar{l}}.$$

By the same argument as in the proof of Theorem 2 — the one used to derive (22) there — it is easy to see that

$$\rho^4 (\log v)_{i\bar{j}k\bar{l}} \in \mathcal{A}^m.$$

On the other hand, similarly as with (29),

$$(\log\rho)_{i\bar{j}k\bar{l}} = \frac{\text{a polynomial in } \rho \text{ and its derivatives of order} \leq 4}{\rho^4}$$

so $\rho^4(\log\rho)_{i\bar{j}k\bar{l}} \in C^\infty(\overline{\Omega})$. Thus

$$\rho^4 g_{i\bar{j}k\bar{l}} \in \mathcal{A}^m. \tag{36}$$

In order to deal with the second term in (35), let us again use (27) to get

$$g_{ik\bar{p}} = q\,(\log\rho)_{ik\bar{p}} + (\log v)_{ik\bar{p}}.$$

Employing once more the argument from the proof of Theorem 2 — the one used in deriving (22) — we find that

$$\rho^3 (\log v)_{ik\bar{p}} = \rho B_{ik\bar{p}} + B\rho_i\rho_k\rho_{\bar{p}}, \qquad B_{ik\bar{p}}, B \in \mathcal{A}^m,$$

with $[B]_0|_{\partial\Omega} = 0$ (cf. (25)). On the other hand, by direct calculation

$$\rho^3(\log\rho)_{ik\bar{p}} = \rho C^\infty(\overline{\Omega}) + 2\rho_i\rho_k\rho_{\bar{p}}.$$

Thus $\rho^3 g_{ik\bar{p}} \in \rho\mathcal{A}^m + A\rho_i\rho_k\rho_{\bar{p}}$, with $A \in \mathcal{A}^m$ satisfying $A|_{\partial\Omega} = 2q$. Similarly for $\rho^3 g_{\bar{j}\bar{l}r}$. Consequently,

$$\begin{aligned} \rho^6 g^{\bar{p}r} g_{ik\bar{p}} g_{\bar{j}\bar{l}r} &= g^{\bar{p}r} \cdot (\rho\mathcal{A}^m + A\rho_i\rho_k\rho_{\bar{p}}) \cdot (\rho\mathcal{A}^m + A\rho_{\bar{j}}\rho_{\bar{l}}\rho_r) \\ &= g^{\bar{p}r} \cdot (\rho^2\mathcal{A}^m + \rho\mathcal{A}^m + A^2\rho_i\rho_{\bar{j}}\rho_k\rho_{\bar{l}}\rho_{\bar{p}}\rho_r). \end{aligned} \tag{37}$$

As $g^{\bar{p}r} \in \rho\mathcal{A}^m$ by Theorem 3(ii), the first two summands give a contribution belonging to $\rho^2\mathcal{A}^m$. In the last summand, we have by Proposition 7 and Lemma 6,

$$\begin{aligned} g^{\bar{p}r}\rho_{\bar{p}}\rho_r &= [\log\rho]^{\bar{p}s} H^r_s \rho_{\bar{p}}\rho_r \\ &= \rho^2 C^\infty(\overline{\Omega}) H^r_s \rho_r \in \rho^2\mathcal{A}^m, \end{aligned}$$

whence also $A^2\rho_i\rho_{\bar j}\rho_k\rho_{\bar l}g^{\bar p r}\rho_{\bar p}\rho_r \in \rho^2\mathcal{A}^m$. Thus

$$\rho^4 g^{\bar p r} g_{ik\bar p} g_{\bar j \bar l r} \in \mathcal{A}^m,$$

which together with (36) implies that, indeed, $\rho^4 R_{i\bar j k\bar l} \in \mathcal{A}^m$.

It remains to compute the leading term. By a simple calculation,

$$\rho^4(\log\rho)_{i\bar j k\bar l} \to -6\rho_i\rho_{\bar j}\rho_k\rho_{\bar l} \qquad \text{as } \rho \to 0.$$

Next, similarly as with (22), one checks that

$$\rho^4(\log v)_{i\bar j k\bar l} = \rho\mathcal{A}^m + \rho_i\rho_{\bar j}\rho_k\rho_{\bar l}B$$

where $B \in \mathcal{A}^m$ satisfies $B|_{\partial\Omega} = 0$ (in fact, $B = O(\rho^m \log\rho)$ as in (25)). Thus

$$\rho^4 g_{i\bar j k\bar l}\,\big|_{\partial\Omega} = -6q\rho_i\rho_{\bar j}\rho_k\rho_{\bar l}. \tag{38}$$

Next, observe that, by Lemma 6 and Proposition 7 again,

$$\begin{aligned} g^{\bar p r}\cdot A\rho_i\rho_k\rho_{\bar p} &= [\log\rho]^{\bar p s} H^r_s A\rho_i\rho_k\rho_{\bar p} \\ &= \rho^2 C^\infty(\overline{\Omega}) H^r_s A\rho_i\rho_k \in \rho^2\mathcal{A}^m, \end{aligned}$$

and similarly for $g^{\bar p r}\cdot A\rho_{\bar j}\rho_{\bar l}\rho_r$; thus in (37), the only contribution to $\rho^4 g^{\bar p r} g_{ik\bar p} g_{\bar j\bar l r}|_{\partial\Omega}$ comes from the last summand there:

$$\begin{aligned} \rho^4 g^{\bar p r} g_{ik\bar p} g_{\bar j\bar l r}\,\big|_{\partial\Omega} &= \rho^{-2} g^{\bar p r} A^2 \rho_i\rho_{\bar j}\rho_k\rho_{\bar l}\rho_{\bar p}\rho_r\,\big|_{\partial\Omega} \\ &= A^2\rho_i\rho_{\bar j}\rho_k\rho_{\bar l}\cdot\rho^{-2} H^r_s [\log\rho]^{\bar p s}\rho_{\bar p}\rho_r\,\big|_{\partial\Omega} \\ &= A^2\rho_i\rho_{\bar j}\rho_k\rho_{\bar l}\big|_{\partial\Omega}\cdot H^r_s\big|_{\partial\Omega}\cdot \frac{[\log\rho]^{\bar p s}\rho_{\bar p}\rho_r}{\rho^2}\bigg|_{\partial\Omega} \\ &= 4q^2\rho_i\rho_{\bar j}\rho_k\rho_{\bar l}\cdot\frac{\delta^r_s}{q}\cdot\frac{[\log\rho]^{\bar p s}\rho_{\bar p}\rho_r}{\rho^2}\bigg|_{\partial\Omega} \\ &= 4q^2\rho_i\rho_{\bar j}\rho_k\rho_{\bar l}\cdot\frac{1}{q}\cdot(-1) \qquad \text{by Lemma 8} \\ &= -4q\rho_i\rho_{\bar j}\rho_k\rho_{\bar l}. \end{aligned}$$

Thus by (38)

$$\rho^4 R_{i\bar j k\bar l}\,\big|_{\partial\Omega} = -2q\rho_i\rho_{\bar j}\rho_k\rho_{\bar l},$$

completing the proof of part (iii).

Finally, for the part (v), recall that by Theorem 2

$$w := \rho^{n+1} g \in \mathcal{A}^m_*.$$

Thus

$$\mathrm{Ric}_{i\bar l} = -(n+1)(\log\rho)_{i\bar l} + (\log w)_{i\bar l}$$

and

$$R = g^{\bar{l}i}[-(n+1)(\log\rho)_{i\bar{l}} + (\log w)_{i\bar{l}}].$$

Using (22) again and Proposition 7 and Lemma 6, just replacing v by w, we get as before,

$$\begin{aligned} g^{\bar{l}i}(\log w)_{i\bar{l}} &= g^{\bar{l}i}\Big(\frac{B_{i\bar{l}}}{\rho} + \frac{\rho_i\rho_{\bar{l}}}{\rho^2}\,B\Big) \\ &= \frac{g^{\bar{l}i}}{\rho}\,B_{i\bar{l}} + [\log\rho]^{\bar{l}k}H^i_k\rho_i\rho_{\bar{l}}\,\frac{B}{\rho^2} \\ &= B_{i\bar{l}}\cdot\mathcal{A}^m + \rho^2 C^\infty(\overline{\Omega})H^i_k\rho_i\,\frac{B}{\rho^2} \in \mathcal{A}^m, \end{aligned}$$

and $g^{\bar{l}i}(\log w)_{i\bar{l}}\big|_{\partial\Omega} = 0$ since both $B_{i\bar{l}}$ and B vanish on the boundary by (25). Furthermore,

$$\begin{aligned} g^{\bar{l}i}(\log\rho)_{i\bar{l}} &= [\log\rho]^{\bar{l}k}H^i_k(\log\rho)_{i\bar{l}} \\ &= \delta^k_i H^i_k = \operatorname{tr}\mathbf{H} \end{aligned}$$

belongs to $\mathcal{A}^m$ and equals $\dfrac{-n}{n+1}$ on $\partial\Omega$ by Proposition 7. Thus $R \in \mathcal{A}^m$ and $R|_{\partial\Omega} = n$, completing the proof of the last part, (v), of Theorem 3. □

Proof of parts (c)–(g) *of Theorem 1.* These all follow from the parts (i)–(v), respectively, of Theorem 3 upon taking $F = K$, which is of the form (14), with $m = n+1$ and $q = -n-1$, in view of (10).

Remark. Analogously to Lemma 4, it is possible to prove that

$$v \in \mathcal{A}^m_*, v \neq 0 \implies \log v \in \mathcal{A}^m.$$

(Indeed, pulling out $\log\eta_0$, we may assume that $\eta_0 \equiv 1$, and then the assertion follows from the Taylor expansion $\log(1+x) = \sum_{k=1}^\infty (-1)^k x^k/k$ for the logarithm.) Instead of the factorization $F = \rho^q v$, $v \in \mathcal{A}^m_*$, as in (14), we may therefore use $F = \rho^q e^w$, $w \in \mathcal{A}^m$. This simplifies some of the formulas in the proofs of Theorems 2 and 3 slightly (for instance, instead of $\frac{v_{i\bar{j}}}{v} - \frac{v_i v_{\bar{j}}}{v^2}$ we get just $w_{i\bar{j}}$); unfortunately, it does not seem to make any simpler the proofs themselves.

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