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CONTINUOUS–TIME INPUT–OUTPUT DECOUPLING FOR SAMPLED–DATA SYSTEMS

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The problem of obtaining a continuous-time (i. e., ripple-free) input-output decoupled control system for a continuous-time linear time-invariant plant, by means of a purely discrete-time compensator, is stated and solved in the case of a unity feedback control system. Such a control system is hybrid, since the plant is continuous-time and the compensator is discrete-time. A necessary and sufficient condition for the existence of a solution of such a problem is given, which reduces the mentioned hybrid control problem to an equivalent purely continuous-time decoupling problem. A simple necessary and sufficient condition for the existence of a solution of such a continuous-time decoupling problem is given for square plants (with and without the additional requirement of the asymptotic stability of the over-all control system), together with a parameterisation of all the decoupling controllers. Moreover, for square plants, it is shown that, whenever the hybrid control problem admits a solution, any solution of the corresponding decoupling problem for the discrete-time model of the given continuous-time system is also a solution of the hybrid control problem.

1. INTRODUCTION

The problem of input-output decoupling is one of the most widely investigated for purely continuous-time or purely discrete-time MIMO control systems [3, 4, 6, 10, 11, 12, 15, 16], since it is a very natural control objective (and, in addition, it can be a useful tool for other requirements, e. g. robustness [5]). In this paper, the problem of input-output decoupling is dealt with for sampled-data systems, which are considered in their hybrid nature (both discrete-time and continuous-time). Therefore, the intersample behaviour is explicitly taken into account, in order to avoid undesirable ripple between sampling instants, which may become unacceptable if the sampling rates are small, or if unbounded exogenous signals are involved, since the amplitudes of such a ripple are modulated by the nonzero scalar exogenous signal [13]. Such an approach is now becoming classical in the study of sampled-data systems [8, 9, 14, 17], but, to the best of the authors' knowledge, has never been applied to the input-output decoupling problem. The mentioned contributions [8, 9, 14, 17] recognise that a continuous-time subcompensator may be needed in order to achieve continuous-time asymptotic tracking and regulation;

Fig. 1. The hybrid control system S.

in this paper it will be shown that this applies also when a continuous-time input-output decoupling is required. Here a unity feedback control scheme is assumed, as in several contributions on input-output decoupling for purely continuous-time plants [10, 12, 15]: a motivation of this choice is that unity feedback might be required in order to achieve further control objectives like, for example, asymptotic tracking, thus involving the presence of an internal model of the exogenous signals in the forward path of the feedback control system.

2. PROBLEM FORMULATION

In this section the problem of the continuous-time input-output decoupling will be formally stated for both the hybrid open-loop control system S in Figure 1 and the closed-loop system Σ obtained from S under a unity feedback (see Figure 3), since the continuous-time input-output decoupling holds for Σ if and only if it holds for S (see the subsequent Proposition 2), as it happens for purely continuous-time or purely discrete-time systems.

The hybrid control system S is constituted by the following components:

– the continuous-time linear time-invariant plant P, to be controlled, having $x(t) \in \mathbb{R}^{n_P}$ as state at time $t \in \mathbb{R}$, and the strictly proper rational matrix $P(s)$ as transfer matrix between the input $u(t) \in \mathbb{R}^p$ and the output $y(t) \in \mathbb{R}^q$, $q \le p$; system P is described, in state space form, by the equations:

$$\dot{x}(t) = A\,x(t) + B\,u(t), \tag{1a}$$

$$y(t) = C\,x(t); \tag{1b}$$

– the zero-order holder H_{δ_T}, with holding period δ_T, $\delta_T \in \mathbb{R}$, $\delta_T > 0$, having $u_D(k) \in \mathbb{R}^p$ as discrete-time input, and $u(t)$ as continuous-time output, expressed by:

$$u(0) = 0, \tag{2a}$$

$$u(t) = u_D(k), \quad k\,\delta_T < t \le (k+1)\,\delta_T, \quad k \in \mathbf{Z}^+,\ t \in \mathbb{R}; \tag{2b}$$

– the sampling device S_{δ_T}, with sampling period δ_T, having $y(t)$ as continuous-time input and $y_D(k) \in \mathbb{R}^q$ as discrete-time output, expressed by:

$$y_D(k) = y(k\,\delta_T), \quad k \in \mathbf{Z}^+; \tag{3}$$

– the discrete-time linear time-invariant compensator K, having $x_K(k) \in \mathbb{R}^{n_K}$ as state, $r_D(k) \in \mathbb{R}^q$ as discrete-time input, $u_D(k)$ as discrete-time output and the proper rational matrix $K(z)$ as transfer matrix between $r_D(k)$ and $u_D(k)$.

It is assumed that S_{δ_T} and H_{δ_T} are synchronised, as it is implied by (2) and (3).

It is well known that the behaviour of the series connection P_H of the holder H_{δ_T}, system P and the sampler S_{δ_T} can be modelled at the sampling instants by a purely discrete-time system P_D, whose state $x_D(k) \in \mathbb{R}^{n_P}$ is defined by

$$x_D(k) := x(k\,\delta_T), \quad k \in \mathbf{Z},$$

and whose transfer matrix between its input $u_D(k)$ and its output $y_D(k)$, denoted by $P_D(z)$, is a strictly proper rational matrix.

In the following, $r_{D,\,j}(k)$ and $y_j(t)$ will denote the jth scalar component of $r_D(k)$ and $y(t)$, respectively, $j = 1, 2, \ldots, q$.

The following definition can be referred to both the hybrid control systems represented in Figures 1 and 3.

Definition 1. (Continuous-time input-output decoupling) A hybrid system having a discrete-time input $r_D(k)$ and a continuous-time output $y(t)$ is said to be continuous-time input-output decoupled if it satisfies the following conditions:

(i) for each $i = 1, 2, \ldots, q$, its output response $y(\cdot)$ from its zero initial state to an input function $r_D(\cdot)$ with $r_{D,\,j}(\cdot) = 0$, for $j = 1, 2, \ldots, q$, $j \neq i$, is such that $y_j(\cdot) = 0$ for $j = 1, 2, \ldots, q$, $j \neq i$;

(ii) the transfer matrix between $r_D(k)$ and the sampling $y_D(k)$ of its continuous-time output $y(t)$ is nonsingular over the rational field.

In Definition 1, in order to avoid trivial solutions, the discrete-time condition (ii) has been used; it is expressed in terms of the transfer matrix of the discrete-time model of the hybrid system under consideration (i. e., the matrix $P_D(z)\,K(z)$ for the open-loop system in Figure 1). A different "continuous-time" condition is the following one:

(iii) in condition (i) the component $y_i(\cdot)$ of the output response $y(\cdot)$ is nonzero, for each function $r_{D,\,i}(\cdot)$ having a nonzero rational z-transform.

It is easy to see that, if condition (i) holds, then condition (ii) implies (iii), whereas the following counterexample, involving a continuous-time system P having $p > q$, shows that the opposite is not true, in general.

Example 1. Let the matrices describing system (1) be given by:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} \frac{1}{1-\mathrm{e}^{-1}} & 0 & 0 \\ 0 & \frac{2}{1-\mathrm{e}^{-2}} & 0 \\ 0 & 0 & \frac{1}{1-\mathrm{e}^{-1}} \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $\delta_T = 1$, the transfer matrix $P_D(z)$ can be easily expressed as follows:

$$P_D(z) = \begin{bmatrix} \frac{1}{z-\mathrm{e}^{-1}} & \frac{1}{z-\mathrm{e}^{-2}} & 0 \\ 0 & 0 & \frac{1}{z-\mathrm{e}^{-1}} \end{bmatrix}.$$

If the dynamic compensator K has the following transfer matrix:

$$K(z) = \begin{bmatrix} 1 & 0 \\ -\frac{z-\mathrm{e}^{-2}}{z-\mathrm{e}^{-1}} & 0 \\ 0 & 1 \end{bmatrix},$$

then the following transfer matrix between $r_D(k)$ and $y_D(k)$ is obtained:

$$P_D(z)\,K(z) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{z-\mathrm{e}^{-1}} \end{bmatrix},$$

which is diagonal, but does not satisfy condition (ii). In order to prove that conditions (i) and (iii) are satisfied, it is convenient to compute the output responses $y^{(1)}(t)$ and $y^{(2)}(t)$, from null initial conditions, corresponding to the input signals $r_D^{(1)}(k) = \begin{bmatrix}\delta_0^D(k) & 0\end{bmatrix}'$ and $r_D^{(2)}(k) = \begin{bmatrix}0 & \delta_0^D(k)\end{bmatrix}'$, respectively, where $\delta_0^D(k)$ denotes the discrete-time unit impulse function. By means of standard computations, described in detail in [13], such output responses can be expressed by:

$$\begin{aligned} y^{(1)}(\varepsilon) &= \begin{bmatrix} \eta_1(\varepsilon) - \eta_2(\varepsilon) \\ 0 \end{bmatrix}, \quad \forall\,\varepsilon \in [0,\ 1), \\ y^{(1)}(k\,\delta_T + \varepsilon) &= \begin{bmatrix} \mathrm{e}^{-(k+\varepsilon-1)} - \mathrm{e}^{-(k+2\,\varepsilon-1)} - \eta_2(\varepsilon)\frac{\mathrm{e}-1}{\mathrm{e}}\mathrm{e}^{-k} \\ 0 \end{bmatrix}, \\ &\qquad \forall\,k > 0,\ k \in \mathbf{Z},\ \forall\,\varepsilon \in [0,\ 1), \\ y^{(2)}(\varepsilon) &= \begin{bmatrix} 0 \\ \eta_1(\varepsilon) \end{bmatrix}, \quad \forall\,\varepsilon \in [0,\ 1), \\ y^{(2)}(k\,\delta_T + \varepsilon) &= \begin{bmatrix} 0 \\ \mathrm{e}^{-(k+\varepsilon-1)} \end{bmatrix}, \ \forall\,k > 0,\ k \in \mathbf{Z},\ \forall\,\varepsilon \in [0,\ 1), \end{aligned}$$

where:

$$\eta_1(\varepsilon) := \frac{1-\mathrm{e}^{-\varepsilon}}{1-\mathrm{e}^{-1}}, \quad \eta_2(\varepsilon) := \frac{1-\mathrm{e}^{-2\,\varepsilon}}{1-\mathrm{e}^{-2}}, \quad \forall\,\varepsilon \in [0,\ 1).$$

This is readily seen to imply that conditions (i) and (iii) hold (see also the plots reported in Figure 2). However, the singularity of $P_D(z)\,K(z)$ makes $y^{(1)}(k\,\delta_T) = 0$, $\forall\,k \in \mathbf{Z}^+$, so that $y^{(1)}(t)$ can be considered as a mere ripple. □

Condition (ii) has been preferred to the weaker condition (iii), since, as it is evident from Example 1, the latter would allow input-output decoupled systems in which the output response to some input functions (namely those having all the nonzero components in correspondence to the zeros in the main diagonal of the (singular) transfer matrix from $r_D(k)$ to $y_D(k)$) would be constituted only by ripple.

Fig. 2. The nonzero components of the output responses $y^{(1)}(t)$ and $y^{(2)}(t)$ considered in Example 1.

Fig. 3. The hybrid control system Σ considered in Proposition 2.

However, the subsequent Proposition 1, whose proof is reported in the next section, states that, for square systems, i. e., for $p = q$, both conditions (ii) and (iii) can be used to define continuous-time input-output decoupling, under the following assumption.

Assumption 1. For each eigenvalue λ of matrix A, none of the complex numbers $\lambda + \jmath 2\pi i/\delta_T$, $i \neq 0$, $i \in \mathbf{Z}$, is an eigenvalue of A.

Remark 1. Assumption 1 is commonly used in order to guarantee that the structural properties of the continuous-time system P are preserved for P_D [2], can be used for guaranteeing the reverse implication for stability [7], and can be easily satisfied by the choice of δ_T (e. g., it is satisfied if δ_T is small enough).

Proposition 1. Under Assumption 1, if $p = q$, conditions (i) and (iii) imply condition (ii).

On the basis of Definition 1, it is now possible to state and proof the following proposition.

Proposition 2. The hybrid control system Σ depicted in Figure 3 is continuous-time input-output decoupled if and only if system S in Figure 1 is.

P r o o f . (if) If system S is continuous-time input-output decoupled, then, considering the control system in Figure 3, the transfer matrix from $e_D(k)$ to $y_D(k)$ of the discrete-time model S_D of S is diagonal and nonsingular, hence also the discrete-time model Σ_D of the hybrid control system Σ has a diagonal and nonsingular transfer matrix from $r_D(k)$ to $y_D(k)$, so that condition (ii) is satisfied by system Σ. Hence, for every $i = 1, \ldots, q$, for every reference input $r_D(\cdot)$ such that $r_{D,j}(k) = 0$ for each $k \in \mathbf{Z}^+$ and for each $j \neq i$, one has $y_{D,j}(k) = 0$ for each $k \geq 0$ and for each $j \neq i$. This implies that the discrete-time signal $e_D(\cdot)$, defined as $e_D(k) := r_D(k) - y_D(k)$, $k \geq 0$, is such that $e_{D,j}(k) = 0$ for all $k \geq 0$ and for each $j \neq i$; this fact, since system S satisfies condition (i), implies $y_j(t) = 0$ for all $t \geq 0$ and for each $j \neq i$, so that condition (i) is satisfied by Σ.

(only if) If system Σ is continuous-time input-output decoupled, its discrete-time model Σ_D is input-output decoupled, whence, by means of purely discrete-time reasonings, wholly similar to those used in the (if) part of this proof, it is easy to see that the transfer matrix from $e_D(k)$ to $y_D(k)$ of the discrete-time model S_D of S, is diagonal and nonsingular, thus implying that condition (ii) is satisfied by S. Hence, for every signal $e_D(\cdot)$ such that $e_{D,j}(k) = 0$ for each $j \neq i$ and for each $k \geq 0$, since S_D is (discrete-time) input-output decoupled, one has that $y_{D,j}(k) = 0$ for each $k \geq 0$ and for each $j \neq i$; this implies that the discrete-time signal $r_D(\cdot)$ defined by $r_D(k) := e_D(k) + y_D(k)$ is such that $r_{D,j}(k) = 0$ for all $k \geq 0$ and for each $j \neq i$. For such a reference signal, $y_j(t) = 0$ for all $t \geq 0$ and for each $j \neq i$, since system Σ satisfies condition (i) of Definition 1. Since such a reasoning holds for every $e_D(\cdot)$ such that $e_{D,j}(\cdot) = 0$ for each $j \neq i$, and for each $i = 1, \ldots, q$, then it is proved that the hybrid system S satisfies (i). □

3. A SOLVABILITY CONDITION OF THE PROBLEM

The following theorem provides a necessary and sufficient condition for the existence of a solution of the continuous-time input-output decoupling problem, for the hybrid control system in Figure 1 or in Figure 3, by reducing such a problem to a purely continuous-time control problem, namely that of the existence of a static precompensator achieving input-output decoupling for the continuous-time plant P.

Theorem 1. Under Assumption 1, there exists a discrete-time compensator K that achieves continuous-time input-output decoupling for the hybrid control system S in Figure 1 [or for the hybrid control system Σ in Figure 3], if and only if there exists a constant matrix $M \in \mathbb{R}^{p \times q}$ such that $P(s)\,M$ is diagonal and nonsingular. If this condition holds, under the same Assumption 1, the static compensator K having $K(z) = M$ as transfer matrix achieves continuous-time input-output decoupling for the same system S in Figure 1 [for the same system Σ in Figure 3].

Remark 2. The latter statement of Theorem 1 stresses that, under Assumption 1, if continuous-time input-output decoupling can be achieved for system S [or for system Σ], a solution can be obtained in form of a static discrete-time controller K, and that its constant transfer matrix M can be designed by means of purely continuous-time techniques. However, if plant P is not asymptotically stable but is stabilisable and detectable, in order to obtain an asymptotically stable control system Σ it can be more convenient to design the discrete-time controller K as the series connection of a dynamic subcompensator K_S and the static subcompensator having M as transfer matrix, and to choose the transfer matrix of K_S in form of a square non-singular and diagonal rational matrix, if any, such that the discrete-time model Σ_D of the hybrid control system Σ in Figure 3 is asymptotically stable, thus implying the asymptotic stability of Σ (see Theorem 4 in [7]), and still guaranteeing for Σ the continuous-time input-output decoupling (for square plants see the subsequent Remarks 5 and 6 and Theorem 3).

Proof of Theorem 1. Just the part of the statements concerning the control system S in Figure 1 will be proven, since the part concerning the control system Σ in Figure 3 (which is stated in square brackets) can be derived from the former part by virtue of Proposition 2.

(if) It will be shown that the hybrid system S in Figure 1 is continuous-time input-output decoupled, if the discrete-time compensator K is static and has transfer matrix $K(z) = M$.

In fact, notice that, with this choice, the series connection H_{DM} of K and H_{δ_T} in Figure 1, has the same input-output behaviour as the series connection H_{CM} of a q-dimensional zero-order holder followed by the static continuous-time compensator K_{CM} having transfer matrix equal to M. With such a replacement, the resulting control system certainly satisfies condition (i), since the underlying continuous-time system, having $P(s)\,M$ as transfer matrix, is input-output decoupled. In order to prove that condition (ii) holds, define

$$r_D^i(k) = \delta_{-1}^D(k)\, e_i, \quad i = 1, 2, \ldots, q,$$

where e_i denotes the ith column of the q-dimensional identity matrix, and $\delta_{-1}^D(k)$ denotes the discrete-time unit step function, and apply the input function $r_D(\cdot) = r_D^i(\cdot)$ to the hybrid control system obtained from system S with the mentioned replacement; then, from the zero initial state, the following continuous-time output response $y(t) = y^i(t)$ is obtained:

$$y^i(t) := \mathcal{L}^{-1}\left\{ \left[\frac{P(s)\,M}{s}\right]_{i,i} e_i \right\}, \quad t \in \mathbb{R},\ t \geq 0, \tag{4}$$

since it is the output response of the series connection of K_{CM} and P to the continuous-time input function

$$u_c^i(t) = \delta_{-1}(t)\, e_i, \tag{5}$$

where $\delta_{-1}(t)$ is the continuous-time unit step function. By the diagonality of $P(s)\,M$, $y^i(t)$ can be expressed as

$$y^i(t) = y_i^i(t)\, e_i, \tag{6}$$

for some scalar function $y_i^i(t)$, and, by the nonsingularity of $P(s)\,M$, $y_i^i(\cdot)$ is nonzero. Notice that, for each $i = 1, 2, \ldots, q$, the input function (5) can be seen as a free response of a system K_I constituted by the parallel of q integrators, to be connected to K_{CM}. Then, the discrete-time function $y_D^i(k)$, obtained by sampling $y^i(t)$, cannot be identically zero, because this would imply a loss of observability, due to sampling, of the series connection of K_I, K_{CM} and P, which cannot take place, in view of Assumption 1, as can be easily shown. Hence, the z-transform $y_D^i(z)$ of $y_D^i(k)$ is nonzero, and by (6) can be expressed as:

$$y_D^i(z) = y_{D,\,i}^i(z)\,e_i, \tag{7}$$

where $y_{D,\,i}^i(z)$ is the z-transform of the sampling of $y_i^i(t)$. On the other hand, by the mentioned equivalence between H_{CM} and H_{DM}, it is easy to see that

$$y_D^i(z) = P_D(z)\,M\frac{z}{z-1}e_i. \tag{8}$$

By (7) and (8) the ith column of $P_D(z)\,M$ is

$$\frac{z-1}{z}y_{D,\,i}^i(z)\,e_i,$$

for each $i = 1, 2, \ldots, q$, hence $P_D(z)\,M$ is (diagonal and) nonsingular over the field of rational functions.

(only if) Now, given a discrete-time precompensator K such that the hybrid control system S in Figure 1 is continuous-time input-output decoupled, a constant matrix $M \in \mathbb{R}^{p\times q}$ will be determined, such that

$$P(s)\,M = \operatorname{diag}\left(m_1(s),\, m_2(s),\, \ldots,\, m_q(s)\right),$$

with $m_i(s)$, $i = 1, 2, \ldots, q$, being nonzero rational functions.

For each $i = 1, 2, \ldots, q$, for zero initial state of system S, if the input of S is chosen as $r_D(k) = r_D^{(i)}(k)$, with

$$r_D^{(i)}(k) := r_i(k)\,e_i, \quad k \in \mathbf{Z}^+, \tag{9}$$

where $r_i(\cdot)$ is an arbitrary nonzero scalar function, then, in view of the obtained continuous-time input-output decoupling, the output $y(t)$ can be written as:

$$y(t) = y^{(i)}(t) = y_i^{(i)}(t)\,e_i, \quad \forall\, t \geq 0, \tag{10}$$

for some scalar function $y_i^{(i)}(t)$. In this situation, the corresponding input $u(t)$ of system P can be written as:

$$u(t) = u^{(i)}(t) = \sum_{r=0}^{\infty} a_r^{(i)}\delta_{-1}(t - r\,\delta_T),$$

for some $a_r^{(i)} \in \mathbb{R}^p$, $r = 0, 1, \ldots, \infty$. Then $P(s)a_r^{(i)}$ cannot be zero for all $r = 0, 1, \ldots, \infty$, since otherwise $y_i^{(i)}(\cdot)$ would be zero, thus implying a contradiction with condition (ii), which is satisfied by hypothesis. Hence, define the integer $\overline{r}_i$ as follows

$$\overline{r}_i := \min \left\{ r \in \mathbf{Z}^+ : \; P(s)\, a_r^{(i)} \neq 0 \right\} .$$

Then, $y^{(i)}(t)$ can be decomposed as follows:

$$y^{(i)}(t) = y_f^{(i)}(t - \overline{r}_i\, \delta_T)\, \delta_{-1}(t - \overline{r}_i\, \delta_T) + \tilde{y}^{(i)}(t), \quad t \geq 0,$$

where $y_f^{(i)}(t)$ denotes the output response of system P from zero initial state to the input $a_{\overline{r}_i}^{(i)} \delta_{-1}(t)$, expressed by $y_f^{(i)}(t) = \mathcal{L}^{-1} \left\{ P(s)\, \frac{a_{\overline{r}_i}^{(i)}}{s} \right\}$, and $\tilde{y}^{(i)}(t) = 0$ for all $t \leq (\overline{r}_i + 1)\, \delta_T$, so that

$$y^{(i)}(t) = y_f^{(i)}(t - \overline{r}_i \delta_T)\, \delta_{-1}(t - \overline{r}_i \delta_T), \quad \forall\, t \in [\overline{r}_i \delta_T,\, (\overline{r}_i + 1)\, \delta_T] \,. \tag{11}$$

By (10) and (11), and by the meaning of $y_f^{(i)}(t)$, for each $j = 1, 2, \ldots, q$, $j \neq i$, the jth component of $y_f^{(i)}(t)$ is zero for all $t \geq 0$. Then, denoting by $y_f^{(i)}(s)$ the Laplace transform of $y_f^{(i)}(t)$, it satisfies

$$y_f^{(i)}(s) = P(s)\, \frac{a_{\overline{r}_i}^{(i)}}{s} = y_{f,\,i}^{(i)}(s)\, e_i ,$$

for some nonzero scalar rational function $y_{f,\,i}^{(i)}(s)$. Since the above reasoning can be repeated for each $i = 1, 2, \ldots, q$, the following relation holds:

$$\begin{aligned} & P(s) \left[\; a_{\overline{r}_1}^{(1)} \;\middle|\; a_{\overline{r}_2}^{(2)} \;\middle|\; \cdots \;\middle|\; a_{\overline{r}_q}^{(q)} \; \right] \\ = \;\; & s\, \mathrm{diag} \left(y_{f,\,1}^{(1)}(s),\, y_{f,\,2}^{(2)}(s),\, \ldots,\, y_{f,\,q}^{(q)}(s) \right) . \end{aligned}$$

By defining $m_i(s) := s\, y_{f,\,i}^{(i)}(s)$, for $i = 1, 2, \ldots, q$, and

$$M := \left[\; a_{\overline{r}_1}^{(1)} \;\middle|\; a_{\overline{r}_2}^{(2)} \;\middle|\; \cdots \;\middle|\; a_{\overline{r}_q}^{(q)} \; \right] ,$$

the proof is completed. □

Remark 3. Notice that Assumption 1 is not needed in the necessity proof of Theorem 1; then, the existence of a static precompensator, which achieves input-output decoupling for the given continuous plant P is needed, in order to solve the continuous-time input-output decoupling problem for the hybrid control system S in Figure 1 or for the hybrid control system Σ in Figure 3, even if such an assumption is not satisfied. Hence, if it does not exist, a continuous-time subcompensator must be inserted both in Figure 1 and in Figure 3 between H_{δ_T} and P in order to achieve the continuous-time input-output decoupling.

Proof of Proposition 1. Notice that, if condition (ii) does not hold, then either of the square matrices $K(z)$ and $P_D(z)$ has to be singular over the rational field.

In order to show that matrix $P_D(z)$ cannot be singular, for each $i = 1, 2, \ldots, q$, consider the same input function $r_D^{(i)}(k)$, vectors $a_r^{(i)}$, $r \in \mathbf{Z}^+$ and integer $\overline{r}_i$, defined in the necessity proof of Theorem 1. It follows that the output response $\overline{y}^{(i)}(t)$ of system P, from the zero initial state, to the input function $u(t) = \overline{u}^{(i)}(t)$, with

$$\overline{u}^{(i)}(t) := a_{\overline{r}_i}^{(i)}\, \delta_{-1}(t), \quad t \geq 0,$$

is nonzero. Letting $\overline{y}_D^{(i)}(k)$ be the sampling of $\overline{y}^{(i)}(t)$, its z-transform $\overline{y}_D^{(i)}(z)$ is given by

$$\overline{y}_D^{(i)}(z) = P_D(z)\, a_{\overline{r}_i}^{(i)} \frac{z}{z-1}, \tag{12}$$

and, by property (i), can be expressed as

$$\overline{y}_D^{(i)}(z) = \overline{y}_{D,\,i}^{(i)}(z)\, e_i,$$

for some scalar function $\overline{y}_{D,\,i}^{(i)}(z)$. Now, notice that for each $i = 1, 2, \ldots, q$, $\overline{u}^{(i)}(t)$ can be seen as a free response of a system K_I constituted by the parallel of q integrators to be connected to P. Therefore the function $\overline{y}_D^{(i)}(k)$ cannot be identically zero, since otherwise this would imply a loss of observability, due to sampling, for the series connection of K_I and P, which cannot take place under Assumption 1, as it can be easily shown. Hence, the function $\overline{y}_{D,\,i}^{(i)}(z)$ is nonzero, for each $i = 1, 2, \ldots, q$. By defining the matrix M as in the necessity proof of Theorem 1, in view of (12), the following relation holds:

$$P_D(z)\, M = \frac{z-1}{z} \operatorname{diag}\left(\overline{y}_{D,\,1}^{(1)}(z), \ldots, \overline{y}_{D,\,q}^{(q)}(z)\right), \tag{13}$$

which implies that $P_D(z)$ is nonsingular.

If, vice-versa, $K(z)$ is singular, then, if $K_i(z)$ denotes the ith column of $K(z)$, $i = 1, 2, \ldots, q$, there exists an integer j, $1 \leq j \leq q$, such that

$$K_j(z) = \sum_{\substack{h=1 \\ h \neq j}}^{q} c_h(z)\, K_h(z), \tag{14}$$

$c_h(z)$ being suitable scalar rational functions. Let $\psi(z)$ a polynomial function of z such that, for each $h = 1, 2, \ldots, q$, $h \neq j$, the function $\overline{c}_h(z)$ defined by $\overline{c}_h(z) := c_h(z)/\psi(z)$ is proper. Then, consider the input function $\overline{r}_D(k)$ having $\overline{r}_D(z) := [\overline{c}_1(z) \ \ldots \ \overline{c}_{j-1}(z) \ \ 0 \ \ \overline{c}_{j+1}(z) \ \ldots \ \overline{c}_q(z)]'$ as z-transform; let $u_D(z)$ be the z-transform of the output response $u_D(k)$ of K to the input $\overline{r}_D(k)$ from the zero initial state. In view of (14), one has

$$u_D(z) = K_j(z) \frac{1}{\psi(z)}. \tag{15}$$

By property (i), it follows that the jth component $\overline{y}_j(t)$ of the output response $\overline{y}(t)$ of S to the function $\overline{r}_D(k)$, from zero initial state, is zero for all $t \in \mathbb{R}$, $t \geq 0$. But, in view of (15), $u_D(z)$ is also the z-transform of the output response of K, from zero initial state, to the input $\tilde{r}_D(k)$ having

$$\tilde{r}_D(z) := \frac{1}{\psi(z)} e_j,$$

as z-transform, so that $\overline{y}_j(t)$ must be nonzero, in view of condition (iii), yielding a contradiction. □

Remark 4. As a byproduct of the proof of Proposition 1, it is easy to see that, for arbitrary p and q, $p \geq q$, if condition (i) of Definition 1, and condition (iii), given right after the same definition, hold, then:

– $K(z)$ has full column rank;

– under Assumption 1, $P_D(z)$ has full row rank.

Notice that such properties, which are obvious if (ii) holds, are implied also by the weaker condition (iii). □

4. THE CASE OF SQUARE SYSTEMS

The following theorem states a relevant property of the class of square systems that can be continuous-time input-output decoupled and satisfy Assumption 1: for such systems, a (possibly dynamic) compensator K such that the discrete-time model of the hybrid control system in Figure 1 or in Figure 3 is input-output decoupled, without achieving the continuous-time input-output decoupling, cannot exist.

Theorem 2. Under Assumption 1, if for plant P $p = q$ and a discrete-time compensator K (either static or dynamic) exists, such that the hybrid control system S in Figure 1 (and, hence, the hybrid control system Σ in Figure 3) is continuous-time input-output decoupled, then any discrete-time compensator K (either static or dynamic) that decouples the discrete-time model P_D of P, also achieves the continuous-time input-output decoupling for S (and for Σ).

Proof. First, a useful relationship between any two discrete-time compensators, K_1 and K_2, both achieving the discrete-time input-output decoupling for system P_D is proven. If $K_1(z)$ and $K_2(z)$ are the transfer matrices of such compensators, define

$$D_1(z) := P_D(z)\, K_1(z), \tag{16a}$$

$$D_2(z) := P_D(z)\, K_2(z), \tag{16b}$$

$D_1(z)$ and $D_2(z)$ being diagonal and nonsingular strictly proper rational matrices. Therefore

$$P_D(z)\, (K_1(z)\, D_2(z) - K_2(z)\, D_1(z)) = 0;$$

Fig. 4. Factorisation of compensator K.

hence, since $P_D(z)$ is nonsingular,

$$K_1(z)\, D_2(z) = K_2(z)\, D_1(z).$$

By the nonsingularity of $D_1(z)$, this implies:

$$K_2(z) = K_1(z)\, D_2(z)\, D_1^{-1}(z). \tag{16}$$

This holds, in particular, for $K_1(z) = M$, M being a constant matrix, whose existence is guaranteed by Theorem 1, and $K_2(z)$ as the transfer matrix of any compensator K mentioned in the statement of the theorem.

Then, denoting $K_2(z)$ by $K(z)$, (16) can be rewritten as

$$K(z) = M\, D_2(z)\, D_1^{-1}(z),$$

thus implying the special property, for this special choice of K_1, that the rational matrix $D_2(z)\, D_1^{-1}(z)$ is proper. Hence, the compensator K is equivalent to the series connection of a (possibly dynamic) discrete-time system L, having $L(z) := D_2(z)\, D_1^{-1}(z)$ as transfer matrix, and of a static precompensator having M as transfer matrix (see Figure 4). Since matrix $L(z)$ is diagonal and nonsingular, and the hybrid control system S appearing in Figure 4 is, by hypothesis, continuous-time input-output decoupled, then it can be easily seen that the whole compensator K achieves continuous-time input-output decoupling. □

Remark 5. It is stressed that, by Theorem 2, either the problem of obtaining continuous-time input-output decoupling for a square plant satisfying Assumption 1 is not solvable (the solvability of the problem can be checked easily, for square plants, by means of the condition reported in the subsequent Proposition 3), or any precompensator which decouples the discrete-time model P_D of the given plant P solves the problem of continuous-time input-output decoupling too. However, the use of a merely static precompensator, in order to achieve continuous-time input-output decoupling, can be more convenient, since it obviously preserves the two key structural properties of stabilisability and detectability of system P, if system P has such properties, and, therefore, it allows to satisfy also the requirement of asymptotic stability, which appears unrenunciable for the overall control system.

The latter part of Remark 5 yields the following theorem.

Theorem 3. (Continuous-time input-output decoupling with stability) Under Assumption 1, if for plant P $p = q$, then there exists a discrete-time compensator K that achieves both continuous-time input-output decoupling and asymptotic stability for the unit feedback hybrid control system Σ in Figure 3, if and only if plant P is stabilisable and detectable and it satisfies the condition stated in Theorem 1.

Proof. The necessity is yielded by Theorem 1. The sufficiency is derived from the proof of Theorem 2 by putting in the control scheme in Figure 3 the same compensator K appearing in Figure 4 (where M is a static, square and nonsingular linear map such that system S in Figure 4 is continuous-time input-output decoupled), and by choosing the entries of the diagonal rational matrix $L(z) := D_2(z)D_1^{-1}(z)$ so that (by virtue of Assumption 1) the discrete-time model Σ_D of the hybrid system Σ in Figure 3 is asymptotically stable, and, hence, Σ too is (see Theorem 4 in [7]). □

Remark 6. The proof of Theorem 3 suggests that, in order to obtain a hybrid system Σ which is both asymptotically stable and continuous-time input-output decoupled, the scheme in Figure 3 can be used, under the hypotheses and conditions of the theorem, with the compensator K constituted by the cascade connection of a discrete-time dynamic stabilising compensator K_S having the diagonal matrix $L(z)$ defined in the proof of the same theorem as transfer matrix, and of a static compensator K_D having the matrix M appearing in the statement of Theorem 1 as constant transfer matrix. The latter can be easily obtained by means of the subsequent Proposition 3.

Theorems 1 and 3 and the proof of the latter motivate the interest in the solution of the problem of input-output decoupling for wholly continuous-time systems by means of static precompensators. Such a problem seems not to have received enough attention, since, at the best authors' knowledge, the only available results are concerned with square systems: an explicit condition for the existence of a static precompensator achieving input-output decoupling for the continuous-time plant P together with a formula for the computation of the solution can be found in [1], expressed in terms of $P(s)$, or (if the state space description (1) of P is given) can be easily derived from [6]. Therefore it seems worth to explicitly state the following proposition, which expresses a condition for the solvability of the problem that seems to be simpler than the above mentioned ones, since it can be checked by direct inspection of the transfer matrix $P(s)$ of P; it gives also a parametrisation of all the constant precompensators constituting a solution of the problem.

Proposition 3. If for plant P $p = q$, then there exists a square constant matrix M such that $P(s)\,M$ is diagonal and nonsingular if and only if there exist a square, rational, diagonal, nonsingular matrix $D(s)$ and a square constant nonsingular matrix Ξ such that

$$P(s) = D(s)\,\Xi. \tag{17}$$

If (17) holds, all such matrices M can be expressed by

$$M = \Xi^{-1}\Delta, \tag{18}$$

where Δ is any square, constant, diagonal, and nonsingular matrix.

Proof. The necessity follows by defining $D(s) := P(s)\,M$ and $\Xi := M^{-1}$, and the sufficiency by defining $M := \Xi^{-1}\Delta$, where Δ is any square, constant diagonal and nonsingular matrix.

In order to show that if there exist a constant square matrix M such that $P(s)\,M$ is diagonal and nonsingular, then (18) holds for some square, constant, diagonal and nonsingular Δ, define $D_M(s) := P(s)\,M$ for such a M, which, together with (17), implies $D(s)\,\Xi = D_M(s)\,M^{-1}$, so that $D^{-1}(s)D_M(s) = \Xi M$ is diagonal, nonsingular and constant. Hence, $M = \Xi^{-1}D^{-1}(s)D_M(s)$, that is (18) with $\Delta = D^{-1}(s)\,D_M(s)$. □

Remark 7. Notice that, by Theorem 1 and Proposition 3, the existence of a square rational diagonal and nonsingular matrix $D(s)$ and of a square constant nonsingular Ξ such that (17) holds is, under Assumption 1, the solvability condition for the problem of obtaining continuous-time input-output decoupling for the given plant P, if it is square (*i. e.*, if $p = q$). Such a condition can be easily checked, since it is equivalent to the fact that, for each $i = 1, 2, \ldots, q$, all the entries of the ith row of the transfer matrix $P(s)$ are multiple, through the constant coefficients $\xi_{i1}, \xi_{i2}, \ldots, \xi_{ip}$, respectively, of the same rational function $d_i(s)$.

5. CONCLUDING REMARKS

The results here reported imply that the problem of the continuous-time input-output decoupling for sampled-data systems may need the use of a continuous-time dynamic subcompensator, in addition to the discrete-time one. In particular, by Theorem 1 and Proposition 3, this can be avoided for square plants (*i. e.*, for $p = q$) only if (17) holds for some square constant and nonsingular matrix Ξ and some square rational diagonal and nonsingular $D(s)$ – and for nonsquare plants only if the condition of Theorem 1 holds –, that is (in any case) a very severe condition. It is stressed that, if it is not satisfied, the use of the control schemes in Figures 1 and 3, involving a purely discrete-time compensator K, can yield a merely discrete-time input-output decoupling, that is, for some $i = 1, \ldots, q$, a nonzero ripple will unavoidably appear in the scalar continuous-time output responses $y_j(t)$, for some $j \neq i$, when $r_{D,j}(\cdot) = 0$ for all $j \neq i$ and $r_{D,i}(\cdot)$ is some nonzero scalar reference signal, and the amplitudes of such a ripple can be unacceptable for small sampling frequencies and/or unbounded signals $r_{D,i}(k)$.

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