KYBERNETIKA — VOLUME *31* (1995), NUMBER 2, PAGES 141 – 150

TOPOLOGICAL EQUIVALENCE AND TOPOLOGICAL LINEARIZATION OF CONTROLLED DYNAMICAL SYSTEMS

SERGEJ ČELIKOVSKÝ[1]

The general, differential-equation-independent definition of a continuous-time controlled dynamical system as well as of the state space transformation and static state feedback are introduced. This approach makes it possible to consider transformations that are not smooth and introduce the so-called topological equivalence of controlled dynamical systems. It is shown that this approach generalizes the usual definitions based on the notion of the smooth ordinary differential equation with the control parameter. Topological equivalence is then used to introduce and investigate the problem of exact topological feedback linearization of a given nonlinear system. Sufficient conditions for the topological linearizability of planar systems are obtained. They particularly show that there do exist smooth systems that are topologically linearizable, but not smoothly linearizable. Finally, we indicate possible application of the topological linearization to the nonsmooth stabilization. Illustrative examples are included.

1. INTRODUCTION

The basic object of this paper is a continuous-time controlled dynamical system (shortly "system" where no confusion arises). Beginning with the Brockett [3], the extensive attention is paid to various kinds of the exact linearization of nonlinear systems (cf. [17, 10] for the detailed exposition). The term "exact" is used to distinguish this approach from the approximate (first order) linearization. The basic goal here is to find (if they exist) reasonable exact compensations and transformations of a given nonlinear system making its behaviour linear. The area of the exact linearization is very extensive and various additional attributes may be used to characterize it. Dependingly on the transformations used it is called state linearization, static state feedback linearization, dynamical feedback linearization, etc. If the transformations used are globally defined, the corresponding linearization is called as the global one – see [8] for details. The problem of the exact linearizability of a given system is a particular case of the equivalence of systems: two systems are called (state, feedback,…) equivalent if they can be transformed one into another using the appropriate transformations.

[1]Supported by the Grant Agency of the Czech Republic through the grant No. 102/94/0053.

We aim to discuss in this contribution the basic feature of all transformations presently used in nonlinear control – *smoothness*. Unfortunately, removing the smoothness from the definitions of transformations requires principially more general definitions of the continuous-time controlled dynamical system as well as of these transformations, namely, differential-equation-independent definitions of all these notions are inevitable. We will adapt for this sake the known definition of the topological equivalence of uncontrolled dynamical systems ([13]) and differential-equation-independent definition of controlled dynamical systems (see [19, 14, 18]). Our contribution in this respect consists in specifying functional-spaces norms for input and state trajectories and also in giving differential-equation-independent definition of the feedback. As a result we will give the definition of the controlled dynamical system, topological equivalence of systems and topological linearization. As it will be shown, these definitions generalize the usual definitions of smooth system and its transformations defined via the smooth differential equation parametrized by the control parameter.

In Section 3 we investigate topological linearizability of planar single-input systems while in Section 4 we discuss application to the nonsmooth stabilization. Actually, topological linearization provides practically realizable simple algorithm for the nonsmooth stabilization – see [7, 9].

Throughout the paper we concentrate ourselves on dynamical systems evolving in $\mathbb{R}^n$ and all definitions will have global character. This is due to the main purpose of this short paper: to introduce, illustrate and underline the key features of this rather novelty approach to the understanding of controlled dynamics and their transformations. Local definitions are available with some additional technicalities as well as the case of a general manifold. For similar reasons we consider only state space transformations and static state feedbacks.

Notations. For any $T > 0$ we consider $\mathcal{I}^m(T)$ as the normed space of all Lebesgue integrable functions $u : [-T, T] \to \mathbb{R}^m$ with the norm[1]

$$\|u\| = \max_{t\in[-T,T]} \sum_{i=1}^{m} \left| \int_{-T}^{t} u_i(\tau)\mathrm{d}\tau \right| .$$

Further, $\mathcal{I}^m$ stands for the space of all functions $u : \mathbb{R} \to \mathbb{R}^m$ such that $u_{[-T,T]} \in \mathcal{I}^m(T)$ for any $T > 0$, where $u_{[-T,T]}$ denotes the restriction of $u \in \mathcal{I}^m$ to the interval $[-T, T]$. For any $T > 0$ we consider $\mathcal{A}^n(T)$ as the normed space of all absolutely continuous functions $[-T, T] \to \mathbb{R}^n$ with the supremum norm and we define $\mathcal{A}^n$ analogously as the $\mathcal{I}^m$. We consider $S_\tau, \tau \in \mathbb{R}$, to be a shift operator: $S_\tau : \mathcal{I}^m \to \mathcal{I}^m$ (or $\mathcal{A}^n \to \mathcal{A}^n$), $S_\tau u = v,\ v(t) = u(t + \tau),\ t \in \mathbb{R}$, and $\mathrm{Pr}_T(\cdot), T \in \mathbb{R}$, to be a "projection" operator: $\forall\, u \in \mathcal{I}^m(\text{or } \mathcal{A}^n)\ \mathrm{Pr}_T(u) = u_{[-T,T]} \in \mathcal{I}^m(T)$ (or $\mathcal{A}^n(T)$).

Further, the standard notation from differential geometry will be used, e. g. smooth vector field f is the smooth map from the manifold M into its tangent bundle $T(M)$ such that $\forall\, x\ f(x) \in T_x(M)$; $V(M)$ is the Lie algebra of the smooth

[1] We left to the reader checking all axioms of the norm (when identifying a. e. equal functions). See [4, 5, 6] for interesting properties and applications of this norm.

vector fields on a smooth manifold, $[f,g]$ and $\mathrm{ad}_f g$, $f,g \in V(M)$, stand for the Lie bracket and adjoint operator, respectively. Finally, $\mathrm{Lie}(L)$ stands for the Lie algebra generated by the $L \subset V(M)$.

2. TOPOLOGICAL EQUIVALENCE OF CONTROLLED SYSTEMS

Throughout the paper we consider time-invariant, continuous-time controlled dynamical systems without outputs; such a system will be further referred to as the controlled dynamical system, or simply system. Usually, such a system is described via ordinary differential equation with control parameter. Following freely ideas from (uncontrolled) dynamical systems theory (cf. [13]), as well as the ideas from control theory foundations (cf. [14], [19], [18]), more general description is possible.

Definition 1. A controlled dynamical system is the quadruple $\sum = (\Omega, X, \mathcal{X}, \Phi)$ given by: i) the space of input functions $\Omega = \mathcal{I}^m$; ii) the state space $X = \mathbb{R}^n$; iii) the space of the state trajectories $\mathcal{X} = \mathcal{A}^n$; iv) the controlled dynamics, i. e. the map $\Phi : X \times \Omega \rightarrow \mathcal{X}$, such that $\forall T > 0$, $\forall x_0 \in X$ it holds

a) $\sigma(0) = x_0$ and $S_\tau \sigma = \Phi(\sigma(\tau), S_\tau u)$ where $\sigma = \Phi(x_0, u), u \in \Omega,\ \tau \in \mathbb{R}$,

b) $\mathrm{Pr}_T(\sigma_1) = \mathrm{Pr}_T(\sigma_2)$ if $\mathrm{Pr}_T(u_1) = \mathrm{Pr}_T(u_2)$, where $\sigma_i = \Phi(x_0, u_i),\ i = 1, 2$,

c) $\Phi^T : X \times \mathcal{I}^m(T) \rightarrow \mathcal{A}^n(T)$, $\Phi^T(x_0, u) = \mathrm{Pr}_T \circ \Phi(x_0, \mathrm{Pr}_T^{-1} u_{[-T,T]})$ is a continuous map for all $T > 0$.

Remark 1. The maps $\Phi^T, T > 0$, in iv) c) of the previous definition are well defined; in spite of the fact that $\mathrm{Pr}_T^{-1}(u_{[-T,T]})$ is a set, property iv) b) guarantees that all elements of this set are mapped into the same element of $\mathcal{A}^n(T)$.

Remark 2. Property iv) a) justifies the terms state and state trajectory, i. e. it guarantees that the future of the state trajectory depends only on the present state and on the future input. Property iv) b) express the nonanticipativity of the system: the past of the state trajectory does not depent on the future input.

Definition 2. The controlled dynamical system in the sense of Definition 1 $\sum = (\Omega, X, \mathcal{X}, \Phi)$ will be called smooth if $\Phi(x_0, u) \in C^\infty(\mathbb{R}, X)\ \forall\, u \in C^\infty(\mathbb{R}, \mathbb{R}^m)$, $x_0 \in X$, and $\sigma(t)$ depends $\forall\, t \in \mathbb{R}$ on $x_0 \in X$ in C^∞ manner, where $\sigma = \Phi(x_0, u)$, $\forall\, u \in C^\infty(\mathbb{R}, \mathbb{R}^m)$.

Example 1. The system given by

$$\Phi(x_0, u) = \left(x_0^{1/3} + \int_0^t u(\tau) \mathrm{d}\tau \right)^3, \quad m = 1,\ n = 1,$$

produce for any smooth input the smooth state trajectory. Nevertheless, it is not the smooth system in the sense of Definition 2 since the dependence on x_0 is not smooth.

Theorem 1. $\sum = (\Omega, X, \mathcal{X}, \Phi)$ is the smooth dynamical system iff it is described by the smooth differential equation on X; i.e. there exists C^∞ vector field $f : X \times \mathbb{R}^m \to T(X)$, $f(x,u) \in T_x(X)$, $x \in X$, such that for $\sigma = \Phi(x_0, u)$ it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma(t) = f(\sigma(t), u(t)), \quad \sigma(0) = x_0, \tag{2.1}$$

Moreover, the vector field $\hat{f}(x,r) = (f(x,u(r)),1)^{\mathrm{T}} \in V(X \times \mathbb{R})$ is complete for any smooth $u(r)$.

Remark 3. Due to the smoothness of $u(t)$, $t \in \mathbb{R}$, one can avoid the time dependence in (2.1) by introducing in the well-known fashion the differential equation on $X \times \mathbb{R}$ and the completeness of the vector field $\hat{f}(x,r) \in V(X \times \mathbb{R})$ follows directly from the global character of Definition 1. The local version of this definition that removes the requirement of the completenes from Theorem 1 is available without any principial obstacles and is omitted only for the sake of simplicity.

P r o o f o f T h e o r e m 1. It is an easy exercise to show that (2.1) defines smooth controlled dynamical system in the sense of Definitions 1, 2. Conversely, consider $\sigma = \Phi(x_0, u)$ and $\dot{\sigma}(\tau)$. It follows from iv) a) – b) (see Def. 1) that $\dot{\sigma}(\tau)$ depends only on the current state $\sigma(\tau) \in X$ and on the values of input u and its derivatives at time τ. Actually, the dependence on derivatives is excluded by i) and iii) of Definition 1, so we have arrived to the desired form (2.1). Smoothness of the right hand side of (2.1) follows directly from Definition 2, while completeness is a consequence of the global character of Definition 1. □

Remark 4. It is interesting to underline that without considering the norms of the input trajectories space as the $\mathcal{I}^m$ norm and absolutely continuous state space trajectories it would not be possible to exclude input derivatives from the right hand side of (2.1) (remind in this context the known discussion Kalman versus Zadeh–Desoer in [14, 19]). This can be illustrated by the following example: $\sigma(t) = \Phi(x_0, u) = u(t) + x_0 - u(0)$, where $\dot{u}(t) = \dot{\sigma}(t)$. All requirements of Definition 1 are valid here except i) and iii).

Notice also that for $u \in \mathcal{I}^m$ a weaker understanding of (2.1) in the sense of almost everywhere is necessary. We omit here standart details from function and measure theory.

The more general definitions of the state and feedback equivalence of the controlled dynamical systems are also available.

Definition 3. Controlled dynamical systems $\sum_i = (\Omega, X_i, \mathcal{X}_i, \Phi_i)$, $i = 1, 2$, are called topologically state equivalent if there exist homeomorphism $H : X_2 \to X_1$, $H(X_2) = X_1$, such that $\mathcal{H} \circ \Phi_2(x_0^2, u) = \Phi_1(H(x_0^2), u)$ for any $u \in \Omega, x_0^2 \in X_2$, where $\mathcal{H} : \mathcal{X}_2 \to \mathcal{X}_1$ is the homeomorphism naturally induced by $H : X_2 \to X_1$, i.e. $\mathcal{H}(\sigma)(t) = H(\sigma(t))$, $\forall\, t \in \mathbb{R}$.

Definition 4. Controlled dynamical systems $\sum_i = (\Omega, X, \mathcal{X}, \Phi_i),\ i = 1, 2$, are called topologically static state feedback equivalent if there exists map $F : X \times \Omega \rightarrow \Omega$ such that

i) $\Phi_2(x_0, \cdot) = \Phi_1(x_0, F(x_0, \cdot))\ \forall x_0 \in X$ and $F(\sigma(\tau), S_\tau u) = S_\tau F(x_0, u)$, where $\tau \in \mathbb{R}, \sigma = \Phi_2(x_0, u), u \in \Omega$;

ii) $\forall T > 0, x_0 \in X :\ \mathrm{Pr}_T(F(x_0, u_1)) = \mathrm{Pr}_T(F(x_0, u_2))$ if $\mathrm{Pr}_T(u_1) = \mathrm{Pr}_T(u_2)$;

iii) $F^T(\cdot, \cdot) = \mathrm{Pr}_T \circ F(\cdot, \mathrm{Pr}_T^{-1}(\cdot)),\ F^T : X \times \mathcal{I}^m(T) \rightarrow \mathcal{I}^m(T)$ is a continuous map and $F^T(x_0, \cdot) : \mathcal{I}^m(T) \rightarrow \mathcal{I}^m(T)$ is a homeomorphism for any $T > 0, x_0 \in X$.

Definition 3*. Smooth systems $\sum_i = (\Omega, X_i, \mathcal{X}_i, \Phi_i),\ i = 1, 2$, are called smoothly state equivalent if there exists diffeomorphism $D : X_2 \rightarrow X_1,\ D(X_2) = X_1$, such that $\mathcal{D} \circ \Phi_2(x_0^2, u) = \Phi_1(D(x_0^2), u),\ u \in \Omega, x_0 \in X_2$. Here $\mathcal{D} : \mathcal{X}_2 \rightarrow \mathcal{X}_1$ is naturally induced by D and maps smooth trajectories onto smooth trajectories.

Definition 4*. Smooth systems $\sum_i = (\Omega, X, \mathcal{X}, \Phi_i),\ i = 1, 2$, are called smoothly static state feedback equivalent if they are topologically static state feedback equivalent in the sense of Definition 4, the map $F : X \times \Omega \rightarrow \Omega$ maps any smooth and only smooth function into the smooth function and for any $u \in \Omega,\ T + 0,\ v(T)$ depends smoothly on x_0, where $v = F(x_0, u)$.

Definition 5. (Smooth) controlled dynamical systems $\sum_i = (\Omega, X_i, \mathcal{X}_i, \Phi_i),\ i = 1, 2$, are called topologically (smoothly) state and static state feedback equivalent if there exists $\sum_3$ that is topologically (smoothly) state equivalent to $\sum_1$ and static state feedback equivalent to $\sum_2$.

The following theorem is stated without proof which is rather straightforward.

Theorem 2. Smooth dynamical systems systems $\sum_1, \sum_2$ described by the differential equations

$$\sum\nolimits_1 : \dot{x} = f(x, u),\ x \in X,\ u \in \mathbb{R}^m,$$

$$\sum\nolimits_2 : \dot{y} = g(y, v),\ y \in Y,\ v \in \mathbb{R}^m,$$

are smoothly state and static feedback equivalent iff there exists diffeomorphism $D : Y \rightarrow X,\ D(Y) = X$ and the smooth map $\alpha : Y \times \mathbb{R}^m \rightarrow \mathbb{R}^m,\ \alpha(y, \cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ being one-to-one for any $y \in Y$, such that

$$D_y g(y, v) = f(D(y), \alpha(y, v)). \tag{2.2}$$

Remark 5. Notice that the transformation (2.2) gives an implicit and pointwise expression of the feedback map F from Definition 4, namely $F(y_0, v)(t) = \alpha(y(t), v(t))$, where $y(t)$ is the solution of $\dot{y} = g(y, v)$ with $y(0) = y_0$. This is possible due to the second part of i) of Definition 4: in fact, F is defined there implicitly as a solution of a certain functional equality. Up to our knowledge this is the original and unique definition of the feedback for continuous-time systems that does not need the notion of the ordinary differential equation.

Remark 6. Analogously as in the case of Definition 1 it is necessary to underline the global character of Definitions 3–5 resulting into the global character of Theorem 2. With some additional technicalities their local versions would be available.

Both Theorem 1 and 2 illustrate that the introduced more general definitions of controlled dynamical systems and their equivalence are not only the generalization of the (2.1) and (2.2), but even more: the only thing that is weakened in these definitions is the smoothness. This fact seems to justify such an approach.

3. STATE AND FEEDBACK TOPOLOGICAL LINEARIZATION OF NONLINEAR SYSTEMS

The main result gives a sufficient condition for the state and feedback topological linearization of planar single-input systems. Despite the fact that we are omitting throughout the paper local definitions, we formulate also local version of this result. As indicated earlier, local versions of all definitions are easily available: in this case all mappings should be restricted to small time intervals containig zero and depending on initial state. For the local version a comparision with the known smooth results is more apparent (see Remarks 7, 8).

Theorem 3. The smooth system

$$\dot{x}_1 = f(x_1, x_2) \quad , \quad \dot{x}_2 = u, \tag{3.1}$$

$f(0) = 0$, is locally (globally) state and static state feedback topologically linearizable at $0 \in \mathbb{R}^2$ (on $\mathbb{R}^2$) if $(x_1, x_2)^{\mathrm{T}} \to (x_1, f(x_1, x_2))^{\mathrm{T}}$ is a local (global) homeomorphism at $0 \in \mathbb{R}^2$ (of $\mathbb{R}^2$ onto itself).

Proof. Let us denote $\xi = H(x_1, x_2) = (x_1, f(x_1, x_2))^{\mathrm{T}}$ the homeomorphism between neighbourhoods of the origin (global homeomorphism of $\mathbb{R}^n$) and let $H^{-1}(\xi_1, \xi_2) = (\xi_1, \psi(\xi_1, \xi_2))^{\mathrm{T}}$. Consider integral equivalent of (3.1)

$$x_1 = x_1^0 + \int_0^t f(x_1, x_2)\,\mathrm{d}s, \qquad x_2 = x_2^0 + \int_0^t u(s)\,\mathrm{d}s \tag{3.1*}$$

The above homeormorphism takes (3.1*) into the form

$$\xi_1 = \xi_1^0 + \int_0^t \xi_2\,\mathrm{d}s, \qquad \xi_2 = f(\xi_1, \psi(\xi_1^0, \xi_2^0) + \int_0^t u(s)\,\mathrm{d}s). \tag{3.2*}$$

For sufficiently small $|\xi_1^0|, |\xi_2^0|$, and $T > 0$ ($\forall\, \xi_{1,2}^0,\ T \in \mathbb{R}$ in the global case) we have that the map $F : \mathcal{I}^1(T) \times V_0 \to \mathcal{I}^1(T), u = F(\xi_0, v)$, given by

$$\int_0^t u(s)\,\mathrm{d}s = \psi\left(\xi_1^0 + \int_0^t \left(\xi_2^0 + \int_0^\tau v(\alpha)\,\mathrm{d}\alpha\right)\mathrm{d}\tau, \xi_2^0 + \int_0^t v(\alpha)\,\mathrm{d}\alpha\right) - \psi(\xi_1^0, \xi_2^0),\ \ |t| \le T, \tag{3.3}$$

is the feedback in the sense of Definition 4 and its application to (3.2*) leads in a straightforward way to the following linear system

$$\xi_1 = \xi_1^0 + \int_0^t \xi_2 \mathrm{d}s, \qquad \xi_2 = \xi_2^0 + \int_0^t v(\alpha)\,\mathrm{d}\alpha. \qquad \square$$

Remark 7. Let us consider the smooth (but nonanalytic) system (1) with $f(x_1, x_2) = x_1 + \mathrm{sign}(x_2)\exp(-|1/x_2|)$. By Theorem 3 this system is locally topologically linearizable and therefore it is also small time locally controllable. On the other hand, note that $\dim \mathrm{Lie}\left(\left\{(f,0)^{\mathrm{T}}, (0,1)^{\mathrm{T}}\right)\right\}(0) = 1$!

Remark 8. It is well known (cf. e.g. [17]) that the system $\dot{x} = \hat{f}(x) + ug(x),\ x \in \mathbb{R}^n, u \in \mathbb{R}$, is locally smoothly state and static state feedback linearizable iff $g, \mathrm{ad}_{\hat{f}} g$ are linearly independent at $0 \in \mathbb{R}^2$. For the system in the form (3.1) this is equivalent to $f_{x_2}(0,0) \neq 0$ and in this case $\xi = (x_1, f(x_1, x_2))^{\mathrm{T}},\ v = f_{x_2} u + f_{x_1} f$, are the linearizing diffeomorphism and the usual nonlinear feedback. In case $f_{x_2}(0,0) = 0$, the mapping $\xi = (x_1, f(x_1, x_2))^{\mathrm{T}}$ may be only homeomorphism, while feedback is even not well defined in the usual sense at $(0,0)^{\mathrm{T}}$ and therefore the system (1) may be only topologically linearizable. In other words, the class of topologically linearizable smooth systems is wider than the class of smoothly linearizable smooth systems. In this connection it is reasonable to underline that the nonsmoothness of the linearizing transformations is not caused by the nonsmoothness of the original nonlinear system.

4. EXAMPLE: APPLICATION TO THE NONSMOOTH STABILIZATION

To illustrate the previous approach more clearly as well as to show that the topological feedback is the reasonable notion let us consider the known Aeyel's, example (see [2], [12], [15]), namely, consider the following globally controllable (see [1]) but smoothly nonstabilizable system:

$$\dot{x}_1 = x_1 + x_2^3, \qquad \dot{x}_2 = u. \tag{4.1}$$

In spite of the fact that (4.1) is not smoothly linearizable ($\mathrm{ad}_f^i g(0) = 0, i \ge 1$) we have by Theorem 3 that it is topologically linearizable. Moreover, the system (4.1) can be linearized globally. Let us consider integral form of (4.1)

$$x_1(t) = x_1(0) + \int_0^t (x_1(\tau) + x_2^3(\tau))\,\mathrm{d}\tau, \qquad x_2(t) = x_2(0) + \int_0^t u(\tau)\,\mathrm{d}\tau. \tag{4.1*}$$

Global homeomorphism $\xi = (x_1, x_2^3)^{\mathrm{T}}$ and the feedback given by

$$\int_0^t u(\tau)\,\mathrm{d}\tau = \sqrt[3]{\xi_2^0 + \int_0^t v(\tau)\,\mathrm{d}\tau} \; - \; \sqrt[3]{\xi_2^0}, \quad t > 0, \tag{4.2*}$$

takes (4.1*) into the linear form

$$\xi_1(t) = \xi_1^0 + \int_0^t (\xi_2(\tau) + \xi_2(\tau))\,\mathrm{d}\tau, \qquad \xi_2(t) = \xi_2^0 + \int_0^t v(\tau)\,\mathrm{d}\tau. \tag{4.3}$$

Note that differentiation of (4.2*) gives the expression

$$u = v/3x_2^2 \tag{4.2}$$

that is not well defined at $(0,0)^{\mathrm{T}}$. The system (4.3) can be stabilized using linear feedback $v = -a\xi_1 - b\xi_2$, $a > b > 1$, and the corresponding stabilizing feedback for (4.1*) would lead to the stable system

$$\begin{aligned} x_1(t) &= x_1(0) + \int_0^t (x_1(\alpha) + x_2^3(\alpha))\,\mathrm{d}\alpha \\ x_2(t) &= \left[x_2^3(0) + \int_0^t (-a x_1(\alpha) - b x_2^3(\alpha))\,\mathrm{d}\alpha \right]^{1/3}. \end{aligned} \tag{4.4*}$$

The transformation of (4.1*) leading to (4.4*) seems to be reasonable and simply physically realizable (e. g. for electrical circuits), at least by block diagrams. On the other hand differentiation of (4.4*) leads to the asymptotically stable differential equation with the discontinuous and unbounded right-hand side

$$\dot{x}_1 = x_1 + x_2^3, \qquad \dot{x}_2 = -a\frac{x_1}{3x_2^2} - \frac{b}{3}x_2, \tag{4.4}$$

whose solutions are well defined in the sense of almost everywhere. This is of course not so nice solution of the stabilization problem as that of [15] but due to its integral form (4.1*)–(4.4*) it may be of practical interest.

In [7] a simple practical application of the stabilizer (4.4) is presented: the right hand side is "cutted" (i. e. whenever $|x_2|$ is too small it is replaced by a suitable nonzero constant) to prevent singularity create unboundedness on the right hand side. Such approach corresponds well to the engineering common sense approach (see [16], p. 52 – "Backlash example") and was successfully tested by numerical simulations. Our topological linearization serves therefore as a theoretical justification of this practically based approach, since the continuity of the topological feedback with respect to the $\mathcal{I}^m$-norm guarantees that a "small cutting" may cause small changes in behaviour of the closed loop system only.

5. CONCLUDING REMARKS

An attempt was made to enlarge the class of linearizable nonlinear systems using the differential-equation-independent definiton of the controlled dynamical system as well as the topological (nosmooth) generalizations of the state and feedback transformations.

In spite of the relative success of this approach the principal question arises what should be the method for treating this problem when all known results concerning smooth or analytic linearization are formulated by the language of smooth vector fields, Lie derivations etc. It seems to be appropriate to use for this task integral equations, possibly together with the Volterra series and/or Fliess functional expansions representation of the controlled dynamical systems. Finally, the example in the Section 4 indicates the way based on using smooth transformations with "negligible" singularities.

(Received September 27, 1993.)

REFERENCES

[1] D. Aeyels: Local and global controllability for nonlinear systems. Systems Control Lett. *5* (1984), 19–26.

[2] D. Aeyels: Stabilization of a class of nonlinear systems by a smooth feedback control. Systems Control Lett. *5* (1985), 289–294.

[3] R. W. Brockett: Feedback invariants for nonlinear systems. In: Proc. of the VIIth IFAC World Congress, Helsinki 1978, pp. 1115–1120.

[4] S. Čelikovský: On the representation of trajectories of bilinear systems and its applications. Kybernetika *23* (1987), 3, 198–213.

[5] S. Čelikovský: On the continuous dependence of trajectories of bilinear systems and its applications. Kybernetika *24* (1988), 4, 278–292.

[6] S. Čelikovský: On the Lipschitzean dependence of trajectories of multi-input time dependent bilinear systems on controls. Problems Control Inform. Theory *17* (1988), 4, 231–238.

[7] S. Čelikovský: Topological linearization of nonlinear systems: Application to the nonsmooth stabilization. In: Proc. of the 2nd ECC'93, Groningen 1993, pp. 41–44.

[8] S. Čelikovský: Global linearization of nonlinear systems: A survey. In: Geometry in Nonlinear and Differential Inclusions (Banach Center Publications, Vol. 32), Polish Academy of Sciences, Warszawa 1995.

[9] S. Čelikovský: Numerical algorithm for the nonsmooth stabilization based on topological linearization. In: Optimization-Based Computer-Aided Modelling and Design (J. Doležal and J. Fidler, eds.), ÚTIA AV ČR, Prague 1995.

[10] D. Claude: Everything you always wanted to know about linearization but were afraid to ask. In: Proc. of the Conf. Algebraic and Geometric Methods in Nonlinear Control Theory, Paris 1985 (M. Fliess and M. Hazewinkel, eds.), Reidel, Dordrecht 1986.
[11] W. Dayawansa, W. M. Boothby and D. L. Elliot: Global state and feedback equivalence of nonlinear systems. Systems Control Lett. *6* (1985), 229–234.
[12] M. Fliess and F. Messager: Vers une stabilisation non lineaire discontinue. In: Analysis and Optimization of Systems (A. Bensoussan and J. L. Lions, eds.), (Lecture Notes in Control Inform. Science 144), Springer-Verlag, New York 1990, pp. 778–787.
[13] J. Guckenheimer and P. Holmes: Nonlinear Oscillations, Dynamical systems and Bifurcations of Vector Fields. Springer-Verlag, New York 1986.
[14] R. E. Kalman, P. L. Falb and M. A. Arbib: Topics in Mathematical Systems Theory. McGraw–Hill, New York 1969.
[15] M. Kawski: Stabilization of nonlinear systems in the plane. Systems Control Lett. *12* (1989), 169–175.
[16] R. R. Kadiyala: A tool box for approximate linearization of nonlinear systems. IEEE Control Systems Magazine, April 1993, 47–57.
[17] W. Respondek: Geometric methods in linearization of control systems. In: Mathematical Control Theory, Banach Center Publications *14* (1985), 453–467.
[18] J. C. Willems: Paradigms and puzzles in the theory of dynamical systems. IEEE Trans. Automat. Control *36* (1991), 259–294.
[19] L. A. Zadeh and C. A. Desoer: Linear Systems Theory. McGraw–Hill, New York 1963.

RNDr. Sergej Čelikovský, CSc., Ústav teorie informace a automatizace AV ČR (Institute of Information Theory and Automation – Academy of Sciences of the Czech Republic), Pod vodárenskou věží 4, 182 08 Praha. Czech Republic.