

Kybernetika

VOLUME 43 (2007), NUMBER 1

The Journal of the Czech Society for
Cybernetics and Information Sciences

Published by:

Institute of Information Theory
and Automation of the AS CR, v.v.i.

Editor-in-Chief:

Milan Mareš

Managing Editors:

Karel Sladký
Lucie Fajfrová

Editorial Board:

Jiří Anděl, Sergej Čelikovský, Marie
Demlová, Petr Hájek, Jan Flusser, Martin
Janžura, Jan Ježek, George Klir, Ivan
Kramosil, Tomáš Kroupa, Friedrich Liese,
Jean-Jacques Loiseau, František Matúš,
Radko Mesiar, Jiří Outrata, Jan Štecha,
Olga Štěpánková, Igor Vajda, Jiřina
Vejnarová, Miloslav Vošvrda, Pavel Zítek

Editorial Office:

Pod Vodárenskou věží 4, 182 08 Praha 8

Kybernetika is a bi-monthly international journal dedicated for rapid publication of high-quality, peer-reviewed research articles in fields covered by its title.

Kybernetika traditionally publishes research results in the fields of Control Sciences, Information Sciences, System Sciences, Statistical Decision Making, Applied Probability Theory, Random Processes, Fuzziness and Uncertainty Theories, Operations Research and Theoretical Computer Science, as well as in the topics closely related to the above fields.

The Journal has been monitored in the Science Citation Index since 1977 and it is abstracted/indexed in databases of Mathematical Reviews, Current Mathematical Publications, Current Contents ISI Engineering and Computing Technology.

Kybernetika. Volume 43 (2007)

ISSN 0023-5954, MK ČR E 4902.

Published bimonthly by the Institute of Information Theory and Automation of the Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 4, 182 08 Praha 8. — Address of the Editor: P. O. Box 18, 182 08 Prague 8, e-mail: kybernetika@utia.cas.cz. — Printed by PV Press, Pod vrstevnicí 5, 140 00 Prague 4. — Orders and subscriptions should be placed with: MYRIS TRADE Ltd., P. O. Box 2, V Štíhlách 1311, 142 01 Prague 4, Czech Republic, e-mail: myris@myris.cz. — Sole agent for all “western” countries: Kubon & Sagner, P. O. Box 34 01 08, D-8 000 München 34, F.R.G.

Published in March 2007.

© Institute of Information Theory and Automation of the AS CR, v.v.i., Prague 2007.

ON BAYESIAN ESTIMATION IN AN EXPONENTIAL DISTRIBUTION UNDER RANDOM CENSORSHIP

MICHAL FRIESL AND JAN HURT

The paper gives some basic ideas of both the construction and investigation of the properties of the Bayesian estimates of certain parametric functions of the parent exponential distribution under the model of random censorship assuming the Koziol–Green model. Various prior distributions are investigated and the corresponding estimates are derived. The stress is put on the asymptotic properties of the estimates with the particular stress on the Bayesian risk. Small sample properties are studied via simulations in the special case.

Keywords: exponential distribution, random censoring, survival data analysis, reliability, Koziol–Green model, Bayesian estimates, Bayesian risk, conjugate priors, asymptotic properties, small sample properties, simulation study

AMS Subject Classification: 62F10, 62F15, 62N05, 62F12, 62F11

1.. INTRODUCTION AND PRELIMINARIES

The exponential distribution is still one of the most popular distribution in survival data analysis and has been extensively studied by many authors. The basic ideas are given in [7]. The comparison of various reliability estimates from the confidential point of view has been given in [6]. A nice test of fit with the Koziol–Green model for random censorship has been suggested by Herbst [5]. More advanced models are treated in Franz [2]. A review of the topic can be found in [8]. Since the processes studied in reliability theory and survival data analysis are rather evolutionary than revolutionary, the prior information seems to be useful to improve the inference. The Bayesian approach is one possible way to implement a prior information into the model. In estimating reliability function and parameter of exponential distribution, Sarhan [13] exploits past experiments to approximate prior density. Liang [10] deals with random censorship with exponentially distributed censor, i. e. in the setting (1.3) but with known parameter of censoring distribution and in fact with restriction $\gamma < 1$ ($p > 1/2$) imposed by a prior. In [1] Jeffreys priors under several censoring mechanisms are derived, Bayesian estimates in the case of exponential distribution being treated in detail. Bayesian estimation for parameters of generalized exponential distribution under Type II censorship is dealt with in [12]. The present paper discusses Bayesian estimation in the exponential distribution under the Koziol–Green model of censorship. Several priors are proposed and corresponding

estimators of characteristics of the distribution and the model are derived. Properties of the estimators are expressed in terms of almost sure convergence, asymptotic normality and Bayesian risks. Weak asymptotics of the Bayesian reliability estimator considered as a stochastic process is under the conjugate prior (2.3) studied in [4].

Let X_1, \dots, X_n be independent identically distributed (i.i.d.) random variables (r.v.'s) with the distribution function F , the density function f and let T_1, \dots, T_n be i.i.d. r.v. which are independent of X_j 's and possess the distribution function G and the density function g . In the model of random censorship we can only observe the i.i.d. pairs

$$(W_1, I_1), \dots, (W_n, I_n), \quad (1.1)$$

where $W_j = \min(X_j, T_j)$, $I_j = I\{X_j \leq T_j\}$, $j = 1, \dots, n$. R.v.'s X_j usually represent the lifetimes or times-to-failure while T_j 's represent time censors. The pair (W_1, I_1) has the distribution with the density function

$$h(w, i) = \{f(w)[1 - G(w)]\}^i \{g(w)[1 - F(w)]\}^{1-i}, \quad w \in \mathbb{R}, \quad i = 0, 1 \quad (1.2)$$

with respect to Lebesgue \times counting product measure.

In the *Koziol-Green model* [9] it is supposed that the distributions of X_j 's and T_j 's are connected by

$$1 - G(t) = [1 - F(t)]^\gamma \quad (1.3)$$

for some $\gamma > 0$. In this case W_1 and I_1 are independent (see Herbst [5], e.g.). Instead of γ , we can consider the parameter

$$p = \Pr[X \leq T], \quad (1.4)$$

$p \in (0, 1)$, since $p = 1/(1 + \gamma)$. The density (1.2) then becomes

$$f(w)[1 - F(w)]^\gamma \gamma^{1-i} \quad w \in \mathbb{R}, \quad i = 0, 1. \quad (1.5)$$

In the present paper we suppose that X_j 's have an exponential distribution $\text{Exp}(\theta)$, i.e., that they possess the density function

$$f(x; \theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0, \quad (1.6)$$

with the expectation $E \text{Exp}(\theta) = \theta$ and variance $\text{var} \text{Exp}(\theta) = \theta^2$, or after introducing a new parameter $\lambda = 1/\theta$

$$f(x; \lambda) = \lambda e^{-\lambda x}, \quad x > 0, \quad (1.7)$$

where λ represents the hazard rate of the distribution in question. Note that under the above assumptions W_1 possesses an $\text{Exp}(p/\lambda)$ and I_1 is a zero-one r.v. with the parameter p . Also we will pay attention to the *reliability function*

$$R = e^{-\lambda} \quad (1.8)$$

at the mission time set to $t := 1$. Next we will mention further distributions used in this paper.

Gamma distribution $G(a, q)$. Gamma distribution with the parameters $a > 0$, $q > 0$ has the density function

$$\frac{a^q}{\Gamma(q)} x^{q-1} e^{-ax}, \quad x > 0$$

with the expectation $E G(a, q) = q/a$ and the variance $\text{var } G(a, q) = q/a^2$.

Logarithmic Gamma Distribution $LG(a, q)$. If the distribution of the r.v. X is $G(a, q)$ then the distribution of $Y = e^{-X}$ is $LG(a, q)$ with the density function

$$\frac{a^q}{\Gamma(q)} (-\ln y)^{q-1} y^a \frac{1}{y}, \quad y \in (0, 1)$$

with the expectation

$$E LG(a, q) = \left(\frac{a}{a+1} \right)^q$$

and the variance

$$\text{var } LG(a, q) = \left(\frac{a}{a+2} \right)^q - \left(\frac{a}{a+1} \right)^{2q}.$$

Inverse Gamma distribution $IG(a, q)$. If the distribution of the r.v. X is $G(a, q)$ then the distribution of $Z = 1/X$ is $IG(a, q)$ with the density function

$$\frac{a^q}{\Gamma(q)} \frac{1}{z^{q+1}} e^{-a/z}, \quad z > 0.$$

If $q > 1$ then the expectation is $E IG(a, q) = a/(q-1)$. If $q > 2$ then the variance is

$$\text{var } IG(a, q) = \frac{a^2}{(q-1)^2(q-2)}.$$

Beta distribution $B(r, s)$. Beta distribution with the parameters $r > 0$, $s > 0$ has the density function

$$\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1} (1-x)^{s-1}, \quad x \in (0, 1).$$

with the expectation $E B(r, s) = r/(r+s)$ and the variance

$$\text{var } B(r, s) = \frac{rs}{(r+s)^2(r+s+1)}.$$

Beta distribution of the second order $B2(r, s)$. If the distribution of the r.v. X is $B(r, s)$, then the distribution of $V = 1/X - 1$ is $B2(r, s)$ with the density function

$$\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \frac{v^{s-1}}{(1+v)^{r+s}}, \quad v > 0.$$

If $r > 1$ then the expectation is $E B2(r, s) = s/(r-1)$. If $r > 2$ then the variance is

$$\text{var } B2(r, s) = \frac{(s+1)s}{(r-1)(r-2)} - \left(\frac{s}{r-1} \right)^2 = \frac{s(r+s-1)}{(r-1)^2(r-2)}.$$

Likelihood. In Bayesian inference we deal with the likelihood function which, under the above assumptions, reads

$$L(\lambda, \gamma; (W_1, I_1), \dots, (W_n, I_n)) = \prod_{j=1}^n (\lambda e^{-\lambda W_j} (e^{-\lambda W_j})^\gamma) \gamma^{1-I_j}.$$

If we denote

$$W = \sum_{j=1}^n W_j, \quad I = \sum_{j=1}^n I_j \quad (1.9)$$

then the likelihood becomes

$$L(\lambda, \gamma; W, I) = \lambda^n e^{-\lambda W} e^{-\lambda \gamma W} \gamma^{n-I}, \quad \lambda > 0, \gamma > 0. \quad (1.10)$$

For given λ , p , or γ we obtain the expected values and variances of the conditional distributions of I_1 and W_1 :

$$E[I_1 | \lambda, p] = p, \quad \text{var}[I_1 | \lambda, p] = p(1-p)$$

and

$$E[W_1 | \lambda, p] = p/\lambda, \quad \text{var}[W_1 | \lambda, p] = p^2/\lambda^2.$$

These facts will be utilized for calculating the unconditional expectations of I and W (which are the sums of i.i.d. with the same distributions as I_1 and W_1) utilizing the a priori knowledge. Also we will use them to establish the law of large numbers in particular cases.

Bayesian estimation. We restrict ourselves to the Bayesian estimates under the quadratic loss function, i. e. the estimator minimizing Bayesian risk function $E(\tau - \hat{\tau})^2$ if τ is the parameter in question. Thus the Bayesian estimate is simply the expected value of τ with respect to the posterior distribution, $\hat{\tau} = E(\tau | W, I)$, in our case. The Bayesian risk of this estimate can be expressed in different ways as

$$\begin{aligned} \varrho_\tau^* &= E(E[\tau - E(\tau | W, I)]^2 | W, I) = E[\tau - E(\tau | W, I)]^2 \\ &= E \text{ var}(\tau | W, I) = \text{var } \tau - \text{var}[E(\tau | W, I)] = \text{var } \tau - \text{var } \hat{\tau}. \end{aligned} \quad (1.11)$$

As for computational aspects, the formula

$$\varrho_\tau^* = E \text{ var}(\tau | W, I) \quad (1.12)$$

is helpful if we know the posterior variance. In words, we just take the expectation of it.

2.. PRIORS AND BAYESIAN ESTIMATORS

Conjugate prior

The natural conjugate prior for (1.10) is the system of densities

$$\left\{ \frac{a^{c+1}}{\Gamma(c-b+1)\Gamma(b)} \lambda^c e^{-\lambda a} e^{-\lambda \gamma a} \gamma^{c-b}; a > 0, b > 0, c > b-1 \right\} \quad (2.1)$$

or, after changing the parameters $a = a$, $b = r$, and $c = s + r - 1$

$$\mathbf{K}_{\lambda, \gamma} = \left\{ \frac{a^{r+s}}{\Gamma(r)\Gamma(s)} \lambda^{r+s-1} e^{-\lambda(1+\gamma)a} \gamma^{s-1}; a > 0, r > 0, s > 0 \right\}, \quad (2.2)$$

see also Franz [2].

$$\text{Denote} \quad K_{a,r,s}(\lambda, \gamma) = \frac{a^{r+s}}{\Gamma(r)\Gamma(s)} \lambda^{r+s-1} e^{-\lambda(1+\gamma)a} \gamma^{s-1} \quad (2.3)$$

the corresponding density function.

Theorem 2.1. The system $\mathbf{K}_{\lambda, \gamma}$ is conjugate with $L(\lambda, \gamma|W, I)$, the marginal distribution of λ is $G(a, r)$ and that of γ is $B2(r, s)$.

Proof. The proof is obvious. \square

Remark. It follows from the above Theorem that $E \lambda = r/a$ and for $r > 1$, $E \gamma = s/(r-1)$. Moreover, the conditional distributions are

$$\begin{aligned} K_{a,r,s}(\lambda|\gamma) &\sim \lambda^{r+s-1} e^{-\lambda(1+\gamma)a} \sim G((1+\gamma)a, r+s), \\ K_{a,r,s}(\gamma|\lambda) &\sim \gamma^{s-1} e^{-\gamma\lambda a} \sim G(\lambda a, s). \end{aligned}$$

For $r > 1$ we have $\text{cov}(\lambda, \gamma) = -\frac{s}{a} \frac{1}{r-1}$ and for $r > 2$ we have $\text{corr}(\lambda, \gamma) = -\sqrt{\frac{s(r-2)}{r(r+s-1)}}$.

Theorem 2.2. If we choose the prior density as $q(\lambda, \gamma) = K_{a,r,s}(\lambda, \gamma) \in \mathbf{K}_{\lambda, \gamma}$ then the a posteriori density is $q(\lambda, \gamma|W, I) = K_{a+W, r+I, s+n-I}(\lambda, \gamma) \in \mathbf{K}_{\lambda, \gamma}$. The corresponding Bayesian estimates under the quadratic loss function are

$$\begin{aligned} \hat{\lambda} &= \frac{I+r}{W+a}, \quad \hat{\gamma} = \frac{n-I+s}{I+r-1}, \\ \hat{R} &= \left(\frac{W+a}{W+a+1} \right)^{I+r}, \quad \hat{\theta} = \frac{W+a}{I+r-1} \text{ for } r > 1, \quad \hat{p} = \frac{I+r}{n+r+s}. \end{aligned}$$

Proof. The form of the a posteriori density follows from the construction of the conjugate priors. The first two estimates are simply the expected values with respect to the a posteriori densities. After substitution $R = e^{-\lambda}$, $\theta = \lambda^{-1}$, and $p = (1+\gamma)^{-1}$ the marginal priors are $K_{a,r,s}(R) \sim LG(a, r)$, $K_{a,r,s}(\theta) \sim IG(a, r)$, and $K_{a,r,s}(p) \sim B(r, s)$, respectively. Therefore, the respective posterior distributions are $K_{a+W, r+I, s+n-I}(R)$, $K_{a+W, r+I, s+n-I}(\theta)$, and $K_{a+W, r+I, s+n-I}(p)$. Taking expectations of these distributions we obtain the remaining estimates. \square

Remark. The expectations of I_1 and W_1 are

$$\begin{aligned} E I_1 &= E p = r/(r+s), \\ E W_1 &= E(p/\lambda) = E \frac{1}{\lambda(1+\gamma)} = \frac{a}{r+s-1}, \quad \text{if } r+s > 1. \end{aligned} \quad (2.4)$$

Independent priors

A flexible system of independent priors is given by the density function

$$q(\lambda, \gamma) = \frac{a^r}{\Gamma(r)} \lambda^{r-1} e^{-a\lambda} \frac{b^s}{\Gamma(s)} \gamma^{s-1} e^{-b\gamma}, \quad a > 0, r > 0, b > 0, s > 0, \quad (2..5)$$

i.e., λ and γ are independent with distributions $G(a, r)$ and $G(b, s)$, respectively. The corresponding posterior distribution possesses the density function

$$q(\lambda, \gamma | W, I) \sim \lambda^{n+r-1} e^{-\lambda(a+w)} e^{-\lambda\gamma w} e^{-b\gamma} \gamma^{n-i+s-1}. \quad (2..6)$$

In this case, it is not possible to express the usual Bayesian estimates in a close form with integrals evaluated. Still we are able to give so called *Bayesian estimates of the maximum likelihood type* or *generalized Bayesian estimates* which maximize the posterior density function.

Theorem 2.3. Suppose that λ and γ possess the density function (2..5), $n+r > 1$, and $n - I + s - 1 > 1$. Then

$$\tilde{\lambda} = \frac{1}{2} \left[\frac{I+r-s}{a+W} - \frac{b}{W} + \sqrt{\frac{(I+r-s)^2}{(a+W)^2} + \frac{b^2}{W^2} + \frac{2b(2(n+r-1) - (I+r-s))}{W(a+W)}} \right]$$

and

$$\tilde{\gamma} = \frac{1}{2} \left[-\frac{I+r-s}{b} - \frac{a+W}{W} + \sqrt{\frac{(I+r-s)^2}{b^2} + \frac{(a+W)^2}{W^2} + 2\frac{I+r-s}{b} \frac{a+W}{W} + \frac{4(n-I+s-1)(a+W)}{Wb}} \right]$$

are the generalized Bayesian estimates.

Proof. The logarithm of the investigated posterior density function

$$\ell(\lambda, \gamma) = (n+r-1) \ln \lambda - \lambda(a+w) - \lambda\gamma w - b\gamma + (n-I+s-1) \ln \gamma.$$

takes its maximum at

$$\gamma(\lambda) = \frac{n-I+s-1}{b+W\lambda},$$

for fixed $\lambda > 0$. Further,

$$\frac{d\ell(\lambda, \frac{n-I+s-1}{b+W\lambda})}{d\lambda} = \frac{n+r-1}{\lambda} - (a+W) - W \frac{n-I+s-1}{b+W\lambda} = \frac{\partial \ell(\lambda, \gamma)}{\partial \lambda} \Big|_{\gamma=\gamma(\lambda)}.$$

Multiplying the last expression by $\lambda(b+W\lambda)$ and setting it to zero we get the quadratic equation

$$-\lambda^2 (W(a+W)) + \lambda (W(I+r-s) - b(a+W)) + b(n+r-1) = 0$$

which has two real roots with the opposite signs so that only the positive one, i. e. $\tilde{\lambda}$, is meaningful. Then $\tilde{\gamma} = \gamma(\tilde{\lambda})$ provides the desired estimate for γ . It is not difficult to show that the pair really maximizes the posterior density. \square

Consider another system of independent priors for the pair (λ, p) :

$$q(\lambda, p) \sim \lambda^{q-1} e^{-a\lambda} p^{r-1} (1-p)^{s-1}, \quad \lambda > 0, p \in (0, 1), \quad (2.7)$$

where $a > 0$ and q, r, s are arbitrary prior parameters. In case $a > 0, q > 0, r > 0, s > 0$ we have $\lambda \sim G(a, q), p \sim B(r, s)$ and λ and γ remain independent with distributions $\lambda \sim G(a, q)$ and $\gamma \sim B2(r, s)$, respectively. The density function of the corresponding posterior distribution is

$$q(\lambda, \gamma | W, I) \sim \lambda^{n+q-1} e^{-a\lambda} e^{-\lambda W} e^{-\lambda \gamma W} \frac{\gamma^{n-I+s-1}}{(1+\gamma)^{r+s}},$$

with the marginal density of γ

$$q(\gamma | W, I) \sim \frac{\gamma^{n-I+s-1}}{(1+\gamma)^{r+s} (\gamma W + W + a)^{n+q}},$$

and the conditional distribution of λ given (γ, W, I) is $q(\lambda | \gamma, W, I) = G(\gamma W + W + a, n + q)$.

Under a more specific choice of prior parameters we can get explicit results.

Theorem 2.4. If $a = 0, q > -n, q > -r, r > 0, s > 0$, then

$$\hat{\lambda} = \frac{I + r + q}{W} \frac{n + q}{n + q + r + s}, \quad \hat{p} = \frac{I + r + q}{n + r + q + s}$$

are the Bayesian estimates under the quadratic loss function. If, moreover, $I + r + q > 1$, then

$$\hat{\theta} = \frac{W}{I + r + q - 1} \frac{n + q + s + r - 1}{n + q - 1} \quad \text{and} \quad \hat{\gamma} = \frac{n - I + s}{I + r + q - 1}$$

are the Bayesian estimates under the quadratic loss function.

Proof. The posterior density function of (λ, p) is

$$q(\lambda, p | W, I) \sim \lambda^{n+q-1} e^{-\lambda W/p} p^{-n+I+r-1} (1-p)^{n-I+s-1}$$

so that

$$q(p | W, I) = B(I + r + q, n - I + s), \quad q(\lambda | p, W, I) = G(W/p, n + q).$$

The first two estimates are then calculated as

$$\begin{aligned} \hat{p} &= E(p | W, I) = \frac{I + r + q}{n + r + q + s}, \\ \hat{\lambda} &= E(\lambda | W, I) = E(E[\lambda | p, W, I] | W, I) = E\left(\frac{n + q}{W} p | W, I\right). \end{aligned}$$

Further, γ has the posterior distribution

$$\begin{aligned} q(\gamma|W, I) &\sim \frac{\gamma^{n-I+s-1}}{(1+\gamma)^{r+s}(\gamma W + W)^{n+q}} \sim \frac{\gamma^{n-I+s-1}}{(1+\gamma)^{n+r+s+q}} \\ &\sim B2(I+r+q, n-I+s) \end{aligned}$$

so that $\hat{\gamma} = E(\gamma|W, I)$. The conditional distribution of θ given (p, W, I) is

$$q(\theta|p, W, I) \sim IG(W/p, n+q)$$

so that

$$\begin{aligned} \hat{\theta} &= E(\theta|W, I) = E(E[\theta|p, W, I]|W, I) = E\left(\frac{W}{(n+q-1)p} \middle| W, I\right) \\ &= \frac{W}{n+q-1} \left(1 + E\left[\left(\frac{1}{p} - 1\right)|W, I\right]\right) = \frac{W}{n+q-1} (1 + E[\gamma|W, I]) \\ &= \frac{W}{n+q-1} \left(1 + \frac{n-I+s}{I+r+q-1}\right) \end{aligned}$$

and the result follows. \square

With another restriction put on the prior parameters we can obtain the explicit form of the estimate of R .

Theorem 2.5. If $r = -s$, $s < q$, $s > 0$, $a > 0$, then

$$\hat{\lambda} = \frac{I+q-s}{W+a}, \quad \hat{R} = \left(\frac{W+a}{W+a+1}\right)^{I+q-s}$$

are the Bayesian estimates under the quadratic loss function. If, moreover $I+q-s > 1$ then

$$\hat{\theta} = \frac{W+a}{I+q-s-1}, \quad \hat{\gamma} = \frac{W+a}{W} \frac{n-I+s}{I+q-s-1}$$

are the Bayesian estimates under the quadratic loss function.

Proof. The estimates are simply the posterior expectations again. The only explanation is needed for $\hat{\gamma}$:

$$\begin{aligned} \hat{\gamma} &= E(\gamma|W, I) = \left(\int_0^\infty \frac{\gamma^{n-I+s}}{\left(1+\gamma\frac{W}{W+a}\right)^{n+q}} d\gamma\right) \left(\int_0^\infty \frac{\gamma^{n-I+s-1}}{\left(1+\gamma\frac{W}{W+a}\right)^{n+q}} d\gamma\right)^{-1} \\ &= \frac{W+a}{W} \frac{\int_0^\infty \frac{\gamma^{n-I+s}}{(1+\gamma)^{n+q}} d\gamma}{\int_0^\infty \frac{\gamma^{n-I+s-1}}{(1+\gamma)^{n+q}} d\gamma} = \frac{W+a}{W} E[B2(I+q-s, n-I+s)], \end{aligned}$$

hence the result. \square

Jeffrey's prior

The Fisher information matrix is

$$\mathbf{J}(\lambda, \gamma) = \mathbb{E} \left[- \begin{pmatrix} -\frac{1}{\lambda^2} & -\frac{W_1}{\gamma^2} \\ -\frac{W_1}{\gamma^2} & -\frac{1}{\gamma^2}(1 - I_1) \end{pmatrix} \right] = \begin{pmatrix} \frac{1}{\lambda^2} & \frac{1}{\lambda(1+\gamma)} \\ \frac{1}{\lambda(1+\gamma)} & \frac{1}{\gamma(1+\gamma)} \end{pmatrix}.$$

Hence the Jeffrey's prior density function of the pair (λ, γ) may be expressed as

$$q(\lambda, \gamma) \sim |\det \mathbf{J}(\lambda, \gamma)|^{1/2} = \frac{1}{\lambda(1+\gamma)\sqrt{\gamma}} \quad (2.8)$$

with marginal densities

$$q(\lambda) \sim \frac{1}{\lambda}, \quad q(\gamma) \sim \frac{1}{(1+\gamma)\sqrt{\gamma}}, \quad \lambda \text{ and } \gamma \text{ independent}.$$

Theorem 2.6. Suppose the Jeffrey's prior distribution (2.8). Then

$$\hat{\lambda} = \frac{I + \frac{1}{2}}{W} \frac{n}{n+1}, \quad \hat{\theta} = \frac{W}{I - \frac{1}{2}} \frac{n}{n-1}, \quad \hat{p} = \frac{I + \frac{1}{2}}{n+1}, \quad \hat{\gamma} = \frac{n - I + \frac{1}{2}}{I - \frac{1}{2}}$$

are the Bayesian estimates under the quadratic loss function.

Proof. Jeffrey's prior is the special case of Theorem 2.4 where we put $q = 0$, $r = s = \frac{1}{2}$. \square

3.. ASYMPTOTIC RESULTS

In this Section we present some asymptotic results for the estimates given above. For the purpose of this Section denote $\bar{I}_n = I/n$ and $\bar{W}_n = W/n$.

Bayesian risks

Theorem 3.1. If $r > 8$ then for the natural conjugate prior (2.2)

$$\begin{aligned} \lim_{n \rightarrow \infty} n \varrho_{\lambda}^* &= \frac{r(r+s+1)}{a^2} \\ \lim_{n \rightarrow \infty} n \varrho_{\theta}^* &= \frac{a^2(r+s-3)}{(r-1)(r-2)(r-3)} \\ \lim_{n \rightarrow \infty} n \varrho_{\gamma}^* &= \frac{(r+s-1)(r+s-2)s}{(r-1)(r-2)(r-3)} \\ \lim_{n \rightarrow \infty} n \varrho_p^* &= \frac{rs}{(r+s)(r+s-1)} \\ \lim_{n \rightarrow \infty} n \varrho_R^* &= \frac{ra^{r-1}}{(a+2)^{r+1}} \left(s + \frac{a}{a+2}(r+1) \right) \end{aligned}$$

are the Bayesian risks of the corresponding estimates.

Proof. Since the posterior variances are known, we can compute the Bayesian risk using (1.12). Derivation of the first four formulæ is analogous so that we present the details for ϱ_λ^* only. We make use of Theorem 5, b1) in Hurt [7] to obtain the asymptotic expansion for

$$\varrho_\lambda^* = \mathbb{E} \frac{I+r}{(W+a)^2} = \frac{1}{n} \mathbb{E} \frac{\bar{I}_n + r/n}{(\bar{W}_n + a/n)^2}.$$

Put $g(w, i, n) = (i+r/n)/(w+r/n)^2$, $q = 1$, $u = 0$, $p = 6$ in the mentioned theorem. We have $|g(w, i, n)| \leq C_1 + C_2 n^2$, $\max[\mathbb{E}_*(\bar{I}_n - \mathbb{E} I_1)^2, \mathbb{E}_*(\bar{W}_n - \mathbb{E}_* W_1)^2] = C_3/n$, $\max[\mathbb{E}_*(\bar{I}_n - \mathbb{E} I_1)^6, \mathbb{E}_*(\bar{W}_n - \mathbb{E}_* W_1)^6] = C_4/n^3 + O(1/n^4)$, where we denote $\mathbb{E}_*(\cdot) = \mathbb{E}[\cdot|\lambda, \gamma]$, so that the assumptions of the Theorem 5, loc.cit., are satisfied. If we moreover add the existence of $\mathbb{E}(\mathbb{E}_* W_1)^6$ ($r+s > 6$) we can even conclude (following the proof of the mentioned theorem) $\mathbb{E} g(\bar{W}_n, \bar{I}_n, n) = \mathbb{E} g(\mathbb{E}_* \bar{W}_n, \mathbb{E}_* \bar{I}_n, n) + O(1/n)$ and the result follows.

The calculation of ϱ_R^* is a bit more complicated. If we denote (and apply expansion of $\ln(1+x)$)

$$\begin{aligned} A_1 &= (I+r) \ln \left(1 + \frac{2}{W+a} \right) = 2 \frac{I+r}{W+a} - \frac{2^2}{2} \frac{I+r}{(W+a)^2} + R_1, \\ A_2 &= 2(I+r) \ln \left(1 + \frac{1}{W+a} \right) = 2 \frac{I+r}{W+a} - \frac{2}{2} \frac{I+r}{(W+a)^2} + R_2, \end{aligned}$$

where $0 \leq R_1 \leq \frac{2^3}{3} \frac{I+r}{(W+a)^3}$ and $0 \leq R_2 \leq \frac{2}{3} \frac{I+r}{(W+a)^3}$, the Bayesian risk is

$$n\varrho_R^* = \mathbb{E} n(\exp(-A_1) - \exp(-A_2)).$$

Since

$$\begin{aligned} \exp \left(2 \frac{I+r}{(W+a)^2} \right) &= 1 + 2 \frac{I+r}{(W+a)^2} + \frac{e^{S_1}}{2!} \left(2 \frac{I+r}{(W+a)^2} \right)^2, \\ \exp(-R_1) &= 1 - e^{-T_1} R_1, \end{aligned}$$

where $0 \leq S_1 \leq 2 \frac{I+r}{(W+a)^2}$ and $0 \leq T_1 \leq R_1$, we have

$$\begin{aligned} \exp(-A_1) &= e^{-2 \frac{I+r}{W+a}} (1 - e^{-T_1} R_1) \left(1 + 2 \frac{I+r}{(W+a)^2} \right) \\ &\quad + e^{-2 \frac{I+r}{W+a}} e^{-R_1} \frac{e^{S_1}}{2!} \left(2 \frac{I+r}{(W+a)^2} \right)^2 \\ &= e^{-2 \frac{I+r}{W+a}} \left(1 + 2 \frac{I+r}{(W+a)^2} \right) + Q_1, \end{aligned}$$

where

$$\begin{aligned} |Q_1| &\leq e^{-2 \frac{I+r}{W+a}} e^{-T_1} R_1 \left(1 + 2 \frac{I+r}{(W+a)^2} \right) + \frac{1}{2!} \left(2 \frac{I+r}{(W+a)^2} \right)^2 e^{-2 \frac{I+r}{W+a} - R_1 + S_1} \\ &\leq 1 \cdot 1 \cdot \frac{8}{3} \frac{I+r}{(W+a)^3} \left(1 + 2 \frac{I+r}{(W+a)^2} \right) + 2 \left(\frac{I+r}{(W+a)^2} \right)^2 e^{-A_1} \\ &\leq K_1 \frac{n}{W^3} + K_2 \frac{n^2}{W^5} + K_3 \frac{n^2}{W^4} \cdot 1. \end{aligned}$$

Similarly we could get

$$\exp(-A_2) = e^{-2\frac{I+r}{W+a}} \left(1 + \frac{I+r}{(W+a)^2} \right) + Q_2.$$

Together

$$n\mathcal{Q}_R^* = E \, n e^{-2(I+r)/(W+a)} \left(2\frac{I+r}{(W+a)^2} - \frac{I+r}{(W+a)^2} \right) + E \, n(Q_1 - Q_2).$$

Applying the cited theorem to the first term finishes the proof, since the second term is of the order $O(1/n)$: $E(1/W)^k$ is $O(1/n^k)$ (given $\lambda, \gamma, 1/W \sim IG(\lambda(1+\gamma), n)$). \square

Almost sure convergence

It has been shown that given $\lambda, \gamma, p = 1/(1+\gamma), R = e^{-\lambda}, \theta = 1/\lambda$

$$E I_1 = p, \quad \text{var } I_1 = p(1-p), \quad E W_1 = p/\lambda, \quad \text{var } W_1 = p^2/\lambda^2$$

hold. Since the derived estimates are of the similar form we can state the general theorem concerning the almost sure convergence.

Theorem 3.2. If a, c_1, c_2, c are arbitrary constants then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n - I + c_1}{I + c_2} &= \gamma \text{ a.s.}, & \lim_{n \rightarrow \infty} \frac{I + c_1}{n + c_2} &= p \text{ a.s.}, \\ \lim_{n \rightarrow \infty} \frac{I + c_1}{W + c_2} &= \lambda \text{ a.s.}, & \lim_{n \rightarrow \infty} \frac{W + c_2}{I + c_1} &= \theta \text{ a.s.}, \\ \lim_{n \rightarrow \infty} \left(\frac{W + a}{W + a + 1} \right)^{I+c} &= R \text{ a.s.} \end{aligned}$$

holds.

Proof. The assertion is a simple consequence of the fact that $\bar{I}_n \rightarrow p$ and $\bar{W}_n \rightarrow p/\lambda$ almost surely for $n \rightarrow \infty$. \square

Asymptotic normality

Theorem 3.3. If a, c_1, c_2, c are arbitrary constants then

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(\frac{I + c_1}{n + c_2} - p \right) \right) &= N(0, p(1-p)), \\ \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(\frac{n - I + c_1}{I + c_2} - \gamma \right) \right) &= N(0, \gamma(1+\gamma)^2), \\ \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(\frac{I + c_1}{W + c_2} - \lambda \right) \right) &= N \left(0, \frac{\lambda^2}{p} \right), \\ \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(\frac{W + c_1}{I + c_2} - \theta \right) \right) &= N \left(0, \frac{\theta^2}{p} \right), \end{aligned}$$

$$\lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(\left(\frac{W+a}{W+a+1} \right)^{I+c} - R \right) \right) = N \left(0, \frac{R^2 (\ln R)^2}{p} \right).$$

Proof. It follows from the central limit theorem for i.i.d. with the finite nonzero variance that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(\frac{I}{n} - p \right) \right) &= N(0, p(1-p)), \\ \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(\frac{W}{n} - \frac{p}{\lambda} \right) \right) &= N(0, p^2/\lambda^2). \end{aligned}$$

Using Cramér–Slutsky and Sverdrup theorem we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(\frac{I+c_1}{n+c_2} - p \right) \right) &= \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \frac{n}{n+c_2} \left(\frac{I}{n} - p \right) + \sqrt{n} \frac{c_1}{n+c_2} + \sqrt{n} \left(\frac{n}{n+c_2} - 1 \right) p \right) \\ &= \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(\frac{I}{n} - p \right) \right) = N(0, p(1-p)), \\ \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(\frac{n-I+c_1}{I+c_2} - \gamma \right) \right) &= \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \frac{I}{I+c_2} \left(\frac{n-I}{I} - \gamma \right) + \sqrt{n} \frac{c_1}{I+c_2} + \sqrt{n} \left(\frac{I}{I+c_2} - 1 \right) \gamma \right) \\ &= \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(\frac{n-I}{I} - \gamma \right) \right) = \lim_{n \rightarrow \infty} \mathcal{L} \left(\frac{\sqrt{n}(1-\bar{I}_n - \gamma \bar{I}_n)}{\bar{I}_n} \right) \\ &= \lim_{n \rightarrow \infty} \mathcal{L} \left(\frac{\sqrt{n}(p - \bar{I}_n)}{p\bar{p}} \right) = N \left(0, \frac{p(1-p)}{p^4} \right) \\ &= N \left(0, \frac{(1-p)}{p^3} \right) = N(0, \gamma(1+\gamma)^2). \end{aligned}$$

Also

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(\frac{I+c_1}{W+c_2} - \lambda \right) \right) &= \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \frac{W}{W+c_2} \left(\frac{I}{W} - \lambda \right) + \sqrt{n} \frac{c_1}{W+c_2} + \sqrt{n} \left(\frac{W}{W+c_2} - 1 \right) \lambda \right) \\ &= \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(\frac{I}{W} - \lambda \right) \right) = \lim_{n \rightarrow \infty} \mathcal{L} \left(\frac{\sqrt{n}(\bar{I}_n - \lambda \bar{W}_n)}{\bar{W}_n} \right) \\ &= \lim_{n \rightarrow \infty} \mathcal{L} \left(\frac{\bar{I}_n - \lambda \bar{W}_n}{p/\lambda} \right) = N \left(0, \frac{1}{p^2/\lambda^2} \left(p(1-p) + \lambda^2 \frac{p^2}{\lambda^2} \right) \right) \\ &= N \left(0, \frac{\lambda^2}{p} \right), \end{aligned}$$

since W_1 and I_1 are independent and thus

$$\begin{pmatrix} \bar{I}_n - p \\ \bar{W}_n - p/\lambda \end{pmatrix} \xrightarrow[n \rightarrow \infty]{D} N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} p(1-p) & 0 \\ 0 & p^2/\lambda^2 \end{pmatrix} \right)$$

and the limiting distribution $\lim_{n \rightarrow \infty} \mathcal{L}(\bar{I}_n - \lambda \bar{W}_n)$ may be obtained as the limit of the linear combination of the independent normal distributions. Similarly

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(\frac{W + c_1}{I + c_2} - \theta \right) \right) &= \\ &= \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(\frac{W}{I} - \theta \right) + \sqrt{n} \left(\frac{W}{I + c_2} - \frac{W}{I} \right) + \sqrt{n} \frac{c_1}{I + c_2} \right) \\ &= \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \frac{1}{\bar{I}_n} (\bar{W}_n - \theta \bar{I}_n) \right) \\ &= N \left(0, \frac{1}{p^2} \left(\frac{p^2}{\lambda^2} + \theta^2 p(1-p) \right) \right) = N \left(0, \theta^2 \left(1 + \frac{1-p}{p} \right) \right) \\ &= N \left(0, \frac{\theta^2}{p} \right). \end{aligned}$$

To justify the next relationship let us note that

$$n^{2/3} \frac{I}{(W + a)^2} = n^{-1/3} \frac{\bar{I}_n}{(\bar{W}_n + \frac{a}{n})^2} \xrightarrow[n \rightarrow \infty]{} 0 \text{ a. s.},$$

so that $I/(W + a)^2 = o(n^{-2/3})$ for $n \rightarrow \infty$ with probability 1 which together with

$$Q_1 = \sum_{m=2}^{\infty} \frac{(-1)^{m+1}}{m(W + a)^m} = O \left(\frac{1}{(W + a)^2} \right) \quad \text{for } n \rightarrow \infty \text{ a. s.},$$

(following from the expansion $\ln(1 + x)$ for $x \rightarrow 0$) gives

$$\sqrt{n} I Q_1 \xrightarrow[n \rightarrow \infty]{} 0 \text{ a. s.}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(I \ln \left(1 + \frac{1}{W + a} \right) - \lambda \right) \right) &= \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \left(\frac{I}{W + a} + I Q_1 \right) \right) \\ &= \lim_{n \rightarrow \infty} \mathcal{L} \left(\sqrt{n} \frac{I}{W + a} \right) = N \left(0, \frac{\lambda^2}{p} \right). \end{aligned}$$

It now follows that

$$n^{1/3} \left(I \ln \left(1 + \frac{1}{W + a} \right) - \lambda \right) \xrightarrow[n \rightarrow \infty]{P} 0$$

and

$$\sqrt{n} \left(I \ln \left(1 + \frac{1}{W + a} \right) - \lambda \right)^2 \xrightarrow[n \rightarrow \infty]{P} 0.$$

We use Taylor expansion of the exponential function to derive the last assertion for $c = 0$. Let us denote

$$Q_2 = \sum_{m=2}^{\infty} \frac{I \ln(1 + \frac{1}{W+a}) - \lambda}{m!}.$$

There exist $K > 0$ and $\delta > 0$ such that for $|I \ln(1 + 1/(W+a)) - \lambda| < \delta$ we have $|Q_2| < K(I \ln(1 + 1/(W+a)) - \lambda)^2$ and

$$\Pr[|\sqrt{n}e^{-\lambda}Q_2| > \varepsilon] \leq \Pr[\sqrt{n}e^{-\lambda}K(I \ln(1 + 1/(W+a)) - \lambda)^2 > \varepsilon] \xrightarrow{n \rightarrow \infty} 0$$

for every $\varepsilon > 0$ or

$$\sqrt{n}e^{-\lambda}Q_2 \xrightarrow[n \rightarrow \infty]{P} 0.$$

Hence

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \left(\sqrt{n} \left(\left(\frac{W+a}{W+a+1} \right)^I - R \right) \right) \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left(\sqrt{n} e^{-\lambda} (e^{-I \ln \frac{W+a+1}{W+a} + \lambda} - 1) \right) \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left(\sqrt{n} e^{-\lambda} \left(1 - \frac{I \ln(1 + \frac{1}{W+a}) - \lambda}{1!} + Q_2 - 1 \right) \right) \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left(\sqrt{n} e^{-\lambda} (I \ln(1 + \frac{1}{W+a}) - \lambda) \right) \\ &= N \left(0, e^{-2\lambda} \frac{\lambda^2}{p} \right) = N \left(0, \frac{R^2 (\ln R)^2}{p} \right). \end{aligned}$$

For arbitrary c

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \left(\sqrt{n} \left(\left(\frac{W+a}{W+a+1} \right)^{I+c} - R \right) \right) \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left(\sqrt{n} \left(\frac{W+a}{W+a+1} \right)^c \right. \\ &\quad \times \left. \left(\left(\frac{W+a}{W+a+1} \right)^I - R \right) - \sqrt{n} R \left(1 - \left(\frac{W+a}{W+a+1} \right)^c \right) \right) \\ &= N \left(0, \frac{R^2 (\ln R)^2}{p} \right), \end{aligned}$$

since

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{W+a}{W+a+1} \right)^c &= \left(\frac{\bar{W}_n + \frac{a}{n}}{\bar{W}_n + \frac{a+1}{n}} \right)^c = 1 \text{ a. s.}, \\ \sqrt{n} \left(\left(\frac{W+a}{W+a+1} \right)^I - R \right) &\xrightarrow[n \rightarrow \infty]{D} N \left(0, \frac{R^2 (\ln R)^2}{p} \right) \quad \text{and} \end{aligned}$$

$$\lim_{n \rightarrow \infty} \sqrt{n} R \left(1 - \left(\frac{W+a}{W+a+1} \right)^c \right) = \lim_{n \rightarrow \infty} \sqrt{n} R \left(1 - \left(1 - \frac{1}{W+a+1} \right)^c \right)$$

$$= \lim_{n \rightarrow \infty} \sqrt{n}R \left(1 - \left(1 + O \left(\frac{1}{W + a + 1} \right) \right) \right) = 0 \text{ a.s.} \quad \square$$

4.. SIMULATION

In case of the independent gamma priors (2.5) we were not able to express the Bayesian risks in a close form. To get an idea about the behaviour of the Bayesian risks we performed a limited simulation study. The prior parameters were chosen as to achieve the expected value of λ equal to 1 and the expected value of γ corresponding to the portion of uncensored observations $p = 1/(1 + \gamma)$ equal to 0.5, 0.8, and 0.9. The accuracy of the prior knowledge is expressed by the coefficients of variation (the standard deviation divided by the mean) V_λ and V_γ equal to 0.5, 0.3, and 0.1. The estimated n -multiples of the Bayesian risks for 5000 realizations of the samples of size $n = 20, 50$, and 100 are summarized in the following tables.

$n \cdot \varrho_\lambda^*$	$V_\gamma = 0.5$			n	$V_\gamma = 0.3$			n	$V_\gamma = 0.1$			
V_λ	1.33	1.12	1.01	20	1.22	1.04	1.05	20	1.06	1.01	0.95	V_λ
=	1.72	1.29	1.16	50	1.60	1.23	1.21	50	1.23	1.12	1.12	=
0.5	2.02	1.39	1.24	100	1.88	1.31	1.21	100	1.42	1.19	1.16	0.5
p	0.5	0.8	0.9		0.5	0.8	0.9		0.5	0.8	0.9	p
V_λ	0.86	0.70	0.69	20	0.80	0.71	0.70	20	0.70	0.69	0.69	V_λ
=	1.30	1.01	0.96	50	1.21	0.94	0.90	50	0.93	0.87	0.90	=
0.3	1.62	1.12	1.04	100	1.47	1.09	0.99	100	1.12	1.04	0.96	0.3
p	0.5	0.8	0.9		0.5	0.8	0.9		0.5	0.8	0.9	p
V_λ	0.18	0.16	0.16	20	0.17	0.16	0.17	20	0.16	0.16	0.17	V_λ
=	0.39	0.34	0.33	50	0.37	0.33	0.34	50	0.34	0.33	0.34	=
0.1	0.63	0.55	0.51	100	0.63	0.53	0.50	100	0.53	0.51	0.49	0.1
	$V_\gamma = 0.5$			n	$V_\gamma = 0.3$			n	$V_\gamma = 0.1$			

$n \cdot \varrho_\gamma^*$	$V_\gamma = 0.5$			n	$V_\gamma = 0.3$			n	$V_\gamma = 0.1$			
V_λ	2.154	0.193	0.047	20	1.219	0.089	0.020	20	0.184	0.012	0.002	V_λ
=	3.240	0.297	0.085	50	2.154	0.164	0.042	50	0.434	0.028	0.006	=
0.5	4.082	0.346	0.109	100	2.865	0.234	0.062	100	0.783	0.051	0.011	0.5
p	0.5	0.8	0.9		0.5	0.8	0.9		0.5	0.8	0.9	p
V_λ	2.114	0.191	0.048	20	1.134	0.091	0.020	20	0.181	0.011	0.002	V_λ
=	3.227	0.286	0.082	50	2.029	0.172	0.042	50	0.426	0.028	0.006	=
0.3	3.677	0.340	0.109	100	2.755	0.233	0.062	100	0.768	0.052	0.011	0.3
p	0.5	0.8	0.9		0.5	0.8	0.9		0.5	0.8	0.9	p
V_λ	1.712	0.184	0.050	20	1.047	0.087	0.020	20	0.183	0.012	0.002	V_λ
=	2.293	0.259	0.079	50	1.639	0.155	0.041	50	0.389	0.028	0.006	=
0.1	2.812	0.311	0.101	100	2.152	0.237	0.063	100	0.738	0.052	0.011	0.1
	$V_\gamma = 0.5$			n	$V_\gamma = 0.3$			n	$V_\gamma = 0.1$			

We can see that under the accurate knowledge of λ the Bayesian risk is not too much influenced either by the accuracy of the knowledge of p or the value p itself. Similarly, under the accurate knowledge of γ the risk is not influenced by the accuracy of λ . The risk ϱ_γ^* substantially depends on the portion of uncensored

observations p , however. Obviously, with the increasing portion of the uncensored observations and increasing sample sizes the risks decrease.

ACKNOWLEDGEMENT

The first author was supported by the Research Plan MSM 4977751301 of the Ministry of Education, Youth and Sports of the Czech Republic. The second author was supported by the Ministry of Education, Youth and Sports of the Czech Republic under Project MSM 0021620839 “Methods of Modern Mathematics and Its Applications”.

(Received February 3, 2006.)

REFERENCES

-
- [1] F. De Santis, J. Mortera, and A. Nardi: Jeffreys priors for survival models with censored data. *J. Statist. Plann. Inference* 99 (2001), 2, 193–209.
 - [2] J. Franz: On Estimation Problems in Random Censored Repair Models. *Econom. Quality Control* 9 (1994), No. 3, 125–142.
 - [3] M. Friesl: Bayesian Methods in Censored Samples. Charles University, Faculty of Mathematics and Physics, Prague 1995. (In Czech)
 - [4] M. Friesl: Weak asymptotics of the Bayes estimator of the reliability function in the Koziol–Green model. *Statist. Decisions* 19 (2001), 1, 83–87.
 - [5] T. Herbst: Test of fit with the Koziol–Green model for random censorship. *Statist. Decisions* 10 (1992), 163–171.
 - [6] J. Hurt: Comparison of some reliability estimators in the exponential case under random censorship. In: *Proc. 5th Pannonian Symp. on Math. Statist.* (W. Grossmann, J. Mogyoródi, and W. Wertz, eds.), Visegrád 1985, pp. 255–266.
 - [7] J. Hurt: Asymptotic expansions for moments of functions of stochastic processes and their Applications. *Statist. Decisions* 4 (1992), 251–271.
 - [8] J. Hurt: On Statistical Methods for Survival Data Analysis. In: *Proc. Summer School ROBUST’92* (J. Antoch and G. Dohnal, eds.), Union of the Czech Mathematicians and Physicists, Prague 1992, pp. 54–74.
 - [9] J. A. Koziol and S. B. Green: A Cramér–von Mises statistic for randomly censored data. *Biometrika* 63 (1976), 465–474.
 - [10] T. Liang: Empirical Bayes estimation with random right censoring. *Internat. J. Inform. Manag. Sci.* 15 (2004), 4, 1–12.
 - [11] H. F. Martz and R. A. Waller: *Bayesian Reliability Analysis*. Wiley, New York 1982.
 - [12] M. Z. Raqab and M. T. Madi: Bayesian inference for the generalized exponential distribution. *J. Statist. Comput. Simulation* 75 (2005), 10, 841–852.
 - [13] A. M. Sarhan: Empirical Bayes estimates in exponential reliability model. *Appl. Math. Comput.* 135 (2003), 2–3, 319–332.

Michal Friesl, Department of Mathematics, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 22, 306 14 Plzeň. Czech Republic.

e-mail: friesl@kma.zcu.cz

Jan Hurt, Department of Probability and Mathematical Statistics, Faculty of Mathematics and Physics – Charles University, Sokolovská 83, 186 75 Praha 8. Czech Republic.

e-mail: hurt@karlin.mff.cuni.cz