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HYBRID VARIATIONAL FORMULATION OF AN ELLIPTIC STATE EQUATION APPLIED TO AN OPTIMAL SHAPE PROBLEM

Jan Chleboun

A variational formulation of the Poisson equation with homogeneous boundary condition is considered as a state equation on a two-dimensional domain. A part of the boundary has to be found to minimize a smooth cost functional. The primal hybrid formulation of the state problem is used to obtain not only a solution of the original state equation but also its derivative with respect to the outward unit normal to the boundary of the domain. Simple approximative spaces are introduced and a convergence of approximate state solutions as well as approximate optimal domains are proved.

INTRODUCTION

To make a theoretical analysis of the optimal shape problem studied in this paper the primal variational formulation of the state equation is quite sufficient (cf. Begis and Glowinski [1]). However, in practice, computational methods and algorithms have to be taken into account to maximize effectiveness and accuracy of computation.

There are different methods used in the field of sensitivity analysis and some of them require to know the derivative of the solution of a state and adjoint problem with respect to the unit outward normal vector to the boundary of an optimized domain, see Haug, Choi, Komkov [5]. The derivative computed by means of the primal finite element method (FEM) is inaccurate. That is why we use the primal hybrid formulation. It directly gives the derivative we need.

The goal of the paper is to apply the primal hybrid formulation of a simple elliptic boundary problem to an optimal shape problem given by a smooth cost functional.

We extend the results of Raviart and Thomas [9] to a family of domains with a variable boundary and incorporate them into the methodological frame of [1], using results attained by Hlaváček [6], Hlaváček and Mäkinen [7].

An optimal design problem is formulated in Section 1 and reformulated in Section 2, where an existence result is given, too. In Section 3, we introduce approximate problems. An error and convergence analysis is given in Section 4.

1. PRIMAL FORMULATION OF AN OPTIMIZATION PROBLEM

Let us introduce the following set

$$\mathcal{U}^0_{ad} = \left\{ v \in C^{(0),1}([0,1]);\, 0 < \hat{C}_1 \le v(x_2) \le \hat{C}_2,\, |v'| \le \hat{C}_3 \text{ a.e. in } (0,1) \right\},$$

where $C^{(0),1}([0,1])$ denotes the space of Lipschitz functions and $\hat{C}_1$, $\hat{C}_2$, $\hat{C}_3$ are given positive constants. The prime symbolizes the derivative with respect to x_2.

Next, we consider a family of domains $\Omega(v)$, $v \in \mathcal{U}^0_{ad}$ (Fig. 1), where

$$\Omega(v) = \left\{ (x_1, x_2) \in \mathbb{R}^2;\, 0 < x_1 < v(x_2),\, 0 < x_2 < 1 \right\}.$$

Fig. 1.

Let us define the *state equation:* Find a function $y(v) \in H^1_0(\Omega(v))$ such that

$$\forall\, w \in H^1_0(\Omega(v)) \qquad \int_{\Omega(v)} \nabla y(v) \cdot \nabla w \,\mathrm{d}x = \int_{\Omega(v)} f w \,\mathrm{d}x, \tag{1}$$

where $H^1_0(\Omega(v))$ denotes the Sobolev space of functions with vanishing traces, $f \in L^2(\Omega_\beta)$ is a given function, $\Omega_\beta = (0, \hat{C}_2) \times (0,1)$. In virtue of the Lax–Milgram theorem there exists a unique solution of 1.

We introduce the *cost functional* $\mathcal{J}$ on the set $\mathcal{U}^0_{ad}$:

$$\mathcal{J}(v, y(v)) = \int_{\Omega(v)} (y(v) - y_p)^2 \,\mathrm{d}x, \tag{2}$$

where $y(v)$ solves 1 and the function $y_p \in L^2(\Omega_\beta)$ is prescribed.

Finally, we define the set of admissible *design variables* (see [7])

$$\mathcal{U}_{ad} = \left\{ v \in C^{(1),1}([0,1]);\, v \in \mathcal{U}^0_{ad},\, |v''(x_2)| \le \hat{C}_4 \text{ a.e.in } (0,1),\, \int_0^1 v(x_2)\,\mathrm{d}x_2 = \hat{C}_5 \right\},$$

where $C^{(1),1}([0,1])$ stands for the space of functions with Lipschitz-continuous derivatives and $\hat{C}_4$, $\hat{C}_5$ are positive constants. The constraint imposed on $v''(x_2)$ is based on numerical tests and recommended by [7] to reduce oscillations of the designed boundary.

Then the *domain optimization problem* $\mathcal{P}$ reads: Find $v_{opt} \in \mathcal{U}_{ad}$ such that

$$\mathcal{J}(v_{opt}, y(v_{opt})) = \min_{v \in \mathcal{U}_{ad}} \mathcal{J}(v, y(v)). \tag{3}$$

2. THE PRIMAL HYBRID FORMULATION OF THE STATE PROBLEM AND THE EXISTENCE OF AN OPTIMAL DOMAIN

Let $\overline{\Omega(v)} = \bigcup_{r=1}^{R(v)} \overline{\Omega_r(v)}$ be a decomposition of the closure of the domain $\Omega(v)$, $v \in \mathcal{U}_{ad}^0$, into a finite number $R(v)$ of disjoint subdomains $\Omega_r(v)$ with a Lipschitz-continuous boundary. We introduce the space

$$X(v) = \left\{ w \in L^2(\Omega(v));\ w_r = w|_{\Omega_r(v)} \in H^1(\Omega_r(v)),\ 1 \le r \le R(v) \right\}$$

provided with the norm derived from the Sobolev norm $\|\cdot\|_{1,\Omega_r(v)}$ on $H^1(\Omega_r(v))$

$$\|w\|_{X(v)} = \left(\sum_{r=1}^{R(v)} \|w_r\|^2_{1,\Omega_r(v)} \right)^{1/2}.$$

The seminorm $|\cdot|_{1,\Omega_r(v)}$ is defined analogously.

Having $H(\mathrm{div};\Omega(v)) = \left\{ \mathbf{q} \in \left[L^2(\Omega(v))\right]^2;\ \mathrm{div}\ \mathbf{q} \in L^2(\Omega(v)) \right\}$, we define the space

$$\begin{aligned} M(v) \quad = \quad & \Bigg\{ \mu \in \prod_{r=1}^{R(v)} H^{-1/2}(\partial\Omega_r(v));\ \exists\, \mathbf{q} \in H(\mathrm{div};\Omega(v)) \ : \\ & \qquad \mu = \mathbf{q}|_{\Omega_r(v)} \cdot \nu_r \text{ on } \partial\Omega_r(v),\ 1 \le r \le R(v) \Bigg\}, \end{aligned}$$

where ν_r is the unit outward normal along $\partial\Omega_r(v)$. Generally, a functional $\mathbf{q}\cdot\nu \in H^{-1/2}(\partial\Omega)$ is defined by the Green formula

$$\forall\, w \in H^1(\Omega) \quad \langle \mathbf{q}\cdot\nu, w\rangle_{\partial\Omega} = \int_\Omega (\nabla w \cdot \mathbf{q} + w\, \mathrm{div}\, \mathbf{q})\, \mathrm{d}x\,,$$

where $\langle\cdot,\cdot\rangle_{\partial\Omega}$ represents the duality between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$.

A norm over the space $M(v)$ will be defined in Section 4.2.

For any function $v \in \mathcal{U}_{ad}^0$, we consider the continuous bilinear forms $a(v;\cdot,\cdot)$: $X(v) \times X(v) \to \mathbb{R}$ and $b(v;\cdot,\cdot)$: $X(v) \times M(v) \to \mathbb{R}$ defined by

$$a(v; w, z) = \sum_{r=1}^{R(v)} \int_{\Omega_r(v)} \nabla w \cdot \nabla z \, \mathrm{d}x \qquad \text{and} \qquad b(v; w, \mu) = -\sum_{r=1}^{R(v)} \langle \mu, w\rangle_{\partial\Omega_r(v)}\,.$$

The *primal hybrid formulation of the state problem* (1.1) (see [9]) reads:
Find a pair $(y(v),\ \lambda(v)) \in X(v) \times M(v)$ such that

$$\forall\, w \in X(v) \quad a(v; y(v), w) + b(v; w, \lambda(v)) = \int_{\Omega(v)} f w \,\mathrm{d}x, \tag{4}$$

$$\forall\, \mu \in M(v) \qquad b(v; y(v), \mu) = 0. \tag{5}$$

It is known [9, Lemma 1] that

$$H_0^1(\Omega(v)) = \{w \in X(v);\ \forall\, \mu \in M(v) \quad b(v; w, \mu) = 0\} \tag{6}$$

and that [9, Theorem 1] the problem (2.1) – (2.2) has a unique solution $(y(v),\ \lambda(v)) \in X(v) \times M(v)$ for any $v \in \mathcal{U}_{ad}^0$. Moreover, $y(v) \in H_0^1(\Omega(v))$ is the solution of the problem (1.1) and we have the equality (though in a weak sense, still useful — see the Introduction)

$$\frac{\partial y(v)}{\partial \nu_r} = \lambda(v)|_{\partial\Omega_r(v)}\,, \quad 1 \le r \le R(v).$$

As a consequence, both (1.1) and (2.1) – (2.2) can be considered as the state problem for the optimal design problem $\mathcal{P}$ (see (1.3)).

If we need we shall extend any function belonging to the space $H_0^1(\Omega)$ or $L^2(\Omega)$ by zero on the set $\mathbb{R}^2 \setminus \Omega$. For the sake of simplicity, the extension will not be denoted by a new symbol. Particularly, any solution $y(v)$ of the state problem (1.1), $v \in \mathcal{U}_{ad}^0$, can be extended to the domain $\Omega_\delta = (0, \delta) \times (0, 1)$, $\delta > \hat{C}_2$ (see Fig. 1).

Lemma 2.1. Let $\{v_n\}_{n=1}^\infty$ be a sequence of functions $v_n \in \mathcal{U}_{ad}$. Then a subsequence $\{v_{n_m}\} \subset \{v_n\}$ and a function $v \in \mathcal{U}_{ad}$ exist such that $v_{n_m} \to v$ in $C^{(1)}([0, 1])$ and

$$y(v_{n_m}) \longrightarrow y(v) \quad \text{in } H^1(\Omega_\delta), \tag{7}$$

where $y(v_{n_m})$ and $y(v)$ are the first components of the solution of the state problem (2.1) – (2.2) on the domain $\Omega(v_{n_m})$ and $\Omega(v)$, respectively. If a sequence $\{v_n\}_{n=1}^\infty \subset \mathcal{U}_{ad}^0$ converges in $C([0, 1])$, then $v \in \mathcal{U}_{ad}^0$ and (2.4) holds again.

P r o o f. Let us notice that subdomains $\Omega_r(v_n)$, $\Omega_r(v)$ are unimportant here. The solutions $y(v_n)$ belong to $H_0^1(\Omega(v_n))$ and are $H^1(\Omega_\delta)$-bounded. Thus a weakly convergent subsequence can be extracted. The set $\mathcal{U}_{ad}$ is compact in $C^{(1)}([0, 1])$. The strong convergence of $y(v_{n_m})$ can be proved like in the proof of [6, Lemma 2.1]. □

Theorem 2.1. There exists at least one solution of the optimization problem $\mathcal{P}$.

P r o o f. The proof is standard — cf. e. g. [1], [6]. Suppose $\{v_n\}_{n=1}^\infty$, $v_n \in \mathcal{U}_{ad}$, is a minimizing sequence, i. e., $\lim_{n\to\infty} \mathcal{J}(v_n, y(v_n)) = \inf_{v \in \mathcal{U}_{ad}} \mathcal{J}(v, y(v))$. A subsequence $\{v_{n_m}\} \subset \{v_n\}$ exists (Lemma 2.1) such that $\lim_{m\to\infty} v_{n_m} = v_0$, $\quad v_0 \in \mathcal{U}_{ad}$.

It is easy to see that the convergence (2.4) implies $\lim_{m\to\infty} \mathcal{J}(v_{n_m}, y(v_{n_m})) = \mathcal{J}(v_0, y(v_0))$. Hence, $\inf_{v \in \mathcal{U}_{ad}} \mathcal{J}(v, y(v)) = \mathcal{J}(v_0, y(v_0))$ and $v_0 \equiv v_{opt}$. □

3. APPROXIMATION BY HYBRID FINITE ELEMENTS

We shall proceed briefly, since the section summarizes known results. For details we refer to e. g. [6], [7] (triangulations) and [9] (the primal hybrid FEM).

Let N be a positive integer and $h = 1/N$. Denoting by e_j the subintervals $[(j-1)h, jh]$, $j = 1, 2, \ldots, N$, we define the set

$$\mathcal{U}^h_{ad} = \Big\{ v_h \in \mathcal{U}^0_{ad};\ v_h|_{e_j} \in P_1(e_j),\ j = 1, \ldots, N,\\ |\delta^2_h v_h(jh)| \le \hat{C}_4,\ j = 1, \ldots, N-1,\ \int_0^1 v_h(x_2)\,\mathrm{d}x_2 = \hat{C}_5 \Big\},$$

where $P_1(e_j)$ is the set of linear functions defined on e_j and

$$\delta^2_h v_h(jh) = \frac{1}{h^2}\left[v_h((j+1)h) - 2v_h(jh) + v_h((j-1)h)\right].$$

Any domain $\Omega(v_h)$ is subdivided into triangles (Fig. 2) and we suppose that the resulting family $\tau = \{\mathcal{T}_h(v_h);\ v_h \in \mathcal{U}^h_{ad},\ h \to 0+\}$ of triangulations $\mathcal{T}_h(v_h)$ is strongly regular in the sense of [3] (see [6] for details). If h is fixed the triangulation of the rectangle $[0, \hat{C}_0] \times [0, 1]$ is independent of v_h, $\hat{C}_0$ is a fixed positive constant less than $\hat{C}_1$.

We prescribe a unique correspondence between $\Omega(v_h)$ and $\mathcal{T}_h(v_h)$. Coordinates of the nodal points are governed by $N+1$ values $v_h(jh)$, $j = 0, \ldots, N$. The triangles of a triangulation $\mathcal{T}_h(v_h)$ serve as subdomains $\Omega_r(v_h)$.

Fig. 2.

To be prepared to use the rectangle Ω_δ we extend each triangulation $\mathcal{T}_h(v_h) \in \tau$ to Ω_δ uniquely and denote by $\mathcal{T}_{h\delta}(v_h)$. The resulting family τ_δ of triangulations is assumed to be strongly regular. Like in Section 2 we define the space $X_\delta(v_h)$ with the norm $\|\cdot\|_{X_\delta(v_h)}$ and the seminorm $|\cdot|_{X_\delta(v_h)}$.

Let us consider the approximate subspaces

$$X_h(v_h) = \{w \in L^2(\Omega(v_h));\ \forall K \in \mathcal{T}_h(v_h)\ w|_K \text{ is a linear function}\} \subset X(v_h),$$

$$M_h(v_h) = \Big\{\mu \in \prod_{K\in\mathcal{T}_h(v_h)} L^2(\partial K);\ \forall K \in \mathcal{T}_h(v_h)\ \forall S\ \ \mu|_S = \text{constant},$$

$$\mu|_{\partial K_1} + \mu|_{\partial K_2} = 0 \text{ on } S = K_1 \cap K_2,\ K_1 \text{ and } K_2 \text{ are adjacent triangles}\Big\} \subset M(v_h),$$

where S denotes a side of the triangle K.

Elements of $M_h(v_h)$ comply with a natural condition which follows from opposite directions of outward normals along a common side of adjacent triangles.

The *approximate state problem*:

Find a pair $(y_h(v_h),\ \lambda_h(v_h)) \in X_h(v_h) \times M_h(v_h)$ such that

$$\forall\, w_h \in X_h(v_h) \quad a(v_h; y_h(v_h), w_h) + b(v_h; w_h, \lambda_h(v_h)) = \int_{\Omega(v_h)} f w_h \,\mathrm{d}x, \tag{8}$$

$$\forall\, \mu_h \in M_h(v_h) \qquad b(v_h; y_h(v_h), \mu_h) = 0. \tag{9}$$

The *approximate domain optimization problem* $\mathcal{P}_h$: Find $v^h_{opt} \in \mathcal{U}^h_{ad}$ such that

$$\mathcal{J}(v^h_{opt}, y_h(v^h_{opt})) = \min_{v_h\in\mathcal{U}^h_{ad}} \mathcal{J}(v_h, y_h(v_h)), \tag{10}$$

where $y_h(v_h)$, $y_h(v^h_{opt})$ are the first components of the solution of (3.1) – (3.2) on the domains $\Omega(v_h)$, $\Omega(v^h_{opt})$, respectively.

As an analogy to (2.3) we introduce the space

$$V_h(v_h) = \{w_h \in X_h(v_h);\ \forall\, \mu_h \in M_h(v_h) \quad b(v_h; w_h, \mu_h) = 0\}$$

and reformulate (3.1) – (3.2) into the problem to find $y_h(v_h) \in V_h(v_h)$ such that

$$\forall\, w_h \in V_h(v_h) \quad a(v_h; y_h(v_h), w_h) = \int_{\Omega(v_h)} f w_h \,\mathrm{d}x. \tag{11}$$

According to [9], the mapping

$$w_h \mapsto [a(v_h; w_h, w_h)]^{1/2} = |w_h|_{X(v_h)}$$

is a norm on $V_h(v_h)$, the problem (3.1) – (3.2) has a unique solution $(y_h(v_h), \lambda_h(v_h)) \in X_h(v_h) \times M_h(v_h)$, the component $y_h(v_h)$ belongs to $V_h(v_h)$ and solves (3.4) uniquely.

Remark 3.1. A general treatment of the primal hybrid FEM applied to the Poisson equation can be found in [9]. Let us note that $V_h(v_h)$ is an external approximation of the space $H^1_0(\Omega(v_h))$, i.e., $V_h(v_h) \not\subset H^1_0(\Omega(v_h))$. A function $w_h \in X_h(v_h)$ belongs to $V_h(v_h)$ if and only if [9, Section 4]

$$w_h \text{ is continuous at midpoints of the sides of triangles contained in } \Omega(v_h); \tag{12}$$

$$w_h \text{ vanishes at midpoints located on } \partial\Omega(v_h). \tag{13}$$

□

Theorem 3.1. There exists at least one solution of the problem $\mathcal{P}_h$ given by (3.3).

Proof. Introducing basis functions of the space $V_h(v_h)$, we rewrite the equation (3.4) into a matrix form. It is not difficult to prove that the solution of the linear system continuously depends on the design variables. The cost functional is continuous, too. Hence, the existence problem is reduced to a minimization of a continuous function on a compact set. □

4. CONVERGENCE ANALYSIS

In this section we study a convergence of the approximate state solutions and the approximate optimal domains, respectively. We emphasize the convergence analysis of the first component of the primal hybrid state solution, since it is the point of the optimal domain problem. Let us note that $C_0^\infty(\Omega(v))$ stands for the set of infinitely continuously differentiable functions with a compact support contained in $\Omega(v)$.

4.1. Approximate solutions of the optimal domain problem

The convergence analysis will be based on the following equality (see [9, Theorem 3]).

Theorem 4.1. Let $y_h(v_h) \in V_h(v_h)$, $v_h \in \mathcal{U}_{ad}^h$, be the solution of the problem (3.4). Then

$$|y(v_h) - y_h(v_h)|^2_{X(v_h)} = \left(\inf_{w_h \in V_h(v_h)} |y(v_h) - w_h|_{X(v_h)} \right)^2 + \tag{14}$$
$$+ \left(\inf_{\mu_h \in M_h(v_h)} \sup_{w_h \in V_h(v_h) \setminus \{0\}} \frac{b(v_h; w_h, \lambda(v_h) - \mu_h)}{|w_h|_{X(v_h)}} \right)^2,$$

where $(y(v_h), \lambda(v_h)) \in H_0^1(\Omega(v_h)) \times M(v_h)$ is the solution of the problem (2.1) – (2.2) on the domain $\Omega(v_h)$.

Lemma 4.1. Let $\{v_h\}$, $h \to 0+$, be a sequence of functions $v_h \in \mathcal{U}_{ad}^h$ such that $\lim_{h \to 0+} v_h = v$ in $C([0,1])$. Then $v \in \mathcal{U}_{ad}$.

Proof. Through a sequence of continuous piecewise linear interpolates of the derivatives v_h' can be proved that $v_h' \to v' \in C^{(0),1}([0,1])$. In addition, $v \in \mathcal{U}_{ad}$ can be shown. See the proof of [7, Lemma 3.2] for details. □

Lemma 4.2. Let us denote $\varepsilon(v_h) = \inf_{w_h \in V_h(v_h)} |y(v_h) - w_h|_{X(v_h)}$, where $y(v_h)$ is the solution of (3.4), $v_h \in \mathcal{U}_{ad}^h$. Let a sequence $\{v_h\}$, $h \to 0+$, $v_h \in \mathcal{U}_{ad}^h$, converge to a function v in $C([0,1])$. Then $\lim_{h \to 0+} \varepsilon(v_h) = 0$.

P r o o f. Let n be a positive integer. For any sufficiently small h (see Lemma 2.1)

$$\|y(v_h) - y(v)\|_{1,\Omega_\delta} \le \frac{1}{n}. \tag{15}$$

Also a function $\varphi_n \in C_0^\infty(\Omega(v))$ exists such that

$$\|y(v) - \varphi_n\|_{1,\Omega(v)} \le \frac{1}{n}\,. \tag{16}$$

The uniform convergence of v_h guarantees the existence of a parameter $h(n) > 0$ such that we have $\operatorname{supp}\varphi_n \subset \Omega(v_h)$ for all $h \in (0, h(n)]$.

We define the space ($P_1(K)$ is the set of linear functions on K)

$$Z_h(v_h) = \Big\{ z_h \in C(\overline{\Omega(v_h)});\ \forall\, K \in \mathcal{T}(v_h)\ \ z_h|_K \in P_1(K),\ z_h|_{\partial\Omega(v_h)} = 0 \Big\} \subset H_0^1(\Omega(v_h)).$$

Denoting by $r_h\varphi_n \in Z_h(v_h)$ the $Z_h(v_h)$-interpolation of φ_n, we estimate (see e. g. [3, Theorem 3.1.6]) $\|\,(\varphi_n - r_h\varphi_n)|_K\,\|_{1,K} \le Ch|\,(\varphi_n|_K)\,|_{2,K}$ for any triangle K. The constant $C > 0$ is independent of h and v_h, since the set of triangulations is strongly regular. Taking the root of the sum of the squared inequality, we obtain (h is small)

$$\|\varphi_n - r_h\varphi_n\|_{1,\Omega(v_h)} \le Ch|\varphi_n|_{2,\Omega(v_h)} \le \frac{1}{n}. \tag{17}$$

The inclusion $Z_h(v_h) \subset V_h(v_h)$ and (4.2), (4.3), (4.4) lead to

$$\varepsilon(v_h) \le |y(v_h) - r_h\varphi_n|_{X(v_h)} = |y(v_h) - r_h\varphi_n|_{1,\Omega_\delta} \le \frac{3}{n}$$

but n has been chosen arbitrarily. □

The next analysis is based on an auxiliary problem similar to (1.1).

Let the domain $\Omega_n(v)$ be given by $v \in \mathcal{U}_{ad}$ and a positive integer n,

$$\Omega_n(v) = \left\{ x \in \mathbb{R}^2;\ \operatorname{dist}(x, \Omega(v)) < \frac{1}{n} \right\}.$$

We define the *auxiliary problem*: Find $y_n(v) \in H_0^1(\Omega_n(v))$ such that

$$\forall\, w \in H_0^1((\Omega_n(v)) \quad \int_{\Omega_n(v)} \nabla y_n(v) \cdot \nabla w\, dx = \int_{\Omega_n(v)} f w\, \mathrm{d}x, \tag{18}$$

the function f (see (1.1)) is extended by zero outside Ω_β.

We remind that for any n and $v \in \mathcal{U}_{ad}$ the equation (4.5) has a unique solution.

Lemma 4.3. Let $v \in \mathcal{U}_{ad}$ be given and Ω_Q be a domain with a Lipschitz boundary, $\overline{\Omega_\delta} \subset \Omega_Q$. Assume the parameter n big enough to ensure $\Omega_n(v) \subset \Omega_Q$. Then

$$\lim_{n\to\infty} \|y_n(v) - y(v)\|_{1,\Omega_Q} = 0,$$

where $y(v) \in H_0^1(\Omega(v))$ is the first component of the state solution (2.1) – (2.2).

Proof. The method of the proof is identical with that of Lemma 2.1. $\square$

An analysis of the second term of (4.1) is more laborious. We shall exploit the method of function regularization. For any parameter $\varrho > 0$, the mollifier is defined as follows

$$\varphi_\varrho(x) = \begin{cases} \frac{1}{\omega} \exp \frac{\|x\|^2_{\mathbb{R}^2}}{\|x\|^2_{\mathbb{R}^2} - \varrho^2}, & \|x\|_{\mathbb{R}^2} \geq \varrho, \quad \omega = \int\limits_{\|x\|_{\mathbb{R}^2} < 1} \exp \frac{\|x\|^2_{\mathbb{R}^2}}{\|x\|^2_{\mathbb{R}^2} - 1} \, \mathrm{d}x, \\ 0, & \|x\|_{\mathbb{R}^2} \geq \varrho. \end{cases} \tag{19}$$

Lemma 4.4. Let a sequence $\{v_h\}$, $h \to 0+$, $v_h \in \mathcal{U}^h_{ad}$, converge to v in $C([0,1])$. Let the pair $(y(v_h), \lambda(v_h)) \in H_0^1(\Omega(v_h)) \times M(v_h)$ be the solution of (2.1) – (2.2) and let $y_n(v) \in H_0^1(\Omega_n(v))$ be the solution of (4.5). Finally, let the function $y_{n\varrho}(v)$ be given by

$$y_{n\varrho}(v)(x) = \varrho^{-2} \int_{\Omega_n(v)} y_n(v)(t) \varphi_\varrho(x - t) \, \mathrm{d}t = \int_{|t|<1} y_n(v)(x + \varrho t) \varphi_1(t) \, \mathrm{d}t,$$

where $x \in \Omega_n(v)$ and $\varrho > 0$.

Define $\kappa_{n\varrho}(v_h) \in M(v_h)$ by

$$\kappa_{n\varrho}(v_h) = \frac{\partial y_{n\varrho}(v)}{\partial \nu_K} \text{ on } \partial K \quad \forall K \in \mathcal{T}_h(v_h), \;\; \Omega(v_h) \subset \Omega_n(v), \tag{20}$$

where ν_K denotes the unit outward normal vector along the boundary of a triangle $K \in \mathcal{T}_h(v_h)$. Then

$$\begin{aligned} &\exists\, h(n,\varrho) > 0 \quad \forall\, h \in (0, h(n,\varrho)] \quad \exists\, \varepsilon(v_h, n, \varrho) \in \mathbb{R} \quad \forall\, w \in X(v_h) \\ &|b(v_h; w, \lambda(v_h) - \kappa_{n\varrho}(v_h))| \leq \varepsilon(v_h, n, \varrho) \, \|w\|_{X(v_h)}\, ; \end{aligned} \tag{21}$$

for any $\varepsilon_0 > 0$ there exist parameters n and ϱ such that

$$0 \leq \lim_{h \to 0+} \varepsilon(v_h, n, \varrho) \leq \varepsilon_0. \tag{22}$$

Proof. We introduce the function f_ϱ on $\Omega_n(v)$,

$$f_\varrho(x) = \varrho^{-2} \int_{\Omega_n(v)} f(t) \varphi_\varrho(x - t) \, \mathrm{d}t. \tag{23}$$

Supposing $x \in \Omega_n(v)$, $\operatorname{dist}(x, \partial\Omega_n(v)) > \varrho$, is a given point, we have $\operatorname{supp} \varphi_\varrho(x–t) \subset \Omega_n(v)$ and $\Delta_t \varphi_\varrho(x - t) = \Delta_x \varphi_\varrho(x - t)$, (see (4.6) or [8], § 16). The subscripts t, x denote variables used in the process of differentiation.

Applying this and the equality $f = -\Delta y_n(v)$ (in the sense of distribution) to (4.10), we may write

$$\varrho^2 f_\varrho(x) = \int_{\Omega_n(v)} f(t)\varphi_\varrho(x-t)\,\mathrm{d}t = \tag{24}$$
$$= \; -\int_{\Omega_n(v)} y_n(v)(t)\Delta_t\varphi_\varrho(x-t)\,\mathrm{d}t = -\int_{\Omega_n(v)} y_n(v)(t)\Delta_x\varphi_\varrho(x-t)\,\mathrm{d}t =$$
$$= \; -\Delta_x \int_{\Omega_n(v)} y_n(v)(t)\varphi_\varrho(x-t)\,\mathrm{d}t = -\varrho^2\Delta_x y_{n\varrho}(v)(x).$$

If ϱ is small enough a value $h(\varrho,n) > 0$ exists such that

$$\forall\, h \in (0, h(\varrho,n)] \quad \mathrm{dist}(\Omega(v_h), \mathbb{R}^2 \setminus \Omega_n(v)) > \varrho.$$

Then on any domain $\Omega(v_h)$, $0 < h \le h(\varrho,n)$, (4.11) reads

$$f_\varrho(v) = -\Delta y_{n\varrho}(v). \tag{25}$$

Using substitution of the domain of integration, we obtain

$$\frac{\partial y_{n\varrho}(v)}{\partial x_i}(x) = \int_{|t|<1} \frac{\partial y_n(v)}{\partial x_i}(x+\varrho t)\varphi_1(t)\,\mathrm{d}t = \tag{26}$$
$$= \left(\frac{\partial y_n(v)}{\partial x_i}\right)_\varrho (x), \quad i = 1,2;\; x \in \Omega(v_h),\; 0 < h \le h(\varrho,n).$$

For any h sufficiently small, the function $y_{n\varrho}(v)$ is defined on the set $\overline{\Omega(v_h)}$. We can define $\kappa_{n\varrho}(v_h) \in M(v_h)$ by (4.7).

Using (4.12) and (4.13), we get (now $\mathbf{q} = \nabla(y_{n\varrho}(v))$ – see Section 2)

$$\forall\, w \in X(v_h) \qquad b(v_h; w, \kappa_{n\varrho}(v_h)) = -\sum_{K\in\mathcal{T}_h(v_h)} \int_K \Delta(y_{n\varrho}(v))w\,\mathrm{d}x \,-$$
$$- \sum_{K\in\mathcal{T}_h(v_h)} \int_K \nabla(y_{n\varrho}(v))\cdot\nabla w\,\mathrm{d}x = \int_{\Omega(v_h)} f_\varrho w\,dx - \sum_{K\in\mathcal{T}_h(v_h)} \int_K (\nabla y_n(v))_\varrho\cdot\nabla w\,\mathrm{d}x.$$

This and (2.1) lead to

$$|b(v_h; w, \lambda(v_h) - \kappa_{n\varrho}(v_h))| \le \left|\int_{\Omega(v_h)} (f - f_\varrho)w\,\mathrm{d}x\right| + \tag{27}$$
$$+ \left|\sum_{K\in\mathcal{T}_h(v_h)} \int_K [\nabla y(v_h) - (\nabla y_n(v))_\varrho]\cdot\nabla w\,\mathrm{d}x\right|.$$

Let $w_\nabla \in [L^2(\Omega(v_h))]^2$ be a function such that for any triangle $K \in \mathcal{T}_h(v_h)$, $w_\nabla|_K = \nabla w|_K$. Obviously, $\|w_\nabla\|_{0,\Omega(v_h)} = |w|_{X(v_h)}$. Replacing ∇w by w_∇ in (4.14) and summing, we arrive at

$$|b(v_h; w, \lambda(v_h) - \kappa_{n\varrho}(v_h))| \leq \left[\int_{\Omega(v_h)} (f - f_\varrho)^2 \,\mathrm{d}x\right]^{1/2} \|w\|_{X(v_h)} + \tag{28}$$

$$+ \quad \left[\int_{\Omega(v_h)} |\nabla y(v_h) - (\nabla y_n(v))_\varrho|^2 \,\mathrm{d}x\right]^{1/2} \|w\|_{X(v_h)}.$$

We can write

$$\int_{\Omega(v_h)} (f - f_\varrho)^2 \,\mathrm{d}x = \int_{\Omega(v)} (f - f_\varrho)^2 \,\mathrm{d}x + \varepsilon_1(v_h, \varrho) = \varepsilon_1(v_h, \varrho) + \varepsilon_2(\varrho), \tag{29}$$

where the term $\varepsilon_1(v_h, \varrho)$ involves the integrals over the sets $\Omega(v_h) \setminus \Omega(v)$ and $\Omega(v) \setminus \Omega(v_h)$. We easily get

$$\lim_{h \to 0+} \varepsilon_1(v_h, \varrho) = 0\,, \qquad \lim_{\varrho \to 0+} \varepsilon_2(\varrho) = 0. \tag{30}$$

Indeed, the first limit is a consequence of the uniform convergence of the functions v_h and the second one is a well known property of regularized functions (see e. g. [8], § 15).

Making use of $\Omega(v_h) \subset \Omega_\delta \cap \Omega_Q \cap \Omega_n(v)$, we estimate the second term of (4.15)

$$\|\nabla y(v_h) - (\nabla y_n(v))_\varrho\|_{0,\Omega(v_h)} \leq \|\nabla y(v_h) - \nabla y(v)\|_{0,\Omega_\delta} + \tag{31}$$

$$+ \quad \|\nabla y(v) - \nabla y_n(v)\|_{0,\Omega_Q} + \|\nabla y_n(v) - (\nabla y_n(v))_\varrho\|_{0,\Omega_n(v)}.$$

Denoting the terms on the right-hand side of (4.18) by $\varepsilon_3(v_h)$, $\varepsilon_4(n)$ and $\varepsilon_5(n, \varrho)$, respectively, and applying Lemmas 2.1 and 4.3, we have

$$\lim_{h \to 0+} \varepsilon_3(v_h) = 0, \qquad \lim_{n \to \infty} \varepsilon_4(n) = 0, \qquad \lim_{\varrho \to 0+} \varepsilon_5(n, \varrho) = 0. \tag{32}$$

The last equality is valid for any positive parameter n.

Finally, (4.15), (4.16) and (4.18) give

$$\varepsilon(v_h, n, \varrho) = [\varepsilon_1(v_h, \varrho) + \varepsilon_2(\varrho)]^{1/2} + \varepsilon_3(v_h) + \varepsilon_4(n) + \varepsilon_5(n, \varrho)$$

and, by virtue of (4.17), (4.19), the statement (4.9) holds. □

Lemma 4.5. Assume that a function $\varphi \in H^2(\Omega(v_h))$, $v_h \in \mathcal{U}_{ad}^h$, is given. Let us define $\psi \in M(v_h)$ by $\psi = \frac{\partial \varphi}{\partial \nu_K}$ on ∂K, $K \in \mathcal{T}_h(v_h)$. Then

$$\inf_{\mu_h \in M_h(v_h)} \sup_{w \in X(v_h) \setminus \{0\}} \frac{b(v_h; w, \psi - \mu_h)}{|w|_{X(v_h)}} \leq Ch|\varphi|_{2,\Omega(v_h)},$$

where the constant $C > 0$ is independent of h and $v_h \in \mathcal{U}_{ad}^h$.

P r o o f. Lemma 4.5 is, in fact, Lemma 9 of [9].

For any $K \in \mathcal{T}_h(v_h)$ and any side S of K, we set $\mu_h = \pi_S^0 \frac{\partial \varphi}{\partial \nu_K} = \pi_S^0 \psi$ on S, where π_S^0 stands for the orthogonal projector in $L^2(S)$ upon constants $P_0|_S$. We can estimate

$$\forall\, w \in H^1(K) \quad \left| \int_S (\psi - \mu_h) w \,\mathrm{d}\gamma \right| = \left| \int_S \left(\frac{\partial \varphi}{\partial \nu_K} - \pi_S^0 \frac{\partial \varphi}{\partial \nu_K} \right) w \,\mathrm{d}\gamma \right| \leq$$
$$\leq \quad Ch|\varphi|_{2,K} |w|_{1,K},$$

where the constant $C > 0$ is independent of K, h and v_h. The inequality is a direct consequence of the estimate cited by [9]. However, the paper [9] refers to [4]. A detailed proof can be found in [2]. Since the independence of C is a consequence of strongly regular triangulations, we consider [4] as a sufficient reference. Thus

$$\forall\, w \in X(v_h) \quad |b(v_h; w, \psi - \mu_h)| = \left| - \sum_{K \in \mathcal{T}_h(v_h)} \int_{\partial K} (\psi - \mu_h) w \,\mathrm{d}\gamma \right| \leq$$
$$\leq \quad \sum_{K \in \mathcal{T}_h(v_h)} C_1 h |\varphi|_{2,K} |w|_{1,K} \leq C_1 h |\varphi|_{2,\Omega(v_h)} |w|_{X(v_h)},$$

and the constant $C_1 > 0$ does not depend on h and v_h. □

Having Lemma 4.4 and Lemma 4.5, we tend to approximation of $\lambda(v_h)$ by $\kappa_{n\varrho}(v_h)$ and to utilization of (4.1). According to the following lemma, the two norms on $V_h(v_h)$ are uniformly equivalent.

Lemma 4.6. There exists a constant $C > 0$, independent of h and $v_h \in \mathcal{U}_{ad}^h$, such that

$$\forall\, w_h \in V_h(v_h) \qquad \|w_h\|_{X(v_h)} \leq C |w_h|_{X(v_h)}.$$

P r o o f. The family τ of triangulations is strongly regular, so that there exist constants $C_1 > 0$, $C_2 > 0$, independent of h and $v_h \in \mathcal{U}_{ad}^h$, such that for any $K \in \mathcal{T}_h(v_h)$ it holds $\operatorname{diam}(K) \leq C_1 h$ and $C_2 h^2 \leq \operatorname{meas}(K)$.

Suppose $v_h \in \mathcal{U}_{ad}^h$ is given. The interval $[0,1]$ on x_2-axis is subdivided into $N = h^{-1}$ subintervals. We denote their midpoints by z_j, $j = 1, \ldots, N$, and erect perpendiculars p_j — see Fig. 3. Let the set of triangles $K_i \in \mathcal{T}_h(v_h)$ intersected by p_j be denoted by I_j, $j = 1, \ldots, N$. We define the sets $Q_j = \bigcup_{i \in I_j} K_i$, $j = 1, \ldots, N$, and the segments $q_i = p_j \cap K_i$, $i \in I_j$, $j = 1, \ldots, N$. The length of q_i is not greater than $C_1 h$.

Fig. 3.

Let us choose a point $x \in \bigcup_{i \in I} K_i$, $I = \bigcup_{j=1}^{N} I_j$. Then the subscripts i and j exist such that $x \in K_i$, $x \in Q_j$. By (3.5), the function w is continuous at a midpoint $s_i = (t_i, z_j) \in \partial K_i$. Using (3.6), we have

$$w(x) = \int_0^{t_i} \frac{\partial w}{\partial x_1}(x_1, z_j)\, dx_1 + (\nabla\, w|_{K_i}) \cdot (x - s_i).$$

Estimating the argument of the above integral on relevant segments q_i and realizing the fact that the number of elements of the set I_j is not greater than C_3/h, $C_3 > 0$ is a constant independent of $v_h \in \mathcal{U}_{ad}^h$, we arrive at

$$w^2(x) \leq 4C_1^2 C_3 h \sum_{k \in I_j} \|\nabla w|_{K_k}\|_{\mathbb{R}^2}^2. \tag{33}$$

We integrate (4.20) over K_i, taking linearity of w on any triangle into account,

$$\begin{aligned} \int_{K_i} w^2(x)\,\mathrm{d}x \quad &\leq \quad 4C_1^2 C_3 h \,\mathrm{meas}\,(K_i) \sum_{k \in I_j} \|\nabla w|_{K_k}\|_{\mathbb{R}^2}^2 \leq \\ &\leq \quad 4\frac{C_1^4 C_3}{C_2} h \sum_{k \in I_j} \int_{K_k} |\nabla w|_{K_k}|^2\,\mathrm{d}x. \end{aligned} \tag{34}$$

The estimate (4.21) is valid for any K_i, $i \in I_j$. Denoting $C_4 = 4C_1^4 C_3/C_2$, we obtain

$$\int_{Q_j} w^2(x)\,\mathrm{d}x = \sum_{i \in I_j} \int_{K_i} w^2(x)\,\mathrm{d}x \leq C_3 C_4 \sum_{k \in I_j} \int_{K_k} |\nabla w|_{K_k}|^2\,\mathrm{d}x.$$

Finally, we integrate over $\Omega(v_h) = \bigcup_{j=1}^{N} Q_j$

$$\int_{\Omega(v_h)} w^2(x)\,\mathrm{d}x \leq C_3 C_4 \sum_{k \in I} \int_{K_k} |\nabla w|_{K_k}|^2\,\mathrm{d}x = C_3 C_4 |w|_{X(v_h)}^2.$$

The closing estimate is independent of h and $v_h \in \mathcal{U}_{ad}^h$,

$$\forall\, w \in V_h(v_h) \qquad \|w\|^2_{X(v_h)} \le (C_3C_4 + 1)\,|w|^2_{X(v_h)}. \qquad \square$$

Now we are able to prove the following convergence lemma based on (4.1).

Theorem 4.2. Assume that a sequence $\{v_h\}$, $h \to 0+$, $v_h \in \mathcal{U}_{ad}^h$, converges to v in $C([0,1])$. The solutions of (3.1)–(3.2) and (2.1)–(2.2) are denoted by $(y_h(v_h), \lambda_h(v_h)) \in X_h(v_h) \times M_h(v_h)$ and $(y(v), \lambda(v)) \in H_0^1(\Omega(v)) \times M(v)$, respectively. Then

$$\lim_{h\to 0+} \|y_h(v_h) - y(v)\|_{X_\delta(v_h)} = 0.$$

Proof. We know (Lemma 2.1) that

$$y(v_h) \to y(v) \text{ in } H^1(\Omega_\delta),\ h \to 0+, \tag{35}$$

where $(y(v_h), \lambda(v_h)) \in H_0^1(\Omega(v_h)) \times M(v_h)$ solves (2.1)–(2.2) on $\Omega(v_h)$.

Denoting the terms on the right-hand side of (4.1) by $I_1^2(v_h)$ and $I_2^2(v_h)$, respectively, we have (see Lemma 4.2)

$$\lim_{h\to 0+} |y(v_h) - y_h(v_h)|^2_{X(v_h)} = \lim_{h\to 0+} (I_1^2(v_h) + I_2^2(v_h)) = \lim_{h\to 0+} I_2^2(v_h). \tag{36}$$

Let us choose $\varepsilon_0 > 0$ arbitrarily. Then (see (4.9)) parameters n, ϱ and $h_0 > 0$, dependent on ε_0, exist such that $\varepsilon(v_h, n, \varrho) \le \varepsilon_0$ and the function $y_{n\varrho}(v)$ defines $\kappa_{n\varrho}(v_h) \in M(v_h)$, $0 < h \le h_0$. Using (4.8) and Lemma 4.6, we get

$$\begin{aligned} &\forall\, w_h \in V_h(v_h)\ \forall\, \mu_h \in M_h(v_h) \qquad |b(v_h; w_h, \lambda(v_h) - \mu_h)| \le \\ &\le |b(v_h; w_h, \lambda(v_h) - \kappa_{n\varrho}(v_h))| + |b(v_h; w_h, \kappa_{n\varrho}(v_h) - \mu_h)| \le \\ &\le \varepsilon_0 C_1 |w_h|_{X(v_h)} + |b(v_h; w_h, \kappa_{n\varrho}(v_h) - \mu_h)|, \end{aligned} \tag{37}$$

where the constant $C_1 > 0$ is independent of h and v_h.

Applying Lemma 4.5 to (4.24), we obtain for any positive ε_0

$$\begin{aligned} |I_2(v_h)| &\le \varepsilon_0 C_1 + \inf_{\mu_h \in M_h(v_h)} \sup_{w_h \in V_h(v_h)\setminus\{0\}} \frac{b(v_h; w_h, \kappa_{n\varrho}(v_h) - \mu_h)}{|w_h|_{X(v_h)}} \le \\ &\le \varepsilon_0 C_1 + C_2 h |y_{n\varrho}(v)|_{2,\Omega(v_h)} \le \varepsilon_0 C_1 + C_2 h |y_{n\varrho}(v)|_{2,\Omega_\delta} \le 2\varepsilon_0 C_1, \end{aligned}$$

where the constant $C_2 > 0$ does not depend on h and v_h. Thus, limit (4.23) tends to zero.

Choosing $\varepsilon > 0$, we have, if h is small enough, (see (4.22), (4.23))

$$|y_h(v_h) - y(v)|_{X_\delta(v_h)} < \varepsilon. \tag{38}$$

Next, a function $y_\varepsilon(v) \in C_0^\infty(\Omega(v))$ exists such that

$$\|y(v) - y_\varepsilon(v)\|_{1,\Omega_\delta} < \varepsilon. \tag{39}$$

For any triangulation $\mathcal{T}_{h\delta}(v_h) \in \tau_\delta$ we define the set of continuous and piecewise linear functions

$$Z_\delta(v_h) = \left\{ z \in C(\overline{\Omega}_\delta); \forall\, K \in \mathcal{T}_{h\delta}(v_h) \quad z|_K \in P_1(K) \right\}.$$

For any $w \in C(\overline{\Omega}_\delta)$, let $r_h w \in Z_\delta(v_h)$ be the $Z_\delta(v_h)$-interpolation.

The well known estimate (see e. g. [3], Theorem 3.2.1) gives

$$\|y_\varepsilon(v) - r_h y_\varepsilon(v)\|_{1,\Omega_\delta} \le C_3 h |y_\varepsilon(v)|_{2,\Omega_\delta} \le \varepsilon, \tag{40}$$

the constant $C_3 > 0$ is independent of h and v_h since the family τ_δ of triangulations is strongly regular. Combining (4.25), (4.26) and (4.27), we estimate

$$|y_h(v_h) - r_h y_\varepsilon(v)|_{X_\delta(v_h)} < 3\varepsilon.$$

For any h sufficiently small, we have $\operatorname{supp} y_\varepsilon(v) \subset \Omega(v_h)$, hence $r_h y_\varepsilon(v)|_\Omega (v_h) \in V_h(v_h)$.

The function $g_h(v_h) = y_h(v_h) - r_h y_\varepsilon(v)|_{\Omega(v_h)}$ complies with the assumptions of Lemma 4.6, hence a constant $C_4 > 0$, independent of h and v_h, exists such that

$$\|g_h(v_h)\|_{X_\delta(v_h)} \le 3C_4\varepsilon. \tag{41}$$

The inequalities (4.28), (4.26) and (4.27) imply the final estimate

$$\begin{aligned} & \|y_h(v_h) - y(v)\|_{X_\delta(v_h)} \le \|y_h(v_h) - r_h y_\varepsilon(v)\|_{X_\delta(v_h)} + \|r_h y_\varepsilon(v) - y(v)\|_{1,\Omega_\delta} \le \\ \le\ & 3C_4\varepsilon + \|y(v) - y_\varepsilon(v)\|_{1,\Omega_\delta} + \|y_\varepsilon(v) - r_h y_\varepsilon(v)\|_{1,\Omega_\delta} \le \varepsilon(3C_4 + 2). \end{aligned}$$

□

Lemma 4.7. Under the assumptions of Theorem 4.2

$$\lim_{h \to 0+} \mathcal{J}(v_h, y_h(v_h)) = \mathcal{J}(v, y(v)).$$

P r o o f. The proof is easily seen from (1.2) and Theorem 4.2. □

To prove the main convergence result we need the following auxiliary lemma.

Lemma 4.8. For any $v \in \mathcal{U}_{ad}$ there exists a sequence $\{v_h\}$, $h \to 0+$, $v_h \in \mathcal{U}_{ad}^h$, such that

$$\lim_{h \to 0+} v_h = v \quad \text{in } C([0,1]).$$

P r o o f. Following the proofs of [7, Lemma 3.1] and [1, Lemma 7.1], we can construct a piecewise linear function dependent on h and prove that it belongs to $\mathcal{U}_{ad}^h$. A sequence of these functions converges to v uniformly. □

Theorem 4.3. Let $\{v^h_{opt}\}$, $h \to 0+$, $v^h_{opt} \in \mathcal{U}^h_{ad}$, be a sequence of solutions of the approximate domain optimization problem $\mathcal{P}_h$ (see (3.3)). Then a subsequence $\left\{v^{h_n}_{opt}\right\} \subset \{v^h_{opt}\}$ exists such that $v^{h_n}_{opt} \to v_{opt}$ in $C([0,1])$, $h_n \to 0+$, and the design variable $v_{opt} \in \mathcal{U}_{ad}$ solves the problem $\mathcal{P}$ (see (1.3)).

The strong convergence of the first component of the approximate hybrid state solution is ensured by Theorem 4.2.

Proof. We follow the proof of [1, Theorem 7.1].

Let $\eta \in \mathcal{U}_{ad}$ be given. There exists a sequence $\{\eta_h\}$, $h \to 0+$, $\eta_h \in \mathcal{U}^h_{ad}$, such that $\eta_h \to \eta$ in $C([0,1])$ (Lemma 4.8). Let us denote by $y_h(\eta_h) \in X_h(\eta_h)$ the first component of the solution of (3.1)–(3.2) where v_h is replaced by η_h.

The set $\mathcal{U}^0_{ad}$ is compact in $C([0,1])$. We choose a subsequence $\left\{v^{h_n}_{opt}\right\} \subset \{v^h_{opt}\} \subset \mathcal{U}^0_{ad}$ such that (Lemma 4.1) $v^{h_n}_{opt}$ converges to $v^* \in \mathcal{U}_{ad}$ in $C([0,1])$.

By virtue of (3.3), $\mathcal{J}(v^{h_n}_{opt}, y_h(v^{h_n}_{opt})) \le \mathcal{J}(\eta_{h_n}, y_h(\eta_{h_n}))$. Passing to the limit with h_n and using Lemma 4.7, we obtain $\mathcal{J}(v^*, y(v^*)) \le \mathcal{J}(\eta, y(\eta))$ and $v^* \equiv v_{opt}$. □

4.2. Convergence of the second component of the approximate state solution

Essentially, in this subsection we verify results of [9] generalized to a family of domains with a movable boudary.

We provide the space $X(v_h)$ with the new norm $|||w|||_{X(v_h)} = \left(\sum_{K\in\mathcal{T}_h(v_h)} |||w|||^2_{1,K}\right)^{1/2}$, where for any triangle $K \in \mathcal{T}_h(v_h)$,

$$|||w|||_{1,K} = \left(|w|^2_{1,K} + h_K^{-2}\|w\|^2_{0,K}\right)^{1/2}, \quad h_K = \text{diam }(K).$$

Then the space $M(v_h)$ is normed by $|||\mu|||_{M(v_h)} = \sup\limits_{w\in X(v_h)\setminus\{0\}} \frac{b(v_h;w,\mu)}{|||w|||_{X(v_h)}}$.

Lemma 4.9. Let $\mathcal{T}_h(v_h)$ belong to the set τ of strongly regular triangulations. Then there exists a constant $C > 0$, independent of h and v_h, such that

$$\forall\, \mu_h \in M_h(v_h) \qquad \sup_{w_h\in X_h(v_h)\setminus\{0\}} \frac{b(v_h;w_h,\mu_h)}{|||w_h|||_{X(v_h)}} \ge C|||\mu_h|||_{M(v_h)}.$$

Proof. The proof is based on detailed estimates concerning an affine mapping between a triangle K and a reference triangle $\hat{K}$. Since τ is a strongly regular family of triangulations, we find out that the proof of [9, Lemma 10] can be followed step by step. □

Lemma 4.10. Let the pair $(y(v_h),\lambda(v_h)) \in X(v_h) \times M(v_h)$ solve the problem (2.1) – (2.2) on the domain $\Omega(v_h)$. Assume that $(y_h(v_h),\lambda_h(v_h)) \in X_h(v_h) \times M_h(v_h)$ is the solution of the problem (3.1) – (3.2). Then

$$\|\!|\lambda(v_h)-\lambda_h(v_h)\|\!|_{M(v_h)} \le \frac{1}{C}|y(v_h)-y_h(v_h)|_{X(v_h)} + \tag{42}$$
$$+\left(1+\frac{1}{C}\right)\inf_{\mu_h\in M_h(v_h)}\|\!|\lambda(v_h)-\mu_h\|\!|_{M(v_h)},$$

where the constant $C > 0$ is independent of h and $v_h \in \mathcal{U}_{ad}^h$ and equals to the constant C in Lemma 4.9.

P r o o f. We refer to the proof of [9, Theorem 3]. The independence of C is an immediate consequence of Lemma 4.9. □

Theorem 4.4. Let a sequence $\{v_h\}$, $h\to 0+$, $v_h \in \mathcal{U}_{ad}^h$, converge to v in $C([0,1])$. Then

$$\lim_{h\to 0+}\|\!|\lambda(v_h)-\lambda_h(v_h)\|\!|_{M(v_h)} = 0.$$

P r o o f. Let us choose $\hat{\varepsilon} > 0$ arbitrarily. By Lemma 4.4, we have $y_{n\varrho}(v) \in C^\infty(\overline{\Omega(v_h)})$ and $\kappa_{n\varrho}(v_h) \in M(v_h)$ such that

$$\forall\, w \in X(v_h)\setminus\{0\} \quad \frac{b(v_h;w,\lambda(v_h)-\kappa_{n\varrho}(v_h))}{\|\!|w\|\!|_{X(v_h)}} \le \varepsilon(v_h,n,\varrho) \le \hat{\varepsilon}. \tag{43}$$

Using $\|\!|w\|\!|_{X(v_h)} \ge |w|_{X(v_h)}$ and Lemma 4.5 ($\varphi = y_{n\varrho}(v)$, $\psi = \kappa_{n\varrho}(v_h)$), we arrive at

$$\inf_{\mu_h\in M_h(v_h)}\|\!|\kappa_{n\varrho}(v_h)-\mu_h\|\!|_{M(v_h)} = \tag{44}$$
$$= \inf_{\mu_h\in M_h(v_h)}\sup_{w\in X(v_h)\setminus\{0\}}\frac{b(v_h;w,\kappa_{n\varrho}(v_h)-\mu_h)}{\|\!|w\|\!|_{X(v_h)}} \le Ch|y_{n\varrho}(v)|_{2,\Omega(v_h)} \le \hat{\varepsilon},$$

where the last inequality holds for all h sufficiently small because $y_{n\varrho}(v)$ does not depend on h and v_h.

The inequality $\|w\|_{X(v_h)} \le \|\!|w\|\!|_{X(v_h)}$ and (4.30), (4.31) give

$$\inf_{\mu_h\in M_h(v_h)}\|\!|\lambda(v_h)-\mu_h\|\!|_{M(v_h)} = \inf_{\mu_h\in M_h(v_h)}\sup_{w\in X(v_h)\setminus\{0\}}\frac{b(v_h;w,\lambda(v_h)-\mu_h)}{\|\!|w\|\!|_{X(v_h)}} \le$$
$$\le \inf_{\mu_h\in M_h(v_h)}\left[\sup_{w\in X(v_h)\setminus\{0\}}\frac{b(v_h;w,\lambda(v_h)-\kappa_{n\varrho}(v_h))}{\|\!|w\|\!|_{X(v_h)}} + \right.$$
$$\left. + \sup_{w\in X(v_h)\setminus\{0\}}\frac{b(v_h;w,\kappa_{n\varrho}(v_h)-\mu_h)}{\|\!|w\|\!|_{X(v_h)}}\right] \le 2\hat{\varepsilon}.$$

Inserting this into (4.29), we finish the proof by

$$\|\!|\lambda(v_h)-\lambda_h(v_h)\|\!|_{M(v_h)} \le \frac{1}{C}|y(v_h)-y_h(v_h)|_{X(v_h)} + 2(1+\frac{1}{C})\hat{\varepsilon}.$$

Indeed, by virtue of Theorem 4.2 (see (4.23)), the term $|y(v_h) - y_h(v_h)|_{X(v_h)}$ tends to zero. The parameter $\hat{\varepsilon}$ is arbitrarily small. □

Closing remark. The system (3.1) – (3.2) seems to be clumsy to solve. Nevertheless, using the space $V_h(v_h)$ (see (3.5) – (3.6)), we can compute $y_h(v_h)$ from the equation (3.4) easily. Substituting $y_h(v_h)$ into (3.1) and introducing suitable basis functions of the spaces $X_h(v_h)$ and $M_h(v_h)$, we obtain a system of linear equations with a diagonal matrix.

Thus the component $\lambda(v_h)$ can be computed and the desired normal derivative along the boundary of an optimized domain approximated. However, computational effectiveness of the approach given in this paper has not been tested yet.

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RNDr. Jan Chleboun, CSc., Ústav teoretické a aplikované mechaniky AV ČR (Institute of Theoretical and Applied Mechanics – Academy of Sciences of the Czech Republic), Vyšehradská 49, 128 49 Praha 2. Czech Republic.