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MONOGENICITY OF PROBABILITY MEASURES BASED ON MEASURABLE SETS INVARIANT UNDER FINITE GROUPS OF TRANSFORMATIONS

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Let $\mathcal{A}$ denote a σ-algebra of subsets of a set Ω, G a finite group of $(\mathcal{A},\mathcal{A})$-measurable transformations $g:\Omega\to\Omega$, $F(G)$ the set consisting of all $\omega\in\Omega$ such that $g(\omega)=\omega$, $g\in G$, is fulfilled, and let $\mathcal{B}(G,\mathcal{A})$ stand for the σ-algebra consisting of all sets $A\in\mathcal{A}$ satisfying $g(A)=A$, $g\in G$. Under the assumption $f(B)\in\mathcal{A}^{|G|}$, $B\in\mathcal{B}(G,\mathcal{A})$, for $f:\Omega\to\Omega^{|G|}$ defined by $f(\omega)=(g_1(\omega),\dots,g_{|G|}(\omega))$, $\omega\in\Omega$, $\{g_1,\dots,g_{|G|}\}=G$, where $|G|$ stands for the number of elements of G, $\Omega^{|G|}$ for the $|G|$-fold Cartesian product of Ω, and $\mathcal{A}^{|G|}$ for the $|G|$-fold direct product of $\mathcal{A}$, it is shown that a probability measure P on $\mathcal{A}$ is uniquely determined among all probability measures on $\mathcal{A}$ by its restriction to $\mathcal{B}(G,\mathcal{A})$ if and only if $P^*(F(G))=1$ holds true and that $F(G)\in\mathcal{A}$ is equivalent to the property of $\mathcal{A}$ to separate all points $\omega_1,\omega_2\in F(G)$, $\omega_1\neq\omega_2$, and $\omega\in F(G)$, $\omega'\notin F(G)$, by a countable system of sets contained in $\mathcal{A}$. The assumption $f(B)\in\mathcal{A}^{|G|}$, $B\in\mathcal{B}(G,\mathcal{A})$, is satisfied, if Ω is a Polish space and $\mathcal{A}$ the corresponding Borel σ-algebra.

1. INTRODUCTION

The main result of this article concerns characterizations of the property of a probability measure P defined on a σ-algebra $\mathcal{A}$ of subsets of a set Ω to be uniquely determined among all other probability measures defined on $\mathcal{A}$ by its restriction to some sub-σ-algebra $\mathcal{B}$, which consists in this article of all sets $A\in\mathcal{A}$ satisfying $A=g(A)$, $g\in G$, where G denotes a finite group of $(\mathcal{A},\mathcal{A})$-measurable transformations $g:\Omega\to\Omega$. For example the results of the second part of this article might be applied to the special group of permutations acting on $\mathbb{R}^n$ or the finite group consisting of 2^n elements acting on $\mathbb{R}^n$ by changing the sign of the coordinates. In the first case a probability measure P on $\mathcal{B}(\mathbb{R}^n)$, where $\mathcal{B}(\mathbb{R}^n)$ is introduced as the Borel-σ-algebra of $\mathbb{R}^n$, is uniquely determined by its restriction to the sub-σ-algebra of $\mathcal{B}(\mathbb{R}^n)$ consisting of all permutation-invariant Borel subsets of $\mathbb{R}^n$, if and only if $P(\Delta)=1$ is valid, where Δ stands for the diagonal of $\mathbb{R}^n$. In the second case, a probability measure P on $\mathcal{B}(\mathbb{R}^n)$ is uniquely determined by its restriction to the sub-σ-algebra of $\mathcal{B}(\mathbb{R}^n)$ consisting of all sign-invariant Borel subsets of $\mathbb{R}^n$, if and only if P is already the one-point mass at the origin of $\mathbb{R}^n$.

In the sequel the underlying model for the investigation of problems of the preceding type will be introduced and studied in detail.

The starting point is the following generalization of a result concerning groups of permutations (cf. [4]) to arbitrary finite groups of transformations.

Lemma 1. Let $\mathcal{A}$ denote a σ-algebra of subsets of some set Ω, G a finite group of $(\mathcal{A},\mathcal{A})$-measurable transformations $g:\Omega\to\Omega$, $\mathcal{B}(G,\mathcal{A})$ the σ-algebra consisting of all $A\in\mathcal{A}$ satisfying $A=g(A)$, $g\in G$, and $\mathcal{C}$ an algebra of subsets of Ω generating $\mathcal{A}$. Then $\mathcal{B}(G,\mathcal{A})$ is generated by $\{\bigcup_{g\in G} g(C) : C\in\mathcal{C}\}$.

Proof. Let $\mathcal{D}$ denote the σ-algebra generated by $\{\bigcup_{g\in G} g(C) : C\in\mathcal{C}\}$. Then $\mathcal{D}\subset\mathcal{B}(G,\mathcal{A})$ holds true, whereas the inclusion $\mathcal{B}(G,\mathcal{A})\subset\mathcal{D}$ will follow from the observation that $\mathcal{M}$ introduced as the set consisting of all $A\in\mathcal{A}$ such that $\bigcup_{g\in G} g(A)\in\mathcal{D}$ is fulfilled, is a monotone class, since $\mathcal{M}$ already contains the algebra $\mathcal{C}$ generating $\mathcal{A}$. Clearly $\bigcup_n A_n\in\mathcal{M}$ is valid for any increasing sequence $A_n\in\mathcal{M}$, $n\in\mathbb{N}$, because of $\bigcup_n(\bigcup_{g\in G} g(A_n))=\bigcup_{g\in G}(\bigcup_n g(A_n))$. Furthermore, for any decreasing sequence $A_n\in\mathcal{M}$, $n\in\mathbb{N}$, $\omega\in\bigcap_n(\bigcup_{g\in G} g^{-1}(A_n))$ implies that for any $n\in\mathbb{N}$ there exists some $g_n\in G$ satisfying $g_n(\omega)\in A_n$, i.e. there exists a $g\in G$ such that $g(\omega)\in A_n$ for infinite many $n\in\mathbb{N}$ is fulfilled, since G is finite. Hence, $g(\omega)\in\bigcap_n A_n$ holds true, i.e. the inclusion $\bigcap_n(\bigcup_{g\in G} g^{-1}(A_n))\subset\bigcup_{g\in G}(g^{-1}(\bigcap_n A_n))$ has been shown, whereas the inclusion $\bigcup_{g\in G}(g^{-1}(\bigcap_{n\in\mathbb{N}} A_n))\subset\bigcap_n(\bigcup_{g\in G} g^{-1}(A_n))$ is obvious. Therefore, $\bigcap_n(\bigcup_{g\in G} g^{-1}(A_n))\in\mathcal{D}$ has been proved for any decreasing sequence $A_n\in\mathcal{M}$, i.e. $\mathcal{M}$ is a monotone class. □

Remarks.

(i) The assertion of Lemma 1 does not hold longer true, in general, for countable groups of transformations, as the following special case shows:
Let Ω stand for the set $\mathbb{R}$ of real numbers and $\mathcal{A}$ for the Borel σ-algebra of $\mathbb{R}$, which might be generated by the algebra $\mathcal{C}$ consisting of all finite unions of pairwise disjoint intervals of the type $(a,b]$, where a,b, $a<b$, are rational numbers including $-\infty$ and ∞. Furthermore, G is introduced by the countable group consisting of all transformations $g_\rho:\mathbb{R}\to\mathbb{R}$ defined by $g_\rho(x)=x+\rho$, $x\in\mathbb{R}$, where ρ is some rational number. Then $\bigcup_\rho g_\rho(\sum_{i=1}^n (a_i,b_i])$, $n\in\mathbb{N}\cup\{0\}$, is equal to $\mathbb{R}$ in the case $n\in\mathbb{N}$ and empty in the case $n=0$, i.e. the σ-algebra generated by $\bigcup_\rho g_\rho(\sum_{i=1}^n (a_i,b_i])$, $a_i<b_i$, a_i,b_i rational, $i=1,\dots,n$, $n\in\mathbb{N}\cup\{0\}$ is equal to $\{\emptyset,\mathbb{R}\}$, whereas $\mathcal{B}(G,\mathcal{A})\neq\{\emptyset,\mathbb{R}\}$ holds true, since the set consisting of all rational numbers belongs to $\mathcal{B}(G,\mathcal{A})$.

(ii) The special case of Lemma 1, where G is the group acting as permutations on $\mathbb{R}^n$ together with $\mathcal{A}$ as the Borel σ-algebra of $\mathbb{R}^n$ leads to a short proof of the well-known fact that $\mathcal{B}(G,\mathcal{A})$ is induced by the order statistics $T:\mathbb{R}^n\to\mathbb{R}^n$ sending $(x_1,\dots,x_n)\in\mathbb{R}^n$ to the corresponding n-tuple, which is increasingly ordered, i.e. $T^{-1}(\mathcal{A})=\mathcal{B}(G,\mathcal{A})$ is valid in this case.

(iii) Let G_j denote finite groups of transformations with underlying σ-algebras $\mathcal{A}_j$, $j=1,2$, then Lemma 1 implies $\mathcal{B}(G_1\times G_2,\ \mathcal{A}_1\otimes\mathcal{A}_2)=\mathcal{B}(G_1,\mathcal{A}_1)\otimes\mathcal{B}(G_2,\mathcal{A}_2)$.

Further applications of Lemma 1 concern a characterization of the atoms of $\mathcal{B}(G,\mathcal{A})$ and the property of $\mathcal{B}(G,\mathcal{A})$ to be countably generated.

Corollary 1. Let $\mathcal{A}$ denote a σ-algebra of subsets of a set Ω, G a finite group of $(\mathcal{A},\mathcal{A})$-measurable transformations $g:\Omega\to\Omega$, and $\mathcal{B}(G,\mathcal{A})$ the σ-algebra consisting of all the sets $A\in\mathcal{A}$ satisfying $A=g(A),\ g\in G$.

Then the following assertions hold true:

(i) $B\in\mathcal{B}(G,\mathcal{A})$ is an atom of $\mathcal{B}(G,\mathcal{A})$ if and only if $B=\bigcup_{g\in G} g(A)$ is valid for an atom A of $\mathcal{A}$,

(ii) $\mathcal{B}(G,\mathcal{A})$ is countably generated if and only if there exists a countably generated σ-algebra $\mathcal{A}'\subset\mathcal{A}$ such that $g:\Omega\to\Omega$ is $(\mathcal{A}',\mathcal{A}')$-measurable, $g\in G$, and $\mathcal{B}(G,\mathcal{A}')=\mathcal{B}(G,\mathcal{A})$ is valid.

Proof. For the proof of part (i) let $A\in\mathcal{A}$ denote an atom of $\mathcal{A}$. Then $B\in\mathcal{B}(G,\mathcal{A})$ defined by $\bigcup_{g\in G} g(A)$ is an atom of $\mathcal{B}(G,\mathcal{A})$, since $g(A),\ g\in G$, are atoms of $\mathcal{A}$, too. Therefore, $C\cap g(A)$ is equal to $g(A)$ or empty, $g\in G$, where $C\in\mathcal{B}(G,\mathcal{A})$ is some subset of B, i.e. $C=\bigcup_{g\in H} g(A),\ H\subset G$. Now $g(C)=C,\ g\in G$, implies $C=\bigcup_{g\in G} g(A)$, if H is not empty, which shows that $C=B$ is valid or C is empty, i.e. B given by $\bigcup_{g\in G} g(A)$, where A stands for some atom of $\mathcal{A}$, is indeed an atom of $\mathcal{B}(G,\mathcal{A})$.

For the proof of the converse implication let $B\in\mathcal{B}(G,\mathcal{A})$ stand for an atom of $\mathcal{B}(G,\mathcal{A})$. According to Lemma 1 there exists a countable subset $\mathcal{C}$ of $\mathcal{A}$ such that B already belongs to the σ-algebra $\mathcal{B}$ generated by $\{\bigcup_{g\in G} g(C) : C\in\mathcal{C}\}$. Let $B_i,\ i\in I$, stand for the atoms of $\mathcal{B}$ and $A_j,\ j\in J$, for the atoms of the σ-algebra $\mathcal{A}'$ generated by $\{g(C) : C\in\mathcal{C},\ g\in G\}$. Then $g:\Omega\to\Omega,\ g\in G$, is $(\mathcal{A}',\mathcal{A}')$-measurable according to Lemma 1, since one might replace $\mathcal{C}$ by the countable algebra generated by $\{g(C) : C\in\mathcal{C},\ g\in G\}$. Therefore, $\mathcal{B}=\mathcal{B}(G,\mathcal{A}')$ holds true and $\bigcup_{j\in J} A_j=\bigcup_{i\in I} B_i=\Omega$. According to the above considerations $\bigcup_{g\in G} g(A_j),\ j\in J$, is an atom of $\mathcal{B}=B(G,\mathcal{A}')$. Now $\bigcup_{j\in J}\bigcup_{g\in G} g(A_j)=\Omega$ and $\bigcup_{i\in I} B_i=\Omega$ shows that any $B_i,\ i\in I$, is of the type $\bigcup_{g\in G} g(A_j)$ for some $j\in J$. In particular, the atom $B\in\mathcal{B}(G,\mathcal{A})$ is of the type $\bigcup_{g\in G} g(A)$ for a certain set $A\in\{A_j : j\in I\}$. Now $A\in\mathcal{A}$ must be an atom of $\mathcal{A}$, since, otherwise, $B\in\mathcal{B}(G,\mathcal{A})$ would not be an atom of $\mathcal{B}(G,\mathcal{A})$, because $\bigcup_{g\in G} g(A')$ and $\bigcup_{g\in G} g(A\setminus A')$ are disjoint and their union coincides with $\bigcup_{g\in G} g(A)$ for any $A'\in\mathcal{A}$ satisfying $A'\subset A$, i.e. $\bigcup_{g\in G} g(A')=\emptyset$ or $\bigcup_{g\in G} g(A\setminus A')=\emptyset$ is valid, from which $A'=\emptyset$ or $A'=A$ follows.

For the proof of part (ii) let $\mathcal{A}'$ be some countably generated σ-algebra contained in $\mathcal{A}$ sucht that $g:\Omega\to\Omega$ is $(\mathcal{A}',\mathcal{A}')$-measurable, $g\in G$, and $\mathcal{B}(G,\mathcal{A}')=\mathcal{B}(G,\mathcal{A})$ holds true. Then $\mathcal{B}(G,\mathcal{A}')(=\mathcal{B}(G,\mathcal{A}))$ is countably generated according to Lemma 1.

For the proof of the converse implication one might choose $\mathcal{B}(G,\mathcal{A})$ for $\mathcal{A}'$. □

Remarks.

(i) Let $\mathcal{A}$ be a countably generated σ-algebra of subsets of a given set Ω. Then there exists a countably generated sub-σ-algebra $\mathcal{A}_1$ of $\mathcal{A}$ and a sub-σ-algebra

$\mathcal{A}_2$ of $\mathcal{A}$ containing $\mathcal{A}_1$ such that it is not countably generated and that $g : \Omega \to \Omega$, $g \in G$, is both $(\mathcal{A}_1, \mathcal{A}_1)$-measurable and $(\mathcal{A}_2, \mathcal{A}_2)$-measurable; further $\mathcal{B}(G, \mathcal{A}_1) = \mathcal{B}(G, \mathcal{A}_2) = B(G, \mathcal{A})$ holds true if and only if the set $\mathcal{E}$ consisting of all atoms of $\mathcal{A}$ not belonging to $\mathcal{B}(G, \mathcal{A})$ is uncountable, which might be proved as follows:

Starting from the assumption $\mathcal{B}(G, \mathcal{A}_2) = \mathcal{B}(G, \mathcal{A})$, where $\mathcal{A}$ is countably generated and where $\mathcal{A}_2$ is a sub-σ-algebra of $\mathcal{A}$ such that $g : \Omega \to \Omega$ is $(\mathcal{A}_2, \mathcal{A}_2)$-measurable, $g \in G$, it is sufficient to show that $\mathcal{A}_2$ is already countably generated, if $\mathcal{E}$ is countable. For this purpose one observes that $\mathcal{A} \cap \Omega_0^c \subset \mathcal{B}(G, \mathcal{A}) \cap \Omega_0^c = \mathcal{B}(G, \mathcal{A}_2) \cap \Omega_0^c \subset \mathcal{A}_2 \cap \Omega_0^c$ holds true for Ω_0 introduced as $\bigcup_{E \in \mathcal{E}} E$. Therefore, $\mathcal{A} \cap \Omega_0^c = \mathcal{A}_2 \cap \Omega_0^c$ is valid, from which it follows that $\mathcal{A}_2$ is countably generated.

For the proof of the other implication let $\mathcal{A}_2$ stand for the σ-algebra generated by $\mathcal{A}_1$ and the atoms of $\mathcal{A}$, where $\mathcal{A}_1$ coincides with $\mathcal{B}(G, \mathcal{A})$. It will be shown that $\mathcal{A}_2$ is not countably generated, if $\mathcal{E}$ is uncountable. The assumption on $\mathcal{A}_2$ to be countably generated results in an existence of a countable set $\{C_n : n \in \mathbb{N}\}$ of atoms of $\mathcal{A}$ such that, for any $A \in \mathcal{A}_2$, there exists a set $B \in \mathcal{A}_1$ satisfying $A \Delta B \subset \bigcup_{n=1}^{\infty} C_n$. Therefore, any $C_0 \in \mathcal{E} \setminus \{g(C_n) : n \in \mathbb{N}, g \in G\}$ satisfies $C_0 \Delta B_0 \subset \bigcup_{n=1}^{\infty} C_n$ for some $B_0 \in \mathcal{A}_1$, which leads to $C_0 \subset B_0$ because of $C_0 \cap C_n = \emptyset$, $n \in \mathbb{N}$. Finally, $C_0 \neq g_0(C_0)$ is valid for some $g_0 \in G$, which results in $g_0(C_0) \cap C_0 = \emptyset$, i. e. $g_0(C_0) \subset B_0 \cap C_0^c \subset \bigcup_{n=1}^{\infty} C_n$ holds true because of $g_0(C_0) \subset g_0(B_0) = B_0$. Hence, there exists a set C_{n_0} satisfying $g_0(C_0) = C_{n_0}$, i. e. one arrives at the contradiction $C_0 = g_0^{-1}(C_{n_0})$.

(ii) Let $\mathcal{A}$ stand for a σ-algebra of subsets of a set Ω, G for a group not necessarily finite, of $(\mathcal{A}, \mathcal{A})$-measurable transformations $g : \Omega \to \Omega$, and let $\mathcal{P}$ stand for the set consisting of all G-invariant probability measures P on $\mathcal{A}$, i. e. $P = P^g$, $g \in G$, is valid. Then it is well-known (cf. [1], p. 38 − 39) that the extremal points of $\mathcal{P}$ might be characterized by the property of G-ergodicity, i. e. $P \in \mathcal{P}$ is G-ergodic if and only if P restricted to the σ-algebra $\mathcal{A}_P$ consisting of all sets $A \in \mathcal{A}$ satisfying $P(A \Delta g(A)) = 0$, $g \in G$, is already $\{0, 1\}$-valued. In case G is finite, the property of $P \in \mathcal{P}$ to be G-ergodic is equivalent to the property of $P \in \mathcal{P}$ that its restriction to $\mathcal{B}(G, \mathcal{A})$ is $\{0, 1\}$-valued. Under the additional assumption that $\mathcal{A}$ is countably generated, any $P \in \mathcal{P}$ is G-ergodic, according to Corollary 1, if and only if there exist an atom $A \in \mathcal{A}$ and $g_k \in G$, $k = 1, \ldots, n$, such that $g_k(A)$, $k = 1, \ldots, n$, are pairwise disjoint and $P(g_k(A)) = \frac{1}{n}$, $k = 1, \ldots, n$, holds true. This result is not longer valid for infinite groups of transformations, as a special case shows in which the underlying set Ω is a compact, metrizable group G with $\mathcal{A}$ as the corresponding Borel σ-algebra. In this case $\mathcal{P}$ only contains the normalized Haar measure, if G is chosen for the corresponding group of $(\mathcal{A}, \mathcal{A})$-measurable transformations $g : \Omega \to \Omega$.

(iii) The conclusion that the property of $\mathcal{A}$ to be countably generated implies that $\mathcal{B}(G, \mathcal{A})$ is also countably generated might also be drawn from the observation that $\frac{1}{|G|} \sum_{g \in G} I_{g(A)}$, where $|G|$ stands for numbers of elements of G, is for any

$A \in \mathcal{A}$ a regular, proper version of the conditional distribution $P(A|\mathcal{B}(G, \mathcal{A}))$, where P is an arbitrary G-invariant probability measure on $\mathcal{A}$ (cf. [2]).

(iv) Let $\mathcal{A}_j$ denote σ-algebras of subsets of some set $\Omega_j,\ j = 1, \ldots, n$ $(n \geq 2)$. Then the atoms of the n-fold direct product $\mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n$ might be characterized by the property to be of the type $A_1 \times \ldots \times A_n$, where each $A_j \in \mathcal{A}_j$ is an atom of $\mathcal{A}_j,\ j = 1, \ldots, n$. Clearly, sets of this type are atoms of $\mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n$. The converse direction might be proved with the aid of the observation that any countably generated σ-algebra has atoms such that their union coincides with the underlying set. In particular, let G denote the symmetric group of order n acting as $(\mathcal{A}^n, \mathcal{A}^n)$-measurable permutations $g : \Omega^n \to \Omega^n$, where Ω^n stands for the n-fold Cartesian product of the set Ω and $\mathcal{A}^n$ for the n-fold direct product of the σ-algebra $\mathcal{A}$ of subsets of Ω. In this case, the atoms of $\mathcal{B}(G, \mathcal{A}^n)$ are of the type $\bigcup_{\pi \in \gamma_n} A_{\pi(1)} \times \ldots \times A_{\pi(n)}$, where $A_j \in \mathcal{A},\ j = 1, \ldots, n$, are atoms of $\mathcal{A}$ and γ_n is the symmetric group of order n consisting of all permutations $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

The conclusion of part (iii) of the preceding remark, namely that $\mathcal{B}(G, \mathcal{A})$ is countably generated for finite groups of $(\mathcal{A}, \mathcal{A})$-measurable transformations $g : \Omega \to \Omega$, if $\mathcal{A}$ is countably generated, is not in general valid for countable groups as the following example shows:

Example 1. Let Ω stand for the unit circle $\{\exp ix : x \in \mathbb{R}\}$ with the corresponding σ-algebra $\mathcal{A}$ and let P stand for the Haar measure of this compact group Ω with $P(\Omega) = 1$. Furthermore, let G be introduced as the countable group of $(\mathcal{A}, \mathcal{A})$-measurable transformations $g_\rho : \Omega \to \Omega$ defined by $g_\rho(e^{ix}) = e^{i(x+\rho)},\ x \in \mathbb{R},\ \rho \in \mathbb{Q}$, where $\mathbb{Q}$ stands for the set of rational numbers. It will be shown that P restricted to $\mathcal{B}(G, \mathcal{A})$ is $\{0, 1\}$-valued under the assumption that $\mathcal{B}(G, \mathcal{A})$ is countably generated, which results in the contradiction that $P(\{\exp i(x+\mathbb{Q})\}) = 1$ must be valid for some atom $\exp i(x+\mathbb{Q}),\ x \in \mathbb{R}$, of $\mathcal{B}(G, \mathcal{A})$. It remains to prove that one arrives, from the assumption on $\mathcal{B}(G, \mathcal{A})$ to be countably generated, at a $\{0, 1\}$-valued restriction of P to $\mathcal{B}(G, \mathcal{A})$, which might be seen as follows: For any set $\exp(iB) \in \mathcal{B}(G, \mathcal{A})$, where B is a Borel subset of $\mathbb{R}$, the equation $\exp(iB) \cap \exp i(B + \rho) = \exp(iB),\ \rho \in \mathbb{Q}$, yields $P(\exp(iB) \cap \exp i(B + \rho)) = P(\exp(iB)),\ \rho \in \mathbb{Q}$, from which $P(\exp(iB) \cap \exp i(B + x)) = P(\exp(iB)),\ x \in \mathbb{R}$, follows, since the function defined by $x \to P(\exp(iB) \cap \exp i(B + x)),\ x \in \mathbb{R}$, is continuous (cf. [6], p. 191). Therefore, for any $x \in \mathbb{R}$ and all sets $e^{iB} \in \mathcal{B}(G, \mathcal{A})$, where B is a Borel subset of $\mathbb{R}$, there exists a P-zero set N_x such that $I_{\exp(iB)}(\exp iy) \cdot I_{\exp i(B+x)}(\exp iy) = I_{\exp(iB)}(\exp iy)$ for $\exp iy \notin N_x$ and $y \in \mathbb{R}$ holds true, if $\mathcal{B}(G, \mathcal{A})$ is countably generated, since one might start from a countable algebra generating $\mathcal{B}(G, \mathcal{A})$ and apply a monotone class argument. Now $e^{iB} \in \mathcal{B}(G, \mathcal{A})$, where B is a Borel subset of $\mathbb{R}$, implies that $e^{i(B-x)} \in \mathcal{B}(G, \mathcal{A}),\ x \in \mathbb{R}$, which implies $I_{\exp(iB)}(\exp iy) \cdot I_{\exp i(B+x)}(\exp iy) = I_{\exp(iB)}(\exp iy)$ for all $\exp iy \notin N_0$ with $y \in \mathbb{R}$ and all $x \in \mathbb{R}$, from which one derives the equation $I_{\exp(iB)}(\exp iy) P(\exp i(y - B)) = I_{\exp(iB)}(\exp iy),\ \exp iy \notin N_0$ with $y \in \mathbb{R}$. Finally $P(\exp(iB)) > 0$ yields the existence of a value $\exp iy \in \exp iB$ satisfying $\exp iy \notin N_0$ with $y \in \mathbb{R}$, i. e. $P(\exp i(y - B)) = P(\exp(-iB)) = 1$ and, therefore, $P(\exp(iB)) = 1$ is valid, since $P(\exp(iB)) > 0$ implies $P(\exp(-iB)) > 0$, i. e. B might be replaced by $-B$.

2. MAIN RESULTS

In the sequel the property of a probability measure P on the σ-algebra $\mathcal{A}$ to be *monogenic* with respect to the σ-algebra $\mathcal{B}(G, \mathcal{A})$ consisting of all G-invariant sets belonging to $\mathcal{A}$, i. e. $A \in \mathcal{B}(G, \mathcal{A})$ if and only if $A = g(A)$, $g \in G$, holds true, will be characterized by properties of approximation, where P is called *monogenic* with respect to $\mathcal{B}(G, \mathcal{A})$ if and only if P is uniquely determined among all probability measures on $\mathcal{A}$ by its restriction $P|\mathcal{B}(G, \mathcal{A})$ to $\mathcal{B}(G, \mathcal{A})$.

Lemma 2. Let $\mathcal{A}$ denote a σ-algebra of subsets of a set Ω, G a finite group of $(\mathcal{A}, \mathcal{A})$-measurable transformations $g : \Omega \to \Omega$, and $\mathcal{B}(G, \mathcal{A})$ the σ-algebra of all G-invariant sets belonging to $\mathcal{A}$. Then a probability measure P on $\mathcal{A}$ is monogenic with respect to $\mathcal{B}(G, \mathcal{A})$ if and only if $P((\bigcup_{g \in G} g(A)) \setminus (\bigcap_{g \in G} g(A))) = 0$ holds true for any $A \in \mathcal{A}$.

Proof. Clearly, if P has this property of approximation, then P is monogenic with respect to $\mathcal{B}(G, \mathcal{A})$, since $\bigcap_{g \in G} g(A) \subset A \subset \bigcup_{g \in G} g(A)$ and $\bigcap_{g \in G} g(A)$, $\bigcup_{g \in G} g(A) \in \mathcal{B}(G, \mathcal{A})$, $A \in \mathcal{A}$, is valid.

For the proof of the converse implication one might start from the observation that $\bar{P}$ defined by $\frac{1}{|G|} \sum_{g \in G} P^g$ ($|G|$ number of elements of G) is a probability measure on $\mathcal{A}$, whose restriction $\bar{P}|\mathcal{B}(G, \mathcal{A})$ to $\mathcal{B}(G, \mathcal{A})$ coincides with $P|\mathcal{B}(G, \mathcal{A})$. Therefore, the property of P to be monogenic with respect to $\mathcal{B}(G, \mathcal{A})$ implies that P is already G-invariant, i. e. $P^g = P$, $g \in G$, holds true. Furthermore, P is an extremal point of the convex set consisting of all probability measures on $\mathcal{A}$ whose restriction to $\mathcal{B}(G, \mathcal{A})$ coincides with $P|\mathcal{B}(G, \mathcal{A})$. Hence, for any $A \in \mathcal{A}$, there exists a $B \in \mathcal{B}(G, \mathcal{A})$ satisfying $P(A \Delta B) = 0$, where Δ stands for the symmetric difference (cf. [7]). This property of approximation fulfilled by P together with the property of P to be G-invariant results in $P(A \Delta (\bigcup_{g \in G} g(A))) = 0$ and $P(A \Delta (\bigcap_{g \in G} g(A))) = 0$ from which $P((\bigcup_{g \in G} g(A)) \setminus (\bigcap_{g \in G} g(A))) = 0$ follows. □

The remaining part of this article is devoted to the problem of simplifying the monogenicity criterion of Lemma 2. In this connection the set $F(G)$ consisting of all $\omega \in \Omega$ which are kept fixed under all $g \in G$, i. e. $\omega = g(\omega)$, $g \in G$, holds true, plays an essential role.

Lemma 3. Let $\mathcal{A}^n$ denote the n-fold direct product of the σ-algebra $\mathcal{A}$ of subsets of some set Ω and let G denote the finite group of $(\mathcal{A}^n, \mathcal{A}^n)$-measurable transformations $g : \Omega^n \to \Omega^n$, Ω^n being the n-fold Cartesian product of Ω, associated with some subgroups of the symmetric group γ_n of all permutations of $\{1, \ldots, n\}$. Then a probability measure P on $\mathcal{A}^n$ is monogenic with respect to $\mathcal{B}(G, \mathcal{A}^n)$ if and only if $P^*(F(G)) = 1$ holds true, where P^* stands for the outer probability measure of P.

P r o o f. Clearly, $P^*(F(G)) = 1$ is according to Lemma 2 sufficient for the property of P to be monogenic with respect to $\mathcal{B}(G, \mathcal{A}^n)$, since $(\bigcup_{g\in G} g(A))\backslash(\bigcap_{g\in G} g(A)) \subset (F(G))^c$ is valid for all $A \in \mathcal{A}^n$.

For the proof of the converse implication one might introduce the following equivalence relation on $\{1, \ldots, n\}$ defined by $i \sim j$ for $i, j \in \{1, \ldots, n\}$ if and only if there exists some $\gamma \in \Gamma$ such that $i = \gamma(j)$ is valid, where Γ stands for the subgroup of the symmetric group γ_n associated with G. Let $[i_1], \ldots, [i_k]$, $i_1 < \ldots < i_k$, $i_j \in \{1, \ldots, n\}$, $j = 1, \ldots, k$, denote the corresponding equivalence classes. It will now be shown that $F(G) \subset \bigcup_{m=1}^{\infty}(A_{m,1} \times \ldots \times A_{m,n})$ for $A_{m,j} \in \mathcal{A}$, $j = 1, \ldots, n$, $m \in \mathbb{N}$, implies $\sum_{m=1}^{\infty} P(A_{m,1} \times \ldots \times A_{m,n}) \geq 1$, from which the assertion $P^*(F(G, \mathcal{A})) = 1$ follows. For this purpose one should take into consideration that Lemma 2 leads to the following equations up to some P-zero set:

$$\begin{aligned} & I_{A_{m,1}} \times \ldots \times I_{A_{m,n}} \\ = \; & I_{\bigcap_{g\in G} g(A_{m,1}\times\ldots\times A_{m,n})} \\ = \; & I_{\bigcap_{g\in G}(\Omega\times\ldots\times\Omega\times\bigcap_{j\in[i_1]} A_{m,j}\times\Omega\times\ldots\times\Omega\times\bigcap_{j\in[i_2]} A_{m,j}\times\Omega\times\ldots\times\Omega\ldots\times\bigcap_{j\in[i_k]} A_{m,j}\times\Omega\times\ldots\times\Omega)}, \end{aligned}$$

where $[i_1] \cup \ldots \cup [i_k] = \{1, \ldots, n\}$ is valid. Finally, let π denote the projection of Ω^n onto $\Omega^{\{i_1,\ldots,i_k\}}$ introduced as the k-fold Cartesian product of Ω. Then $P(A_{m,1} \times \ldots \times A_{m,n}) = P^{\pi}(\bigcap_{j\in[i_1]} A_{m,j} \times \ldots \times \bigcap_{j\in[i_k]} A_{m,j})$ is implied by the preceding equations. Now $F(G) \subset \bigcup_{m=1}^{\infty}(A_{m,1} \times \ldots \times A_{m,n})$, together with $F(G) = \{(\omega_1, \ldots, \omega_n) \in \Omega^n : \omega_i = \omega_j,\ i, j \in [i_\nu],\ \nu \in \{1, \ldots, k\}\}$, yields the inclusion $\Omega^{\{i_1,\ldots,i_k\}} \subset \bigcup_{m=1}^{\infty}(\bigcap_{j\in[i_1]} A_{m,j} \times \ldots \times \bigcap_{j\in[i_k]} A_{m,j})$, from which $\sum_{m=1}^{\infty} P(A_{m,1} \times \ldots \times A_{m,n}) = \sum_{m=1}^{\infty} P^{\pi}(\bigcap_{j\in[i_1]} A_{m,j} \times \ldots \times \bigcap_{j\in[i_k]} A_{m,j}) \geq P^{\pi}(\Omega^{\{i_1,\ldots,i_k\}}) = 1$ follows, i. e. monogenicity of P with respect to $\mathcal{B}(G, \mathcal{A}^n)$ implies $P^*(F(G)) = 1$. □

Remarks.

(i) If G is associated with the symmetric groups γ_n, then $F(G)$ is equal to the diagonal Δ of Ω^n. It is known that $\Delta \in \mathcal{A}^n$ is equivalent to the property of $\mathcal{A}$ to separate points $\omega \in \Omega$ by a countable system of sets belonging to $\mathcal{A}$. A short proof of this characterization of $\Delta \in \mathcal{A}^n$ might be based on the fact that the atoms of $\mathcal{A}^n$ are of the type $A_1 \times \ldots \times A_n$, where $A_j \in \mathcal{A}$, $j = 1, \ldots, n$, are atoms of $\mathcal{A}$ (cf. part (iv) of the remark following Corollary 1). The assumption $\Delta \in \mathcal{A}^n$ implies $\Delta \in \mathcal{A}_0^n$, where $\mathcal{A}_0$ is a countably generated sub-σ-algebra of $\mathcal{A}$. Therefore, Δ is equal to the union of atoms of $\mathcal{A}_0^n$ of the type $A_1 \times \ldots \times A_n$, where $A_j \in \mathcal{A}_0$, $j = 1, \ldots, n$, are atoms of $\mathcal{A}_0$, i. e. A_j, $j = 1, \ldots, n$, must be singletons. Hence, any countable generator $\mathcal{C}$ of $\mathcal{A}_0$ separates points $\omega \in \Omega$. The converse implication follows easily from the fact that Δ^c is the union of sets of the type $\Omega \times \ldots \times \Omega \times A \times \Omega \times \ldots \times \Omega \times A^c \times \Omega \times \ldots \times \Omega$, where A runs through some countable subsets of $\mathcal{A}$, which might be assumed to be closed with respect to complements. The property of $\mathcal{A}$ to separate points $\omega \in \Omega$ by a countable system of sets belonging to $\mathcal{A}$ implies that the cardinality of the underlying set Ω exceeds the cardinality of the set $\mathbb{R}$ of real numbers. In particular, $\pi_1 - \pi_2$ is not $(\mathcal{A} \otimes \mathcal{A}, \mathcal{A})$-measurable, where $\pi_j : \Omega \times \Omega$, $j = 1, 2$, are the projections associated with the Banach space Ω, if the cardinality of

Ω exceeds the cardinality of $\mathbb{R}$ and $\mathcal{A}$ is the corresponding Borel σ-algebra (cf. [5]).

(ii) The case $P^*(\Delta) = 1$ together with $P_*(\Delta) = 0$ is possible, where P_* stands for the inner probability measure of P as the following special case shows: Let Ω be an uncountable set, let $\mathcal{A}$ be the σ-algebra of subsets of Ω generated by all singletons $\{\omega\}$, $\omega \in \Omega$, i. e. $\mathcal{A} = \{A \subset \Omega : A$ or A^c is a countable subset of $\Omega\}$, and let P stand for the probability measure on $\mathcal{A}$ defined by $P(A) = 0$, if A is a countable subset of Ω, resp. $P(A) = 1$, if A^c is a countable subset of Ω. Then it is not difficult to see that $(P \otimes P)^*(\Delta) = 1$ and $(P \otimes P)_*(\Delta) = 0$ is valid.

In the sequel Lemma 3 will be extended to arbitrary finite groups of transformations. The special case of a finite group G of transformations $g : \Omega \to \Omega$ with $F(G) \notin \{\emptyset, \Omega\}$ together with the σ-algebra $\mathcal{A}$ consisting of the sets $\emptyset, \Omega, F(G)$, and $(F(G))^c$, i. e. $\mathcal{B}(G, \mathcal{A}) = \mathcal{A}$ is valid, shows that some additional assumption must be introduced, which is given in the following

Theorem 1. Let $\mathcal{A}$ denote a σ-algebra of subsets of a set Ω, G a finite group of $(\mathcal{A}, \mathcal{A})$-measurable transformations $g : \Omega \to \Omega$, $\mathcal{B}(G, \mathcal{A})$ the σ-algebra consisting of all G-invariant sets belonging to $\mathcal{A}$, $F(G)$ the set consisting of all $\omega \in \Omega$ satisfying $g(\omega) = \omega$, $g \in G$, $f : \Omega \to \Omega^{|G|}$, where $|G|$ stands for the number of elements of G, the mapping defined by $f(\omega) = (g_1(\omega), \dots, g_{|G|}(\omega))$, $\omega \in \Omega$, $G = \{g_1, \dots, g_{|G|}\}$, $\Omega^{|G|}$ the G-fold Cartesian product of Ω, and $\mathcal{A}^{|G|}$ the $|G|$-fold direct product of $\mathcal{A}$. Under the assumption $f(B) \in \mathcal{A}^{|G|}$, $B \in \mathcal{B}(G, \mathcal{A})$, the following assertions hold true:

(i) A probability measure P on $\mathcal{A}$ is monogenic with respect to $\mathcal{B}(G, \mathcal{A})$ if and only if $P^*(F(G)) = 1$ is valid, where P^* stands for the outer probability measure of P.

(ii) $F(G) \in \mathcal{A}$ holds true if and only if there exists a countable system contained in $\mathcal{A}$ which separates all points $\omega_1, \omega_2 \in F(G)$, $\omega_1 \neq \omega_2$, and $\omega \in F(G)$, $\omega' \notin F(G)$.

P r o o f. The finite group $G = \{g_1, \dots, g_{|G|}\}$ induces a subgroup $\mathcal{S}_G$ of the symmetric group $\gamma_{|G|}$ of permutations of $\{1, \dots, |G|\}$ according to $\pi_g(1, \dots, |G|) = (g_{\pi(1)}, \dots, g_{\pi(|G|)})$, where π stands for the permutation of $\{1, \dots, |G|\}$ associated with $g \in G$ by $(g_1 g, \dots, g_{|G|} g) = (g_{\pi(1)}, \dots, g_{\pi(|G|)})$. In particular, $f^{-1}(A_1 \times \dots \times A_{|G|}) = \bigcap_{g \in G} g(A) \in \mathcal{B}(G, \mathcal{A})$ is valid for $A_1 = \dots = A_{|G|} = A \in \mathcal{A}$ according to Lemma 1, from which $\mathcal{B}(G, \mathcal{A}) = f^{-1}(\mathcal{C})$ follows, where $\mathcal{C}$ stands for the σ-algebra of subsets of $\Omega^{|G|}$ generated by all sets of the type $A_1 \times \dots \times A_{|G|}$, $A_1 = \dots = A_{|G|} = A \in \mathcal{A}$. This observation shows that monogenicity of the probability measure P^f on $\mathcal{A}^{|G|}$ with respect to $\mathcal{B}(\mathcal{S}_G, \mathcal{A}^{|G|})$, where P^f stands for the probability measure on $\mathcal{A}^{|G|}$ induced by the probability measure P on $\mathcal{A}$ and the $(\mathcal{A}, \mathcal{A}^{|G|})$-measurable mapping f, implies that P is monogenic with respect to $\mathcal{B}(G, \mathcal{A})$. This follows, according to Lemma 2, from the equation $P^f(A_1 \times \dots \times A_{|G|} \setminus \bigcap_{\pi \in \mathcal{S}_G} A_{\pi(1)} \times \dots \times A_{\pi(|G|)}) = 0$, $A_j \in \mathcal{A}$, $j = 1, \dots, |G|$, since the special case $A_j = \Omega,\ j = 2,\dots,|G|$ and $A_1 = g_1(A),\ A \in \mathcal{A}$, results in $P(A \setminus f^{-1}(B_1 \times \dots \times B_{|G|})) = 0,\ B_j = A,\ j = 1,\dots,|G|$, if one takes into consideration that the subgroup of $\gamma_{|G|}$ associated with $\mathcal{S}_G$ acts transitively on $\{1,\dots,|G|\}$.

For the converse implication, namely that monogenicity of P with respect to $\mathcal{B}(G,\mathcal{A})$ implies that P^f is monogenic with respect to $\mathcal{B}(\mathcal{S}_G,\mathcal{A}^{|G|})$ one might start from the equation $P(A \setminus B) = 0,\ A \in \mathcal{A},\ B = \bigcap_{g\in G} g(A)$, according to Lemma 2. Now, $f(B) \in \mathcal{A}^{|G|}$ is valid by assumption, from which $P^f(A_1 \times \dots \times A_{|G|} \setminus f(B)) = 0$ follows for $A_j \in \mathcal{A},\ j = 1,\dots,|G|$, where B stands for $\bigcap_{g\in G} g(C)$ and C for $\bigcap_{j=1}^{|G|} g_j^{-1}(A_j) = f^{-1}(A_1 \times \dots \times A_{|G|}) \in \mathcal{A}$. Finally, $f(B) \in \mathcal{B}(\mathcal{S}_G,\mathcal{A}^{|G|})$, which is implied by $B \in \mathcal{B}(G,\mathcal{A})$, shows that P^f is monogenic with respect to $\mathcal{B}(\mathcal{S}_G,\mathcal{A}^{|G|})$ if and only if P is monogenic with respect to $\mathcal{B}(G,\mathcal{A})$.

Now everything is prepared for the proof of part (i) of Theorem 1. For this purpose let P stand for a probability measure on $\mathcal{A}$ being monogenic with respect to $\mathcal{B}(G,\mathcal{A})$. Then P^f is monogenic with respect to $\mathcal{B}(\mathcal{S}_G,\mathcal{A}^{|G|})$, i. e. $(P^f)^*(F(\mathcal{S}_G)) = 1$ holds true according to Lemma 3. Now $f^{-1}(F(\mathcal{S}_G)) = F(G)$ together with the assumption $f(B) \in \mathcal{A}^{|G|},\ B \in \mathcal{B}(G,\mathcal{A})$, leads to $P^*(F(G)) = 1$, since the coverings of $F(G)$ entering into the definition of $P^*(F(G))$ might have been chosen to belong to $\mathcal{B}(G,\mathcal{A})$. Clearly, the property of P to fulfill the last equation $P^*(F(G)) = 1$ implies, with regard to Lemma 2, that P is monogenic with respect to $\mathcal{B}(G,\mathcal{A})$ because of $\bigcup_{g\in G} g(A) \setminus \bigcap_{g\in G} g(A) \subset (F(G))^c,\ A \in \mathcal{A}$, i. e. part (i) of Theorem 1 has been proved.

The proof of part (ii) of Theorem 1 might be based on the observation that the subgroup of $\gamma_{|G|}$ associated with $\mathcal{S}_G$ acts transitively on $\{1,\dots,|G|\}$, from which $F(\mathcal{S}_G) = \{(\omega_1,\dots,\omega_{|G|}) : \omega_1 = \dots = \omega_{|G|} = \omega,\ \omega \in \Omega\}$ follows. Now the assumption $f(B) \in \mathcal{A}^{|G|},\ B \in \mathcal{B}(G,\mathcal{A})$ together with the condition $F(G) \in \mathcal{A}$ results in $f(\Omega) \cap F(\mathcal{S}_G) = f(F(G)) \in \mathcal{A}^{|G|}$. Therefore, $f(F(G)) \in \hat{\mathcal{A}}^{|G|}$ for a certain countably generated sub-σ-algebra $\hat{\mathcal{A}}$ of $\mathcal{A}$ holds true. Now the atoms of $\hat{\mathcal{A}}^{|G|}$ are of the type $A_1 \times \dots \times A_{|G|}$, where $A_j \in \hat{\mathcal{A}},\ j = 1,\dots,|G|$, are atoms of $\hat{\mathcal{A}}$ (cf. part (iv) of the remark following Corollary 1), and the union of all atoms of $\hat{\mathcal{A}}^{|G|}$ coincides with $\Omega^{|G|}$. Hence, the atoms of $\hat{\mathcal{A}}^{|G|}$, whose union coincides with $f(F(G))$, are of the type $A_1 \times \dots \times A_{|G|}$, where $A_j \in \mathcal{A},\ j = 1,\dots,|G|$, are singletons of the type $\{\omega\},\ \omega \in F(G)$, i. e. any countable system of sets generating $\hat{\mathcal{A}}$ separates all points $\omega_1,\omega_2 \in F(G),\ \omega_1 \neq \omega_2$ and $\omega \in F(G),\ \omega' \notin F(G)$. Conversely, the existence of a countable system $\mathcal{C} \subset \mathcal{A}$ with this property of separation results in $f(\Omega) \cap F(\mathcal{S}_G) \in \mathcal{A}^{|G|}$ because the complement of $f(\Omega) \cap F(\mathcal{S}_G) = f(F(G))$ consists of the union of the sets of the type $A_1 \times \dots \times A_{|G|},\ A_j = C \in \mathcal{C},\ A_k = C^c,\ j,k \in \{1,\dots,|G|\},\ j \neq k,\ A_i = \Omega,\ i \in \{1,\dots,|G|\} \setminus \{j,k\}$, since one might assume without loss of generality that $\mathcal{C}$ is already closed with respect to complements. Finally, $f(F(G)) \in \mathcal{A}^{|G|}$ together with $f^{-1}(f(F(G))) = F(G)$ yields $F(G) \in \mathcal{A}$, i. e. part (ii) of Theorem 1 has been proved. □

Remarks.

(i) The condition $f(B) \in \mathcal{A}^{|G|},\ B \in \mathcal{B}(G,\mathcal{A})$, is fulfilled, if Ω is a Polish space and $\mathcal{A}$ the correspondingBorel σ-algebra (cf. [3], p. 276).

(ii) The σ-algebra generated by all sets of the type $A_1 \times \ldots \times A_{|G|}$, $A_1 = \ldots = A_{|G|} = A \in \mathcal{A}$, which occurs in the proof of Theorem 1, has been characterized in [4].

In the final part of this article a further rather simple condition will be introduced, which yields simultaneously $F(G) \in \mathcal{A}$ and the characterization of monogenicity of a probability measure P on $\mathcal{A}$ with respect to $\mathcal{B}(G, \mathcal{A})$ by $P(F(G)) = 1$.

Theorem 2. Let $\mathcal{A}$ denote a σ-algebra of subsets of a set Ω, G a finite group of $(\mathcal{A}, \mathcal{A})$-measurable transformations $g : \Omega \to \Omega$, $\mathcal{B}(G, \mathcal{A})$ the σ-algebra consisting of all G-invariant sets belonging to $\mathcal{A}$, and $F(G)$ the set $\{\omega \in \Omega : g(\omega) = \omega,\ g \in G\}$. Under the assumption that $\mathcal{A}$ separates all points $\omega, g(\omega)$, $\omega \in \Omega$, $g \in G$, $\omega \neq g(\omega)$, by a countable system of sets belonging to $\mathcal{A}$, the following assertions hold true:

(i) $F(G) \in \mathcal{A}$,

(ii) a probability measure P on $\mathcal{A}$ is monogenic with respect to $\mathcal{B}(G, \mathcal{A})$ if and only if $P(F(G)) = 1$ is valid.

Proof. Let $\mathcal{C} \subset \mathcal{A}$ stand for a countable system such that for $\omega \in \Omega$, $g \in G$, $\omega \neq g(\omega)$, there exists a $C \in \mathcal{C}$ satisfying $\omega \in C$, $g(\omega) \notin C$ or $\omega \notin C$, $g(\omega) \in C$. Then $\bigcup_{C \in \mathcal{C}}((\bigcup_{g \in G} g(C)) \setminus (\bigcap_{g \in G} g(C))) = (F(G))^c$ holds true, from which $P(F(G)) = 1$ follows, if P is monogenic with respect to $\mathcal{B}(G, \mathcal{A})$, since this property implies according to Lemma 2 the equation $P((\bigcup_{g \in G} g(C)) \setminus (\bigcap_{g \in G} g(C))) = 0$. Clearly, $P(F(G)) = 1$ yields, by Lemma 2 being applied, that P is monogenic with respect to $\mathcal{B}(G, \mathcal{A})$. □

Remarks.

(i) The property of $\mathcal{A}$ to separate points ω, $g(\omega)$, $\omega \in \Omega$, $g \in G$, $\omega \neq g(\omega)$, by a countable system of sets belonging to $\mathcal{A}$ is shared by all countably generated σ-algebras $\mathcal{A}$ of subsets of Ω satisfying $\{\omega\} \in \mathcal{A}$, $\omega \in \Omega$, since such σ-algebras separates all points $\omega_1, \omega_2 \in \Omega$, $\omega_1 \neq \omega_2$, by a countable system of sets belonging to the corresponding σ-algebra.

(ii) In case G is associated with the symmetric group γ_n of all permutations π of $\{1, \ldots, n\}$ acting $(\mathcal{A}^n, \mathcal{A}^n)$-measurably on Ω^n, the property of $\mathcal{A}^n$ to separate points $\omega, g(\omega)$, $\omega \in \Omega^n$, $g \in G$, $\omega \neq g(\omega)$, by a countable system of sets belonging to $\mathcal{A}^n$, is equivalent to the property of $\mathcal{A}$ to separate all points $\omega_1, \omega_2 \in \Omega$, $\omega_1 \neq \omega_2$, by a countable system of sets belonging to $\mathcal{A}$. This follows from the observation that any σ-algebra generated by some system $\mathcal{C}$ of sets belonging to this σ-algebra and separating a given set of points by some countable system of sets belonging to this σ-algebra, already separates this given set of points by a countable system of sets belonging to $\mathcal{C}$.

An application of Theorem 2 and Lemma 1 results in

Corollary 2. Let $\mathcal{A}_j$ denote σ-algebras of subsets of some set $\Omega_j,\ G_j$ finite groups of $(\mathcal{A}_j, \mathcal{A}_j)$-measurable transformations $g : \Omega \to \Omega,\ B(G_j, \mathcal{A}_j)$ the σ-algebra consisting of all G_j-invariant sets belonging to $\mathcal{A}_j,\ j = 1, 2$, and $\mathcal{B}(G_1 \times G_2,\ \mathcal{A}_1 \otimes \mathcal{A}_2)$ the σ-algebra consisting of all $(G_1 \times G_2)$-invariant sets belonging to $\mathcal{A}_1 \otimes \mathcal{A}_2$. Then $\mathcal{B}(G_1 \times G_2,\ \mathcal{A}_1 \otimes \mathcal{A}_2) = \mathcal{B}(G_1, \mathcal{A}_1) \otimes \mathcal{B}(G_2, \mathcal{A}_2)$ is valid and under the assumption that $\mathcal{A}_j$ separates all points $\omega_j,\ g(\omega_j),\ \omega_j \in \Omega_j,\ g \in G_j,\ \omega_j \neq g(\omega_j),\ j = 1, 2$, the following assertion holds true: A probability measure P on $\mathcal{A}_1 \otimes \mathcal{A}_2$ is monogenic with respect to $\mathcal{B}(G_1 \times G_2,\ \mathcal{A}_1 \otimes \mathcal{A}_2)$ if and only if the corresponding marginal probability measures P_j of P on $\mathcal{A}_j$ are monogenic with respect to $\mathcal{B}(G_j, \mathcal{A}_j),\ j = 1, 2$.

P r o o f. Lemma 1 implies $\mathcal{B}(G_1 \times G_2,\ \mathcal{A}_1 \otimes \mathcal{A}_2) = \mathcal{B}(G_1, \mathcal{A}_1) \otimes \mathcal{B}(G_2, \mathcal{A}_2)$ and monogenicity of the marginal probability measures P_j on $\mathcal{A}_j$ with respect to $\mathcal{B}(G_j, \mathcal{A}_j),\ j = 1, 2$, of some probability measure P on $\mathcal{A}_1 \otimes \mathcal{A}_2$, leads, according to Theorem 2, to $P_j(F(G_j)) = 1,\ j = 1, 2$, from which $P(F(G_1) \times F(G_2)) = P(F(G_1) \times \Omega_2) \cap (\Omega_1 \times F(G_2)) = 1$ follows, i.e. $P(F(G_1 \times G_2)) = 1$ holds true because of $F(G_1 \times G_2) = F(G_1) \times F(G_2)$, i.e. P is monogenic with respect to $\mathcal{B}(G_1 \times G_2,\ \mathcal{A}_1 \otimes \mathcal{A}_2)$. Conversely, $P(F(G_1 \times G_2)) = 1$, which follows by means of Theorem 2 from monogenicity of P with respect to $\mathcal{B}(G_1 \times G_2,\ \mathcal{A}_1 \otimes \mathcal{A}_2)$, implies $P_j(F(G_j)) = 1,\ j = 1, 2$, i.e. P_j is monogenic with respect to $\mathcal{B}(G_j, \mathcal{A}_j),\ j = 1, 2$.□

Remarks.

(i) Theorem 2 remains valid for countable groups, since Lemma 2 holds true for countable groups, too. However, Theorem 2 (and also Theorem 1) is not longer true for uncountable groups even in the case where Ω is an uncountable Polish space and $\mathcal{A}$ is the σ-algebra of Borel subsets of Ω, which might be seen as follows: For any analytic subset $A_0 \notin \mathcal{A}$ of Ω the equation $\bigcap_{B \in \mathcal{A}_0} B = A_0$ is valid, where $\mathcal{A}_0$ stands for all Borel subsets $B \in \mathcal{A}$ containing A_0 and $\mathcal{A}$ denotes the Borel σ-algebra of Ω (cf. [3], Theorem 8.3.1, and [3], Corollary 8.2.17 together with [8], p. 422 in connection with the existence of A_0). Furthermore, let G denote the group of $(\mathcal{A}, \mathcal{A})$-measurable mappings $g : \Omega \to \Omega$ such that there exists a set $B \in \mathcal{A}_0$ with the property $g(x) = x,\ x \in B,\ g(x) \neq x,\ x \in \Omega \setminus B$, where g is a one-to-one transformation of Ω which maps Ω onto Ω. In particular, g^{-1} is $(\mathcal{A}, \mathcal{A})$-measurable (cf. [3], Theorem 8.3.2 and Proposition 8.3.5), $F(G) = A_0 \notin \mathcal{A}$ is valid, and $\mathcal{B}(G, \mathcal{A}) = \{B \in \mathcal{A} : B \subset A_0$ or $B^c \subset A_0\}$ holds true, since for $c_1, c_2 \in \Omega \setminus A_0,\ c_1 \neq c_2$, there exists a mapping $g \in G$ satisfying $g(c_1) = c_2$, i.e. $A_0^c \cap B \neq \emptyset$ for a set $B \in \mathcal{B}(G, \mathcal{A})$ implies $A_0^c \cap B = A_0^c$. In particular, $\mathcal{B}(G, \mathcal{A})$ is not countably generated, since otherwise for any $\omega \in A_0^c$ there would exist an atom C of $\mathcal{B}(G, \mathcal{A})$ containing ω. Now $C \cap A_0^c \neq \emptyset$ implies $C^c \subset A_0$, i.e. $A_0^c \subset C$. Therefore, there exists an element $\omega' \in C$ with the property $\omega' \in A_0$ because of $A_0^c \neq C$. Finally $\{\omega'\} \in \mathcal{B}(G, \mathcal{A})$ results in the fact that $C \setminus \{\omega'\}$ is a proper subset of C, i.e. C would not be an atom of $\mathcal{B}(G, \mathcal{A})$.

(ii) The model described by (i) admits the following characterization in connection with the question whether a probability measure P defined on $\mathcal{A}$ has the property to be an extremal point of the set $\mathcal{P}$ consisting of all probability measures Q defined on $\mathcal{A}$ and satisfying $Q|\mathcal{B}(G,\mathcal{A}) = P|B(G,\mathcal{A}) : P \in \mathcal{P}$ is an extremal point of $\mathcal{P}$ if and only if $\bar{P}(A_0^c \cap B) = \bar{P}(A_0^c)\delta_\omega(B),\ B \in \mathcal{A}$, is valid for some $\omega \in A_0^c$, where $\bar{P}$ stands for the completion of P restricted to the σ-algebra consisting of the universally measurable subsets of Ω (cf. [3], Corollary 8.4.3) and where δ_ω denotes the one-point mass at $\omega,\ \omega \in \Omega$. This observation follows from the fact that for any $B \in \mathcal{A}$ there exists a set $B' \in \mathcal{B}(G,\mathcal{A})$ such that $I_{B'} = I_B$ P-a.e. holds true (cf. [7]), from which either $\bar{P}(A_0^c \cap B) = 0$ in the case $B' \subset A_0$ or $\bar{P}(A_0^c \cap B^c) = 0$ in the case $B'^c \subset A_0$ follows, i.e. the probability measure Q defined on $\mathcal{A}$ by $Q(B) = \bar{P}(A_0^c \cap B)/\bar{P}(A_0^c),\ B \in \mathcal{A}$, in the case $\bar{P}(A_0^c) > 0$ is equal to δ_ω for some $\omega \in A_0^c$, since $\mathcal{A}$ is countably generated and contains all singletons $\{\omega\},\ \omega \in \Omega$. Hence, $\bar{P}(B \cap A_0^c) = \bar{P}(A_0^c)\delta_\omega(B),\ B \in \mathcal{A}$, is valid. Furthermore, $\bar{P}(B \cap A_0) = \bar{P}(B \cap B_0),\ B \in \mathcal{A}$, where $B_0 \in \mathcal{A}$ satisfies $B_0 \subset A_0$ and $\bar{P}(A_0 \setminus B_0) = 0$, shows that the probability measure defined on $\mathcal{A}$ by $B \to \bar{P}(B \cap A_0)/\bar{P}(A_0),\ B \in \mathcal{A}$, is monogenic with respect to $\mathcal{B}(G,\mathcal{A})$, from which the assertion about the characterization of extremal points of $\mathcal{P}$ follows. In particular, P is monogenic with respect to $\mathcal{B}(G,\mathcal{A})$ if and only if $\bar{P}(A_0) = 1$, i.e. $P^*(A_0) = 1$ holds true, since monogenicity of P relative to $\mathcal{B}(G,\mathcal{A})$ implies that $\delta_\omega,\ \omega \in A_0^c$, has the same property in the case $\bar{P}(A_0^c) > 0$.

Example 2. Let $\mathcal{A}$ denote a countably generated σ-algebra of subsets of a set Ω containing all singletons $\{\omega\},\ \omega \in \Omega$, and let G stand for the countable group of $(\mathcal{A}^{\mathbb{N}}, \mathcal{A}^{\mathbb{N}})$-measurable mappings $g : \Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ acting as a permutation for a finite number of coordinates and keeping the remaining coordinates fixed, where $\Omega^{\mathbb{N}}$ resp. $\mathcal{A}^{\mathbb{N}}$ is introduced as the $\mathbb{N}$-fold Cartesian product of Ω resp. $\mathbb{N}$-fold direct product of $\mathcal{A}$. Then $F(G)$ is equal to the diagonal Δ of $\Omega^{\mathbb{N}}$ and a probability measure on $\mathcal{A}^{\mathbb{N}}$ of the type $\bigotimes_{n\in\mathbb{N}} P_n$, where $P_n,\ n \in \mathbb{N}$, are probability measures defined on $\mathcal{A}$, is monogenic with respect to $\mathcal{B}(G,\mathcal{A}^{\mathbb{N}})$ if and only if $P_n = P_1,\ n \in \mathbb{N}$, is valid and P_1 coincides with a one-point mass at a certain element $\omega \in \Omega$. This follows from Theorem 2 together with Fubini's theorem.

Example 3. Let $\mathcal{A}$ stand for a countably generated σ-algebra of subsets of a set Ω containing all singletons $\{\omega\},\ \omega \in \Omega$, and let $G_j,\ j = 1,2$, stand for finite groups of $(\mathcal{A},\mathcal{A})$-measurable mappings $g_j : \Omega \to \Omega,\ g_j \in G_j,\ j = 1,2$. Then the corresponding group G_{12} of $(\mathcal{A},\mathcal{A})$-measurable transformations generated by G_1 and G_2 consists of all elements of the type $h_1 \circ \ldots \circ h_n,\ h_j \in G_1 \cup G_2,\ j = 1,\ldots,n,\ n \in \mathbb{N}$, which implies $F(G_{12}) = F(G_1) \cap F(G_2)$. Now Theorem 2 shows that a probability measure P on $\mathcal{A}$ is monogenic with respect to $\mathcal{B}(G_{12},\mathcal{A})$ if and only if P is monogenic with respect to $\mathcal{B}(G_1,\mathcal{A})$ and $\mathcal{B}(G_2,\mathcal{A})$.

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