KYBERNETIKA — VOLUME *28* (1992), NUMBER 3, PAGES 181 – 190

EXPONENTIALLY DISCOUNTED ESTIMATES AND OSCILLATIONS IN LINEAR CONTROLLED SYSTEMS

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Slow unmodelled oscillations of a system are regarded as a parameter and estimated by the discounted least squares method. The estimate is used to eliminate the oscillations. Properties of the procedure are presented for vanishing discount factor. The application is shown on the example of a computer controlled system.

1. INTRODUCTION

Exponential discounting of information in estimating the parameters of a system is often used in adaptive control (cf. [2]). The present paper deals with a method to eliminate periodical oscillations in linear systems. This method consists in using estimates with small discount factor λ, i. e. the oscillations are assumed to be slow. This makes it possible to analyze the procedure by means of asymptotic expressions as $\lambda \to 0+$. Application and interpretation of the results are shown on the example of a second order system with computer control (see Fig. 1), which is calculated in detail. The formulation of the problem in this paper is related to the results in [3], [4].

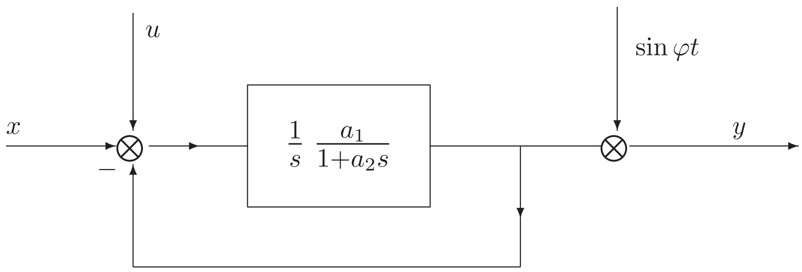

Fig. 1.

We consider a stochastic linear controlled system, which is modelled by the following differential equation

$$\mathrm{d}X_t = fX_t\mathrm{d}t + b(\alpha_t)\mathrm{d}t + U_t\mathrm{d}t + \mathrm{d}W_t, \qquad t \in (-\infty, \infty), \tag{1}$$

where

$$b(\alpha_t) = b_0 + \alpha_t^1\, b_1 + \cdots + \alpha_t^p\, b_p = b_0 + b\alpha_t,$$

and

$$\alpha_t = \left(\alpha_t^1, \ldots, \alpha_t^p\right)'$$

is the p-dimensional vector of parameters. X_t is the n-dimensional state vector and U_t is the m-dimensional control signal. Let f be a stable $(n\times n)$-matrix, $b_0, b_1, \ldots, b_p$ be n-dimensional linearly independent vectors, b be the $(n\times p)$-matrix, the columns of which are formed by $b_i,\ i = 1,\ldots,p$. $W = \{W_t,\, t \in (-\infty,\infty)\}$ is the n-dimensional Wiener process with incremental variance matrix h, i. e.

$$\mathrm{d}W_t\,\mathrm{d}W_t' = h\,\mathrm{d}t.$$

Further, in Remark 1 it is stated that in the case treated here the existence of a solution of (1) for $t \in (-\infty,\infty)$ can be assumed.

The term $b(\alpha_t)$ represents undesirable oscillations, α_t is assumed to be unknown. These oscillations are to be decreased by the control signal.

2. DISTRIBUTION OF ESTIMATE

The quantity α_t is estimated as a constant α, since the oscillations are assumed to be slow. The estimate α_T^* is obtained from the observations of $X_t,\ t \in (-\infty, T)$, by the least squares method with exponential discounting. Slow changes of α are matched by the discounting.

Let λ be the discount factor, $\lambda > 0$. Small discount factor improves the accuracy of the estimate, but reduces its sensibility to parameter changes.

The following expression is minimized

$$\int_{-\infty}^{T} \mathrm{e}^{\lambda t}\left[\left(\dot{X}_t - f\,X_t - b(\alpha) - U_t\right)'\,\ell\left(\dot{X}_t - f\,X_t - b(\alpha) - U_t\right) - \dot{X}_t'\,\ell\dot{X}_t\right]\mathrm{d}t, \quad (2)$$

where ℓ is a positively semidefinite symmetric matrix. In (2) the undefined term $\int_{-\infty}^{T} \dot{X}_t'\,\ell\dot{X}_t\,\mathrm{d}t$ is cancelled and the other terms with $\dot{X}_t$ have $\dot{X}_t\mathrm{d}t$ which is rewritten as $\mathrm{d}X_t$. Equating the gradient of (2) with respect to α to zero we obtain the relation

$$\int_{-\infty}^{T} \mathrm{e}^{\lambda t}\,Q\,\mathrm{d}t\,\alpha_T^* = \int_{-\infty}^{T} \mathrm{e}^{\lambda t} L\left(\mathrm{d}X_t - f X_t\,\mathrm{d}t - b_0\,\mathrm{d}t - U_t\,\mathrm{d}t\right). \quad (3)$$

Q and L are constant matrices,

$$Q = b'\,\ell\,b, \qquad L = b'\,\ell.$$

From (1) we obtain

$$Q\int_{-\infty}^{T/\lambda} \mathrm{e}^{\lambda t}\left(\alpha_{T/\lambda}^* - \alpha_t\right)\mathrm{d}t = Q\int_{-\infty}^{T/\lambda} \mathrm{e}^{\lambda t} L\,\mathrm{d}W_t.$$

Since α_t is assumed to represent slow oscillations we write

$$\alpha_t = a(\lambda\,t),$$

where $a(y),\ y \in (-\infty,\infty)$, is a piecewise continuous periodic function and λ is the discount factor treated as a small parameter. Using the substitution $y = \lambda\,t$ we obtain after rearrangements

$$\frac{1}{\sqrt{\lambda}}\left(\alpha_{T/\lambda}^* - \int_{-\infty}^{T} \mathrm{e}^{(y-T)} a(y)\mathrm{d}y\right) = \sqrt{\lambda}\,Q^{-1}\int_{-\infty}^{T/\lambda} \mathrm{e}^{\lambda(t-T/\lambda)} L\,\mathrm{d}W_t. \quad (4)$$

Denote by $\bar{a}(T)$ the integral on the left-hand side of (4). The distribution of

$$Y_T = \frac{1}{\sqrt{\lambda}}\left(\alpha_{T/\lambda}^* - \bar{a}(T)\right)$$

is seen from (4) to be independent of the control signal U_t. It is independent of the discount factor λ, as well. Namely, calculating the covariance function of the integral on the right-hand side of (4) we obtain the following proposition.

Proposition 1. The process $\{Y_t\}$ is Gaussian with zero mean and covariance function

$$\mathsf{E}\, Y_t\, Y_s' = \frac{1}{2} \mathrm{e}^{-|t-s|}\, Q^{-1}\, L\, h\, L'\, Q^{-1}.$$

Now we aim to derive the differential equation for the estimate α_t^*. To eliminate the oscillations represented in (1) by $b(\alpha_t)$ the control signal

$$U_t = -\left(b_0 + b\, \alpha_t^*\right) \tag{5}$$

is introduced. The true value of α_t is replaced by its estimate α_t^*. (1) is rewritten as

$$\mathrm{d}X_t = f\, X_t\, \mathrm{d}t + b\left(\alpha_t - \alpha_t^*\right)\, \mathrm{d}t + \mathrm{d}W_t. \tag{6}$$

Remark 1. The periodicity of $a(y)$ and the stability of f can be used to establish the existence of a weak solution of (6) on the interval $(-\infty, \infty)$.

From (3) it follows

$$\frac{\mathrm{e}^{\lambda T}}{\lambda}\, \alpha_T^* = \int_{-\infty}^{T} \mathrm{e}^{\lambda t}\, Q^{-1}\, L\left(\mathrm{d}X_t - f X_t\, \mathrm{d}t + b\alpha_t^*\, \mathrm{d}t\right). \tag{7}$$

Differentiating (7) and using the relation $Q^{-1}\, L\, b = I$ one obtains

$$\mathrm{d}\, \alpha_t^* = \lambda\, Q^{-1}\, L\left(\mathrm{d}X_t - f X_t\, \mathrm{d}t\right) = \lambda\, Q^{-1}\, L\left(b(\alpha_t - \alpha_t^*)\mathrm{d}t + \mathrm{d}W_t\right). \tag{8}$$

3. MEAN AND VARIANCE OF STATE VECTOR

The efficiency of the control (5) can be often expressed adequately by means of the average of a quadratic form $X_t'\, r\, X_t$, where r is a suitable positively semidefinite matrix. To investigate the criterion first the asymptotic expansion of the mean and of the variance matrix of $X_{T/\lambda}$ will be obtained in this section.

The solution of (6) can be represented in the form

$$X_t = \int_{-\infty}^{t} \mathrm{e}^{(t-s)f} b\left(\alpha_s - \alpha_s^*\right) \mathrm{d}s + \int_{-\infty}^{t} \mathrm{e}^{(t-s)f} \mathrm{d}W_s.$$

Let $a(y)$ be continuously differentiable. Denote

$$\begin{aligned} A^T &= \int_{-\infty}^{T} \mathrm{e}^{(T-y)f/\lambda} b\left(a(y) - \bar{a}(y)\right) \mathrm{d}y/\lambda, \\ \Phi(T,\lambda) &= \int_{-\infty}^{T} \mathrm{e}^{-yf/\lambda} \mathrm{e}^{-y} \mathrm{d}y/\lambda = \int_{-\infty}^{T/\lambda} \mathrm{e}^{-yf} \mathrm{e}^{-y\lambda} \mathrm{d}y. \end{aligned}$$

Then

$$X_{T/\lambda} = A^T + \mathrm{e}^{Tf/\lambda} \int_{-\infty}^{T/\lambda} \left[\left(\Phi(\lambda s, \lambda) - \Phi(T,\lambda)\right)\, \lambda Q^{-1} L \mathrm{e}^{s\lambda} + \mathrm{e}^{-sf}\right]\, \mathrm{d}W_s.$$

Let $\lambda \to 0+$. Using relations

$$\begin{aligned} \mathrm{e}^{Tf/\lambda}\, \Phi(\lambda_s, \lambda) &= -\mathrm{e}^{(T/\lambda - s)f} \mathrm{e}^{-s\lambda} f^{-1} + O(\lambda), \\ \mathrm{e}^{Tf/\lambda}\, \Phi(T, \lambda) &= -\mathrm{e}^{-T} f^{-1} + O(\lambda), \end{aligned}$$

we obtain after rearrangements the following expansion

$$\begin{aligned} X_{T/\lambda} &= A^T - \lambda \int_{-\infty}^{T/\lambda} f^{-1}\, \mathrm{e}^{(T/\lambda - s)f}\, Q^{-1} L\, \mathrm{d}W_s + \\ &\quad + \sqrt{\lambda} \int_{-\infty}^{T/\lambda} f^{-1} \mathrm{e}^{\lambda(s - T/\lambda)} \sqrt{\lambda}\, Q^{-1} L \mathrm{d}W_s + \int_{-\infty}^{T/\lambda} \mathrm{e}^{(T\lambda - s)f}\, \mathrm{d}W_s + O(\lambda^2), \end{aligned} \tag{9}$$

where $O(\lambda^k)$ denotes a term $\lambda^k R$ with $\mathsf{E}R^2 < \infty$. Using the periodicity and the differentiability of $a(y)$ it is proved that

$$A^T = -f^{-1}\, b\,(a(T) - \overline{a}(T)) + \lambda f^{-2}\, b\,(a'(T) - \overline{a}'(T)) + O(\lambda^2).$$

Thus $X_{T/\lambda}$ has normal distribution with the mean

$$\mathsf{E}X_{T/\lambda} = A^T = A_0^T + \lambda A_1^T + O(\lambda^2), \tag{10}$$

where

$$\begin{array}{rcl} A_0^T &=& -f^{-1}\, b\,(a(T) - \overline{a}(T))\,, \\ A_1^T &=& f^{-2}\, b\,\left(a'(T) - \overline{a}'(T)\right) = f^{-2} b(a'T - a(T) + \overline{a}(T)). \end{array} \tag{11}$$

The last but one term in (9) has the same distribution as the state vector X_t in the case of no oscillations. Hence its variance matrix S satisfies

$$fS + Sf' + h = 0.$$

After calculating the covariance matrices of the separate terms in (9) one obtains the variance matrix of $X_{T/\lambda}$ as

$$\begin{aligned} \mathsf{var}\, X_{T/\lambda} \quad &= \qquad\qquad (12) \\ = S + \lambda \Big[-f^{-2} h L' Q^{-1} - Q^{-1} L h f^{-2'} &+ \frac{1}{2} f^{-1} Q^{-1} L h L' Q^{-1} f^{-1'} - f^{-1} D - D' f^{-1'} \Big] + O(\lambda^2), \end{aligned}$$

where D is the solution of

$$fD + Df' + hL'Q^{-1} = 0.$$

The variance matrix of the state vector $X_{T/\lambda}$ does not depend on oscillations.

4. PERFORMANCE OF CRITERION

Introduce the criterion

$$C = \frac{1}{\tau} \int_0^\tau \mathsf{E}X'_{T/\lambda} r X_{T/\lambda} \mathrm{d}T,$$

where r is a positively semidefinite matrix and τ is the period of the function $a(y)$ representing the oscillations. Then (10), (12) imply

$$\mathsf{E}X'_{T/\lambda} r X_{T/\lambda} = \left(A_0^T + \lambda A_1^T\right)' r \left(A_0^T + \lambda A_1^T\right) + \text{ trace } (r(S + \lambda S_1)) + O(\lambda^2), \tag{13}$$

where S_1 denotes the expression in square brackets in (12).

In what follows we shall investigate the asymptotic behaviour of the criterion as $\lambda \to 0+$ in the case that

$$a(y) = \sin 2\pi\omega y.$$

From (11), (13) performing the calculations it follows

$$C = \omega \int_0^{1/\omega} \mathsf{E}X'_{T/\lambda} r\, X_{T/\lambda} \mathrm{d}T = \frac{1}{2}\left(F_1' r F_1 + F_2' r F_2\right) + \text{ trace } (r(S + \lambda S_1)) + O(\lambda^2), \tag{14}$$

where

$$\begin{array}{rcl} F_1 &=& \dfrac{2\pi\omega}{1 + (2\pi\omega)^2} \left(-f^{-1} b + \lambda (2\pi\omega)^2 f^{-2} b\right), \\ F_2 &=& \dfrac{(2\pi\omega)^2}{1 + (2\pi\omega)^2} \left(-f^{-1} b + \lambda f^{-2} b\right). \end{array}$$

Approximation with an error of first order in λ is given as

$$C = \frac{1}{2} \frac{(2\pi\omega)^2}{1 + (2\pi\omega)^2} b' f^{-1'} r f^{-1} b + \text{ trace}\,(rS) + O(\lambda). \tag{15}$$

5. EXAMPLE

Consider the system the block diagram of which is in Figure 1. To the output of the system with transfer function

$$H(s) = \frac{f_2}{s^2 + f_1 s + f_2},$$

where $f_1 = 1/a_2$, $f_2 = a_1/a_2$, sinusoidal oscillations are added. To reduce the oscillations the control u is added to the input. It results that

$$y = H(s)\,(x+u) + \sin\varphi t,$$

which is equivalent to

$$y'' + f_1 y' + f_2 y = f_2(x+u) + \alpha(t), \tag{16}$$

where

$$\alpha(t) = f_2 \sin\varphi t + \varphi f_1 \cos\varphi t - \varphi^2 \sin\varphi t. \tag{17}$$

$\alpha(t)$ is assumed to be unknown. Since the oscillations are slow, $\alpha(t)$ is estimated as a constant by the least squares method with exponential discounting. The estimate α_T^* at time T is obtained by minimizing the expression

$$\int_{-\infty}^{T} \mathrm{e}^{\lambda t} \left(y'' + f_1 y' + f_2 y - f_2(x+u) - \alpha\right)^2 \,\mathrm{d}t.$$

Equating the derivative with respect to α to zero yields

$$\frac{\mathrm{e}^{\lambda T}}{\lambda}\alpha_T^* = \int_{-\infty}^{T} \mathrm{e}^{\lambda t} \left(y'' + f_1 y' + f_2 y - f_2(x+u)\right) \mathrm{d}t. \tag{18}$$

Differentiating (18) and setting to eliminate the oscillations

$$u_t = -\alpha_t^* \,/\, f_2 \tag{19}$$

one obtains

$$\mathrm{d}\,\alpha_t^* = \lambda\,(y'' + f_1 y' + f_2 y - f_2 x)\,\mathrm{d}t. \tag{20}$$

This equation corresponds to (8). The Laplace transform of (20) has the following form

$$s\,\alpha^* = \lambda\left(s^2 + f_1 s + f_2\right)\,y - \lambda f_2 x. \tag{21}$$

Let us now take the viewpoint of computer control. Namely, let us assume that the input and the output are measured in discrete times with sampling interval Δ and let the control signal be a constant u_k in the interval $[k\Delta,\,(k{+}1)\Delta)$. Backward Euler approximation is applied to (21), i. e. the passage to the z-transform is performed by substituting $s = (z-1)\,/\,\Delta z$. Then after rearrangements one obtains

$$\alpha_k^* = \alpha_{k-1}^* + \left(\frac{\lambda}{\Delta} + \lambda f_1 + \lambda\Delta f_2\right) y_k - \left(\frac{2\lambda}{\Delta} + \lambda f_1\right) y_{k-1} + \frac{\lambda}{\Delta}\,y_{k-2} - \lambda\Delta f_2 x_k, \tag{22}$$

where x_k, y_k denote the values of the input and of the output at time $k\Delta$. Let

$$p_0 = -\left(\tfrac{\lambda}{\Delta f_2} + \lambda\tfrac{f_1}{f_2} + \lambda\Delta\right), \qquad p_1 = \tfrac{2\lambda}{\Delta f_2} + \lambda\tfrac{f_1}{f_2},$$

$$p_2 = -\lambda\,/\,\Delta f_2, \qquad q_0 = \lambda\Delta.$$

(19), (22) yield the recursive relation for the control signal

$$u_k = u_{k-1} + p_0 y_k + p_1 y_{k-1} + p_2 y_{k-2} + q_0 x_k. \tag{23}$$

In what follows the criterion under the control (23) will be investigated. The input is assumed to be colored noise, i. e.

$$\mathrm{d}X_t = -cX_t\mathrm{d}t + \mathrm{d}W_t^3, \qquad c > 0. \tag{24}$$

In addition the white noise W^2 is introduced into equation (16). The stochastic state model for $X = \left(X^1, X^2, X^3\right)'$ is constructed by setting $X^1 = y,\ X^2 = y',\ X^3 = x$. Then from (16), (24) it follows

$$\begin{aligned}\mathrm{d}X_t &= \begin{pmatrix} 0 & 1 & 0 \\ -f_2, & -f_1, & f_2 \\ 0 & 0 & -c \end{pmatrix} X_t\mathrm{d}t + \begin{pmatrix} 0 \\ f_2 \\ 0 \end{pmatrix} U_t\mathrm{d}t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \alpha(t)\,\mathrm{d}t + \mathrm{d}W_t \\ &= \qquad f \qquad X_t\,\mathrm{d}t + \quad g \quad U_t\,\mathrm{d}t + \quad b \quad \alpha(t)\,\mathrm{d}t + \mathrm{d}W_t,\end{aligned} \tag{25}$$

where $W_t = \left(0,\, W_t^2,\, W_t^3\right)'$ is the Wiener process with incremental variance matrix h,

$$\mathrm{d}W_t\,\mathrm{d}W_t' = h\,\mathrm{d}t = \begin{pmatrix} 0, & 0, & 0 \\ 0, & h_2, & 0 \\ 0, & 0, & h_3 \end{pmatrix} \mathrm{d}t, \qquad h_2 > 0,\ h_3 > 0. \tag{26}$$

The control is defined by $U_t = u_k$ for interval $t \in [k\Delta,\, (k+1)\Delta)$, where u_k is given by (23) in recursive form.

To evaluate the precision of formulas (14), (23) we calculate the value of the criterion

$$C = \frac{\varphi}{2\pi} \int_0^{2\pi/\varphi} \mathsf{E}\,\left(X_t^1 - X_t^3\right)^2\,\mathrm{d}t. \tag{27}$$

This criterion expresses the mean quadratic difference between the input and the output. In this case the matrix r in (14) has the form

$$r = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \tag{28}$$

Next we aim to construct the discrete recursive model for $X_{k\Delta}$. The solution of (25) is

$$\begin{aligned} X_{t+\Delta} &= \mathrm{e}^{\Delta t} X_t + \int_0^{\Delta} \mathrm{e}^{(\Delta-s)f} g\,\mathrm{d}s\,U_t + \\ &\quad + \int_0^{\Delta} \mathrm{e}^{(\Delta-s)f} b\alpha(t+s)\,\mathrm{d}s + \int_0^{\Delta} \mathrm{e}^{(\Delta-s)f}\mathrm{d}W_{t+s}. \end{aligned} \tag{29}$$

Denote $\exp(\Delta f)$ by $A = \left(a_{ij}\right)_{i,j=1,2,3}$ and the first integral on the right-hand side of (29) by $B = (b_1, b_2, b_3)'$. Then B fulfils the following equation

$$f\,B = A\,g - g.$$

The second integral in (29) is equal to the term

$$D_1 \cos\varphi t + D_2 \sin\varphi t,$$

where

$$\begin{aligned} D_1 &= f_1\varphi A\,Y + (f_2 - \varphi^2)\,A\,Z, \\ D_2 &= (f_2 - \varphi^2)\,A\,Y - f_1\varphi\,A\,Z, \end{aligned}$$

and it holds

$$\begin{aligned} \left(f + \varphi^2 f^{-1}\right)\,A\,Y &= -b\cos\varphi\Delta + \varphi f^{-1} b \sin\varphi\Delta + A\,b \\ \left(f + \varphi^2 f^{-1}\right)\,A\,Z &= -\varphi f^{-1} b\cos\varphi\Delta - b\sin\varphi\Delta + \varphi f^{-1} A\,b. \end{aligned} \tag{30}$$

The stochastic integral in (29)

$$E_t = \int_0^\Delta \mathrm{e}^{(\Delta-s)f}\,\mathrm{d}W_{s+t}$$

has zero mean and the variance matrix H satisfying

$$f\,H + H\,f' = A\,h\,A' - h, \tag{31}$$

where h is given by (26).

Using the calculated quantities we obtain from (29) the discrete system

$$X_{k+1} = A\,X_k + B\,u_k + D_1\cos\varphi k\Delta + D_2\sin\varphi k\Delta + E_k. \tag{32}$$

X_k, E_k stand for $X_{k\Delta}$, $E_{k\Delta}$.

To obtain the value of criterion (27) the extended discrete model for $\mathbf{X}_k = (y_k,\, y_k',\, x_k,\, y_{k-1},\, y_{k-2},\, u_{k-1})'$ is introduced. Relations (23), (32) imply that

$$\mathbf{X}_{k+1} = \mathbf{F}\,\mathbf{X}_k + \mathbf{D}_1\cos\varphi k\Delta + \mathbf{D}_2\sin\varphi k\Delta + \mathbf{E}_k, \tag{33}$$

where

$$\begin{aligned}
\mathbf{F} &= \begin{pmatrix} a_{11}+b_1p_0, & a_{12}, & a_{13}+b_1q_0, & b_1p_1, & b_1p_2, & b_1 \\ a_{21}+b_2p_0, & a_{22}, & a_{23}+b_2q_0, & b_2p_1, & b_2p_2, & b_2 \\ a_{31}+b_3p_0, & a_{32}, & a_{33}+b_3q_0, & b_3p_1, & b_3p_2, & b_3 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ p_0 & 0 & q_0 & p_1 & p_2 & 1 \end{pmatrix}, \\
\mathbf{D}_i &= (D_i,\, 0,\, 0,\, 0)', \qquad i = 1,2.
\end{aligned}$$

$\{\mathbf{E}_k\} = \{(E_k, 0, 0, 0)'\}$ is the random noise with variance matrix $\mathbf{H}$.

First the mean and the variance matrix of $\mathbf{X}_k$ will be calculated. From (33) it follows

$$\mathbf{X}_k = \sum_{j=0}^{\infty}\mathbf{F}^j\left(\mathbf{D}_1\cos\varphi(k-j)\,\Delta + \mathbf{D}_2\sin\varphi(k-j)\,\Delta\right) + \sum_{j=0}^{\infty}\mathbf{F}^j\,\mathbf{E}_{k-j}, \tag{34}$$

and hence after rearrangements

$$\mathsf{E}\,\mathbf{X}_k = \cos\varphi k\,\left({}^1\mathbf{J}_1 - {}^2\,\mathbf{J}_2\right) + \sin\varphi k\,\left({}^2\mathbf{J}_1 + {}^1\,\mathbf{J}_2\right),$$

where $\mathbf{J}_i' = \left({}^1\mathbf{J}_i',\, {}^2\mathbf{J}_i'\right)$, $i = 1,2$, satisfies the following equation

$$\left(\mathbf{I} - \begin{pmatrix} \mathbf{F}\cos\varphi, & -\mathbf{F}\sin\varphi \\ \mathbf{F}\sin\varphi, & \mathbf{F}\cos\varphi \end{pmatrix}\right)\mathbf{J}_i = \begin{pmatrix} \mathbf{D}_i \\ 0 \end{pmatrix}. \tag{35}$$

From (34)

$$\mathbf{V} = \mathsf{E}\,\left(\mathbf{X}_k - \mathsf{E}\,\mathbf{X}_k\right)\,\left(\mathbf{X}_k - \mathsf{E}\,\mathbf{X}_k\right)' = \sum_{j=0}^{\infty}\mathbf{F}^j\,\mathbf{H}\,\mathbf{F}'^j$$

which implies that $\mathbf{V}$ fulfils

$$\mathbf{F}\,\mathbf{V}\,\mathbf{F}' + \mathbf{H} = \mathbf{V}\,. \tag{36}$$

The criterion (27) has for discrete time system (32) the following equivalent

$$C = \frac{\varphi\Delta}{2\pi}\sum_{k=1}^{2\pi/\varphi\Delta}\mathsf{E}\,X_k'\,r\,X_k = \frac{\varphi\Delta}{2\pi}\sum_{k=1}^{2\pi/\varphi\Delta}\mathsf{E}\,\mathbf{X}_k'\,\mathbf{R}\,\mathbf{X}_k$$

with obvious definition of $\mathbf{R}$. The quantity $2\pi\,/\,\varphi\Delta$ is assumed to be an integer. The value of C is obtained from

$$C = \frac{\varphi\Delta}{2\pi}\sum_{k=1}^{2\pi/\varphi\Delta}\mathsf{E}\,\mathbf{X}_k'\,\mathbf{R}\,\mathsf{E}\,\mathbf{X}_k + \ \mathrm{trace}\,(\mathbf{V}\,\mathbf{R})$$

by solving the linear equations (30), (31), (35), (36).
Using the denotations

$$\mathbf{V} = (v_{ij})_{i,j=1,\dots,6},$$
$$\mathbf{J}'_i = \left({}^1\mathbf{J}'_i,\ {}^2\mathbf{J}'_i\right) = \left(J_i^1,\dots,J_i^6,\ J_i^7,\dots,J_i^{12}\right), \quad i=1,2,$$

we get for r as in (28)

$$C = \frac{1}{2}\left[\left(J_1^1 - J_2^7 - J_1^3 + J_2^9\right)^2 + \left(J_1^7 + J_2^1 - J_1^9 - J_2^3\right)^2\right] + v_{11} - 2v_{13} + v_{33}. \quad (37)$$

We return to the approximation of (37) as it is presented in Section 4. Set in (17)

$$\varphi = 2\pi\omega\lambda.$$

The asymptotic expansion of the criterion as $\lambda \to 0+$ has the same form as (14) with

$$F_1 = \frac{2\pi\omega}{1+(2\pi\omega)^2}\left[-f_2 f^{-1} b + \lambda (2\pi\omega)^2 \left(-f_1 f^{-1} b + f_2 f^{-2} b\right)\right],$$
$$F_2 = \frac{(2\pi\omega)^2}{1+(2\pi\omega)^2}\left[-f_2 f^{-1} b - \lambda\left(f_1 f^{-1} b + f_2 f^{-2} b\right)\right].$$

Since $a(t) = f_2 \sin 2\pi\omega\lambda t + O(\lambda)$, an approximation of C with an error of first order in λ follows from (15),

$$C = \frac{1}{2}\frac{(2\pi\omega)^2}{1+(2\pi\omega)^2} + \frac{1}{2}\left(\frac{f_1^2 + f_2 + c f_1}{c f_1^2 + f_1 f_2 + c^2 f_1}\, h_3 + \frac{1}{f_1 f_2}\, h_2\right) + O(\lambda). \quad (38)$$

Numerical results

For the constants $a_1 = 4,\ a_2 = 0.01$ the values of (37) in dependence on λ, ω, Δ are compared with (38) in the following table

(37)	$\lambda = 0.05$		$\lambda = 0.1$	
	$\omega = 0.5$	$\omega = 0.1$	$\omega = 0.5$	$\omega = 0.1$
$\Delta = 0.5$	0.5623	0.2471	0.5656	0.2479
$\Delta = 0.1$	0.5587	0.2462	0.5586	0.2460
(38)	0.5588	0.2463	0.5588	0.2463

It holds trace $(r\,S) = 0.1048$. This is the value of C if there are no oscillations. The unreduced oscillations increase the quadratic difference between the input and the output by $\sin^2 \varphi t$, hence in average by 0.5. Therefore in this case $C = 0.6048$.

(Received July 9, 1991.)

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