KYBERNETIKA — VOLUME *30* (1994), NUMBER 2, PAGES 199 – 209

ON HODGES–LEHMANN OPTIMALITY OF LR TESTS

FRANTIŠEK RUBLÍK

It is shown that the likelihood ratio test statistics are Hodges-Lehmann optimal for testing the null hypothesis against the whole parameter space, provided that certain regularity conditions are fulfilled. These conditions are verified for the non-singular normal, multinomial and Poisson distribution.

Let $\{\overline{P}_\gamma;\ \gamma \in \Xi\,\}$ be a family of probability measures, defined on $(X, \mathcal{F})$ by means of the densities $\{\,f(x,\gamma);\ \gamma \in \Xi\,\}$ with respect to a measure ν. If we denote the q-fold products

$$S = X^\infty \times \ldots \times X^\infty\,, \quad \mathcal{S} = \mathcal{F}^\infty \times \ldots \times \mathcal{F}^\infty\,, \quad \Theta = \Xi^q \tag{1}$$

then for $\theta = (\theta_1, \ldots, \theta_q) \in \Theta$ the corresponding product measure

$$P_\theta = \overline{P}^\infty_{\theta_1} \times \ldots \times \overline{P}^\infty_{\theta_q}\,, \tag{2}$$

defined on the σ-algebra $\mathcal{S}$, describes independent sampling from the q populations $(X, \mathcal{F}, \overline{P}_{\theta_j})$, $j = 1, \ldots, q$.

Throughout the paper we shall assume that

$$\emptyset \neq\ \Omega_0 \subset\ \Theta\,. \tag{3}$$

In describing asymptotic properties of tests of the null hypothesis we shall use the notation

$$\mathcal{P} = \left\{\ p \in R^q;\ \sum_{j=1}^{q} p_j = 1 \ \text{ and } \ p_j > 0 \ \text{ for all j}\right\} \tag{4}$$

and for θ,$\theta^* \in \Theta$, $p \in \mathcal{P}$ we denote

$$K(\theta^*, \theta, p) = \sum_{j=1}^{q} p_j K(\theta^*_j, \theta_j)\,, \tag{5}$$

$$K(\Omega_0, \theta, p) = \inf\{\,K(\theta^*, \theta, p);\ \theta^* \in \Omega_0\,\}\,, \tag{6}$$

where $K(\theta_j^*, \theta_j) = K(\overline{P}_{\theta_j^*}, \overline{P}_{\theta_j})$ is the Kullback-Leibler information number.

We shall suppose that a test φ_u of Ω_0 against $\Theta - \Omega_0$ depends on

$$s = (\{x_j^{(1)}\}_{j=1}^{\infty}, \dots, \{x_j^{(q)}\}_{j=1}^{\infty}) \in S$$

through

$$x^{(u)} = (y(1, n_u^{(1)}), \dots, y(q, n_u^{(q)})) \tag{7}$$

only, where

$$y(j, n_u^{(j)}) = (x_1^{(j)}, \dots, x_{n_u^{(j)}}^{(j)}) \tag{8}$$

is a sample from the j-th population.The sample sizes will be subjected to the following assumption, which in the one sample case $q = 1$ simply means that the sample size n tends to infinity.

(C1) In the notation

$$n_u = \sum_{j=1}^{q} n_u^{(j)}, \qquad p_u^{(j)} = n_u^{(j)}/n_u \tag{9}$$

the relations

$$\lim_{u\to\infty} n_u = +\infty, \qquad \lim_{u\to\infty} p_u^{(j)} = p_j \in (0,1\rangle, \quad j = 1, \dots, q \tag{10}$$

hold.

If a test φ_u of Ω_0 against $\Theta - \Omega_0$ is based on (7) and for $\beta_u(\theta) = E_\theta(\varphi_u)$ the relation

$$\sup\{\lim_{u\to\infty} \beta_u(\theta^*);\, \theta^* \in \Omega_0\} = \alpha \in (0,1) \tag{11}$$

holds, then according to Lemma 6.1 in [1] and Theorem 2.1 in [8] under validity of *(C1)* for each parameter $\theta \in \Theta - \Omega_0$

$$\liminf_{u\to\infty} \frac{1}{n_u} \log[1 - \beta_u(\theta)] \ge -K(\Omega_0, \theta, p). \tag{12}$$

We remark that an extension of this inequality to the case of stochastic processes and random fields can be found in [11].

In accordance with [4], [7] and [8] we shall say that the tests $\{\varphi_u\}$ are *Hodges-Lehmann optimal (H–L optimal) for testing* Ω_0 *against* $\Theta - \Omega_0$, if *(C1)* and (11) imply that

$$\lim_{u\to\infty} \frac{1}{n_u} \log[1 - \beta_u(\theta)] = -K(\Omega_0, \theta, p) \tag{13}$$

for each $\theta \in \Theta - \Omega_0$.

The H–L optimality was investigated by Brown in [2], where testing Ω_0 against Ω_1 is replaced with testing Ω_0^* against Ω_1^*, and Ω_i^* is the closure of Ω_i in a set Θ^*, into which the original parameter set Θ is embedded. As pointed out in [2], p. 1208, the likelihood ratio statistics $T(x_1, \dots, x_n, \Omega_0, \Omega_1) = \log[L(x_1, \dots, x_n, \Omega_1)/L(x_1, \dots, x_n,$

$\Omega_0)]$ can be essentially different from $T(x_1, \dots, x_n, \Omega_0^*, \Omega_1^*)$, which is in [2] proved to be optimal for the extended problem Ω_0^* against Ω_1^*.

In some particular cases was the H–L optimality of the LR tests proved in [8]. In the case of exponential families was this H–L optimality in the multisample case $q > 1$ proved in [7] under the conditions, including the following assumptions.

(D 1) The effective set

$$B = \left\{ x; \sup_{\gamma} \log f(x, \gamma) < +\infty \right\} \tag{14}$$

is open.

(D 2) The arithmetic mean $\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ *of the i.i.d. observations* $x_1, \dots, x_n$ *belongs to* B *with probability* 1 *for all* $n \geq N$ *and* $\gamma \in \Xi$.

These assumptions are in [7] imposed to assure existence of the MLE with probability 1 for all $n \geq N$. They are fulfilled by the families generated by the exponential reparametrization of the non-singular normal distributions, but for important exponential families—generated by the exponential reparametrization of the Poisson or the multinomial distributions—the set (14) is closed, and the MLE does not exist with positive probability for all n. Moreover, the null hypothesis Ω_0 is in [7] assumed to have the property that the function (25) of the parameter θ^* is continuous. The aim of this paper is to present regularity assumptions, fulfilled by the non-singular normal, the Poisson and the multinomial distributions. As it can be seen from theorem 1, the presented assumptions in difference from [7] facilitate a unified approach to the H–L optimality of the LR tests for all integers $q \geq 1$, without imposing any restrictions on value of the quantity $K(\theta)$ (defined in [7] on p. 7), or on continuity of $K(\cdot, \Omega_0, p)$.

(A I) There exists a σ*-compact metric space* Ξ_1 *such that* Ξ *is dense in* Ξ_1 *(i. e.* $\overline{\Xi} = \Xi_1$*), and the original system* $\{ f(x, \gamma); \gamma \in \Xi \}$ *can be extended to a system* $\{ f(x, \gamma); \gamma \in \Xi_1 \}$ *of densities with respect to the same measure* ν *(by the extension we understand that* $f(x, \gamma)$ *coincides with the original density if* $\gamma \in \Xi$*). Moreover, the extended system* $\{ \overline{P}_\gamma; \gamma \in \Xi_1 \}$ *consists of mutually different probabilities, the Kullback-Leibler information quantity* $K(\theta^*, .)$ *is continuous on* Ξ_1 *for each* $\theta^* \in \Xi_1$ *(i. e.* $\gamma_n^* \to \gamma^*$ *implies* $K(\theta^*, \gamma_n^*) \to K(\theta^*, \gamma^*)$*), and* $f(x, \gamma)$ *is continuous on* Ξ_1 *for each* $x \in X$.

(A II) The function $K(., \theta)$ *is finite and continuous on* Ξ_1 *for each* $\theta \in \Xi$.

(A III) Let $\{\theta_n^*\}_{n=1}^{\infty}$ *be parameters from* Ξ_1, $\gamma \in \Xi$ *and*

$$\limsup_{n \to \infty} K(\theta_n^*, \gamma) < +\infty .$$

(a) If $\theta \in \Xi$, *then* $\limsup_{n \to \infty} K(\theta_n^*, \theta) < +\infty$.

(b) The sequence $\{\theta_n^*\}_{n=1}^{\infty}$ *possesses a limit point in* Ξ_1 *and if the parameters* $\{\theta_n\}_{n=1}^{\infty}$ *belonging to* Ξ *are such that* $\limsup_{n \to \infty} K(\theta_n^*, \theta_n) < +\infty$, *then also the sequence* $\{\theta_n\}_{n=1}^{\infty}$ *possesses a limit point in* Ξ_1 .

(c) Let $\lim_{n\to\infty} \theta_n^* = \theta^*$*. If* $\{\theta_n\}_{n=1}^\infty$ *is a sequence of parameters from* Ξ *and* $\lim\limits_{n\to\infty} \theta_n = \theta$*, then* $\liminf\limits_{n\to\infty} K(\theta_n^*, \theta_n) \geq K(\theta^*, \theta)$ *.*

(A IV) There exist measurable sets $A_n \subset X^n$*, an integer* N *and measurable mappings* $\hat{\theta}_n : A_n \to \Xi_1$*,* $g_n : A_n \to R$ *such that for each* $\gamma \in \Xi$ *and* $n \geq N$

$$\overline{P}_\gamma(A_n) = 1 \tag{15}$$

and on the set A_n *in the notation* $L(x_1, \ldots, x_n, \gamma) = \prod_{i=1}^n f(x_i, \gamma)$ *the equality*

$$\log L(x_1, \ldots, x_n, \gamma) = g_n(x_1, \ldots, x_n) - nK(\hat{\theta}_n, \gamma) \tag{16}$$

holds.

(A V) In the notation from (A IV) and

$$p_u = (p_u^{(1)}, \ldots, p_u^{(q)}) \,, \tag{17}$$

for $\theta = (\theta_1, \ldots, \theta_q) \in \Xi^q$ *let (cf. (8))*

$$K\Big(\hat{\theta}, \theta, p_u\Big) = \sum_{j=1}^q p_u^{(j)} K\Big(\hat{\theta}_{n_u^{(j)}}\big(y(j, n_u^{(j)})\big), \theta_j\Big) . \tag{18}$$

If (10) holds and $\theta \in \Xi^q$*, then for every real* t

$$P_\theta\big[\, n_u K(\hat{\theta}, \theta, p_u) \geq t \,\big] \leq \exp[-t + h_u(t)] \tag{19}$$

where

$$\limsup_{u\to\infty} \frac{h_u(t_u)}{n_u} = 0 \tag{20}$$

whenever the inequality

$$\limsup_{u\to\infty} \frac{t_u}{n_u} < +\infty \tag{21}$$

holds.

We recall that a metric space is said to be σ-compact, if it can be expressed as a countable union of its compact subsets. The reader can easily verify that both this property of Ξ_1 and the continuity of $f(x, .)$, which are postulated in *(A I)*, together with validity of *(A IV)* imply measurability of (22) for any non-empty set $\Omega \subset \Theta = \Xi^q$.

The regularity assumptions imposed for the multisample case in [7] include only the exponential families, for which the MLE exists with probability 1 for all $n \geq N$. Instead of the exponential reparametrization we use an enlargement of the class of probabilities. Since the Kullback-Leibler information number is non-negative, the axiom
(A IV) guarantees that for all $n \geq N$ there exists with probability 1 a MLE $\hat{\theta}_n$ of the unknown parameter $\gamma \in \Xi$, but taking values in the enlarged parameter set Ξ_1,

and having the property that the likelihood functions computed for the original parameters $\gamma \in \Xi$ can be expressed in the specific postulated way. We remark that for the exponential families, satisfying the assumptions *(D 1)* and *(D 2)*, imposed in [7], the assumption *(A IV)* is fulfilled with $\Xi_1 = \Xi$. Validity of *(A IV)* makes possible to use the inequality (2.8) from [3] in the proof of H–L optimality of LR tests.

If we put for $\Omega \subset \Theta$

$$L(x^{(u)}, \Omega) = \sup \left\{ \prod_{j=1}^{q} \prod_{i=1}^{n_u^{(j)}} f(x_i^{(j)}, \theta_j);\ \theta = (\theta_1, \ldots, \theta_q) \in \Omega \right\} , \tag{22}$$

then by the likelihood ratio test statistics T_u for testing Ω_0 against $\Theta - \Omega_0$ we shall understand the statistics

$$T_u = 2 \log \frac{L(x^{(u)}, \Theta)}{L(x^{(u)}, \Omega_0)} . \tag{23}$$

In the proof of the theorem on H–L optimality of (23) the following lemma will be used.

Lemma 1. Let the assumptions *(A I)* – *(A III)* be fulfillled, (3) hold and for $\delta \geq 0$ let

$$D(\delta, p) = \{ \theta^* \in \Xi_1^q; K(\theta^*, \Omega_0, p) \leq \delta \} , \tag{24}$$

where

$$K(\theta^*, \Omega_0, p) = \inf \{ K(\theta^*, \tilde{\theta}, p);\ \tilde{\theta} \in \Omega_0 \} . \tag{25}$$

If $\lim_{u\to\infty} \delta_u = 0$, $\lim_{u\to\infty} p_u = p \in \mathcal{P}$, then for each $\theta \in \Theta$ in the notation (6)

$$d_u = K\big[D(\delta_u, p_u), \theta, p_u\big] \tag{26}$$

is a real number and

$$\liminf_{u\to\infty} d_u \geq K(\Omega_0, \theta, p) . \tag{27}$$

P r o o f. Since $d_u \leq K(\theta^*, \theta, p_u)$ whenever $\theta^* \in \Omega_0$, taking into account *(A II)* we see that

$$\limsup_{u\to\infty} d_u < +\infty . \tag{28}$$

Since the sets (24) are non-empty, there exist $\theta_u^* \in D(\delta_u, p_u)$ such that

$$d_u \leq K(\theta_u^*, \theta, p_u) \leq d_u + u^{-1} . \tag{29}$$

Let $\theta_u^{(0)} \in \Omega_0$ be such that

$$K(\theta_u^*, \Omega_0, p_u) \leq K(\theta_u^*, \theta_u^{(0)}, p_u) \leq K(\theta_u^*, \Omega_0, p_u) + u^{-1} . \tag{30}$$

Let us choose an increasing sequence $\{u_v\}_{v=1}^{\infty}$ of positive integers such that

$$\liminf_{u\to\infty} d_u = \lim_{v\to\infty} d_{u_v} . \tag{31}$$

Fixing $\gamma_0 \in \Omega_0$ and utilizing (28)–(30), $p_u \to p$ and *(A III)(a)* we get

$$\limsup_{u\to\infty} K(\theta_u^*, \theta_u^{(0)}, p_u) \le \limsup_{u\to\infty} K(\theta_u^*, \gamma_0, p_u) < +\infty$$

which together with *(A III)(b)* means that there exists a subsequence $\{u_{v_t}\}_{t=1}^{\infty}$ and points $\theta^* \in \Xi_1$, $\theta^{(0)} \in \overline{\Omega}_0$ such that in the notation

$$z_t = u_{v_t}\,, \qquad \gamma_t^* = \theta_{z_t}^*\,, \qquad \gamma_t^{(0)} = \theta_{z_t}^{(0)}$$

for $t \to \infty$

$$\gamma_t^* \to \theta^*\,, \quad \gamma_t^{(0)} \to \theta^{(0)}\,.$$

If $\tilde{\theta} \in \Omega_0$, then *(A II)*, (30) and *(A III)(c)* yield

$$K(\theta^*, \tilde{\theta}, p) = \lim_{t\to\infty} K(\gamma_t^*, \tilde{\theta}, p_{z_t}) \ge \limsup_{t\to\infty} K(\gamma_t^*, \gamma_t^{(0)}, p_{z_t}) \ge$$

$$\ge \liminf_{t\to\infty} K(\gamma_t^*, \gamma_t^{(0)}, p_{z_t}) \ge K(\theta^*, \theta^{(0)}, p) \ge \; K(\theta^*, \overline{\Omega}_0, p)\,.$$

Since according to *(A I)*

$$K(\theta^*, \overline{\Omega}_0, p) = K(\theta^*, \Omega_0, p)\,, \tag{32}$$

obviously $K(\theta^*, \Omega_0, p) = \lim_{t\to\infty} K(\gamma_t^*, \gamma_t^{(0)}, p_{z_t})$. Combining this with (30) and $\theta_u^* \in D(\delta_u, p_u)$ we see that $K(\theta^*, \Omega_0, p) = 0$. Utilizing *(A III)(b), (A I)* we obtain that $\theta^* \in \overline{\Omega}_0$ and by virtue of (31), (29), *(A II)*

$$\liminf_{u\to\infty} d_u = \lim_{t\to\infty} K(\gamma_t^*, \theta, p_{z_t}) = K(\theta^*, \theta, p) \ge \; K(\overline{\Omega}_0, \theta, p)\,.$$

The inequality (27) now follows from *(A II)*. □

Theorem 1. Let a family of probabilities $\{\overline{P}_\gamma; \gamma \in \Xi\}$, determined by densities $\{f(x,\gamma);$
$\gamma \in \Xi\}$ with respect to a measure ν be such that the assumptions *(A I)–(A V)* are fulfilled. Let

$$\emptyset \ne \Omega_0 \subset \Theta = \Xi^q \tag{33}$$

and in the notation (23)

$$\varphi_u(x^{(u)}) = \begin{cases} 1 & T_u > t_u \\ 0 & T_u \le t_u\,. \end{cases} \tag{34}$$

If validity of *(C1)* and (11) implies that the critical constants t_u in (34) can be chosen in such a way that

$$\limsup_{u\to\infty} t_u < +\infty \tag{35}$$

then the tests (34) are H–L optimal.

Proof. We shall proceed analogously as in the proof of theorem 4.5 in [7]. Let us assume that both *(CI)* and (11) hold. In accordance with *(A IV)* let

$$B_u = A_{n_u^{(1)}} \times \cdots \times A_{n_u^{(q)}}$$

and for $x^{(u)} \in B_u$ put

$$\hat{\theta}_{(u)}(x^{(u)}) = \left(\hat{\theta}_{n_u^{(1)}}(y(1, n_u^{(1)})), \ldots, \hat{\theta}_{n_u^{(q)}}(y(q, n_u^{(q)}))\right) . \tag{36}$$

Making use of *(A IV)*, *(A II)* we get for $u \geq u_0$ in the notation (24) that

$$P_\theta \left[2 \log \frac{L(x^{(u)}, \Theta)}{L(x^{(u)}, \Omega_0)} \leq t_u \right] = P_\theta \left[2 n_u K(\hat{\theta}_{(u)}, \Omega_0, p_u) \leq t_u \right] =$$

$$= P_\theta \left[\hat{\theta}_{(u)} \in D(\delta_u, p_u) \right] \tag{37}$$

where $\delta_u = t_u / 2n_u$. According to the assumptions $\lim_{u \to \infty} \delta_u = 0$. Utilizing the first inequality in (2.8) in [3] and applying the assumption *(A V)* we see that for $u \geq u_1$

$$P_\theta \left[\hat{\theta}_{(u)} \in D(\delta_u, p_u) \right] \leq P_\theta \left[K(\hat{\theta}_{(u)}, \theta, p_u) \geq d_u \right] \leq \exp[-n_u d_u + o_u] \tag{38}$$

where $\lim_{u \to \infty} o_u / n_u = 0$. Combining (37) and (38) with (27) we get

$$\limsup_{u \to \infty} \frac{1}{n_u} \log P_\theta[\varphi_u = 0] \leq -K(\Omega_0, \theta, p)$$

and (13) follows from (12). □

According to Theorem 1 the LR tests (34) based on (23) are H–L optimal for arbitrary non-empty set $\Omega_0 \subset \Xi^q$, having the property (35). However, if $\theta^* \in \Omega_0$, then

$$2 \log \frac{L(x^{(u)}, \Theta)}{L(x^{(u)}, \Omega_0)} \leq \tilde{T}_u(x^{(u)}, \theta^*) , \quad \tilde{T}_u(x^{(u)}, \theta^*) = 2 \log \frac{L(x^{(u)}, \Theta)}{L(x^{(u)}, \theta^*)} . \tag{39}$$

If Ξ is an open subset of R^m and the densities $\{f(x, \gamma); \gamma \in \Xi\}$ satisfy regularity conditions on partial derivatives, then the MLE $\hat{\theta}_n$ of the unknown parameter from Ξ has the property that $\mathcal{L}(\sqrt{n}(\hat{\theta}_n - \gamma) | P_\gamma) \to N(0, J^{-1}(\gamma))$, where $J(\gamma)$ is the Fisher information matrix, and according to the well-known classical results under validity of *(C1)* for $u \to \infty$ the weak convergence of distributions

$$\mathcal{L}(\tilde{T}_u(x^{(u)}, \theta^*) | P_{\theta^*}) \to \chi^2_{mq} \tag{40}$$

takes place, where χ^2_{mq} denotes the chi-square distribution with mq degrees of freedom. From (39) and (40) one can easily find out, that validity of *(C1)* and (11) implies (35). This fact will be used in the proofs of theorems in the further text.

Let $k > 1$ be an integer and $a = (k-1)k/2$. Let us put $m = 2k + a$ and denote

$$\Xi = \{\, \gamma = (\mu', \sigma_1, \dots, \sigma_k, \rho')' \in R^m;\ \mu \in R^k, \min \sigma_i > 0, \rho \in R^a, R(\rho) \text{ is positive definite} \,\} \tag{41}$$

the set of parameters of the non-singular k-dimensional normal distributions, i. e. μ is the vector of means, σ_i^2 are the variances, $\rho = (\rho_{12}, \dots, \rho_{k-1k})'$ are the correlation coefficients and $R(\rho)$ is the symmetric matrix with $R(\rho)_{ij} = \rho_{ij}$ if $i < j$, and $R(\rho)_{ii} = 1$. For $\gamma \in \Xi$ we shall denote by $V(\gamma)$ the covariance matrix of the corresponding normal distribution and $f(x, \gamma)$ its density. In this notation the following theorem holds.

Theorem 2. If Ω_0 is a non-empty subset of $\Theta = \Xi^q$ and T_u are the statistics (23), then the tests (34) are H–L optimal for testing Ω_0 against $\Theta - \Omega_0$.

Proof. Since (40) holds with $m = \frac{k(k+3)}{2}$, the tests (34) have the property (35). We shall prove that also the assumptions *(A I)–(A V)* are fulfilled.

Let us denote $\Xi_1 = \Xi$. For $\gamma, \gamma^* \in \Xi$

$$K(\gamma, \gamma^*) = \frac{1}{2}(\mu - \mu^*)' V(\gamma^*)^{-1} (\mu - \mu^*) + \frac{1}{2} \mathrm{tr}\big[V(\gamma) V(\gamma^*)^{-1}\big] + \frac{1}{2} \log \frac{|V(\gamma^*)|}{|V(\gamma)|} - \frac{k}{2}$$

and *(A I)*, *(A II)* obviously hold.

According to the inequality (3.31) in [6]

$$K(\gamma^*, \gamma) \geq \frac{1}{2}(\mu^* - \mu)' V(\gamma)^{-1} (\mu^* - \mu) + \frac{1}{2} \sum_{j=1}^{k} g\Big(\frac{\lambda_j(\gamma)}{\lambda_j(\gamma^*)}\Big) - \frac{k}{2}$$

where $\lambda_1(\gamma) \geq \ldots \geq \lambda_k(\gamma)$ are characteristic roots of $V(\gamma)$ and $g(z) = z^{-1} + \log z$. Since $g(z)$ attains its minimum in $z = 1$ and $g(z) \longrightarrow +\infty$ if either $z \to +\infty$ or $z \to 0^+$, the assumption *(A III)* holds.

The assumption *(A V)* can be proved either similarly as the theorem 2.1 in [3], or after the exponential reparametrization by means of the lemma 4.4 in [7]. Since *(A IV)* holds with the usual MLE $\hat{\theta}_n$ and with the set A_n of the n-tuples $(x_1, \dots, x_n) \in (R^k)^n$ for which the matrix $\sum_{j=1}^n (x_j - \overline{x})(x_j - \overline{x})'$ is positive definite, the assumptions of Theorem 1 are fulfilled. □

As we have already noted, this theorem 2 can be applied to testing the null hypothesis (3) on parameters of the non-singular normal distribution by means of the test statistics (23). Since in some situations the likelihood ratio test statistics (23) are expressible as monotone transformations $Z_u(T_u^*)$ of some usually used test statistics T_u^*, in such a case also H–L optimality of T_u^* is established. In the following example the statistics T_u^* have not this property.

Example. Testing equality of covariances. Let us denote in accordance with (1) and (41)

$$\Omega_0 = \{\theta = (\theta_1, \dots, \theta_q) \in \Theta;\ V(\theta_1) = \ldots = V(\theta_q)\} \tag{42}$$

the hypothesis that the covariance matrices of the q normal populations are equal. Let

$$\bar{x}_j, \;\hat{\Sigma}_j$$

denote the sample mean and the sample covariance matrix constructed from the sample drawn from the j-th population. If we put $S_j = n_u^{(j)}\hat{\Sigma}_j$, $S = \sum_{j=1}^q S_j$, then

$$T_u = 2\log\frac{L(x^{(u)},\Theta)}{L(x^{(u)},\Omega_0)} = \log\tilde{T}_u\;, \qquad \tilde{T}_u = \left|\frac{1}{n_u}S\right|^{n_u} \Big/ \prod_{j=1}^q \left|\frac{1}{n_u^{(j)}}S_j\right|^{n_u^{(j)}} .$$

As pointed out in [10], p. 225, to obtain an unbiased test, instead of $\tilde{T}_u$ the modified test statistic

$$T_u^* = \left|\frac{1}{n_u-q}S\right|^{n_u-q} \Big/ \prod_{j=1}^q \left|\frac{1}{n_u^{(j)}-1}S_j\right|^{n_u^{(j)}-1} \tag{43}$$

is used. We shall prove the H–L optimality of the statistic T_u^*.

As it is shown in the proof of the theorem 3.1 in [6], if $\gamma \in \Xi$ and a is a real number, then there exist real numbers $0 < \varepsilon < \beta$ such that in the notation

$$B_n = \{\,(x_1,\dots,x_n);\; \lambda_k(\hat{\Sigma}) \geq \varepsilon\,, \quad \lambda_1(\hat{\Sigma}) \leq \beta\,\}\,, \tag{44}$$

where $\lambda_k(\hat{\Sigma})$ is the smallest and $\lambda_1(\hat{\Sigma})$ is the largest characteristic root of $\hat{\Sigma}$, the inequality

$$\limsup_{n\to\infty}\frac{1}{n}\log\left[1-\overline{P}_\gamma(B_n)\right] \leq -a \tag{45}$$

holds. Let (10) hold. If $a = K(\Omega_0,\theta,p)$ and $B_n^{(j)}$ are the sets (44) satisfying (45) with
$\gamma_j = \theta_j$ and $a_j^* = a/p_j$, then in the notation $X = R^k$, $C_u = B^{(1)}_{n_u^{(1)}} \times \ldots \times B^{(q)}_{n_u^{(q)}}$

$$\limsup_{u\to\infty}\frac{1}{n_u}\log P_\theta\Big[X^{n_u} - C_u\Big] \leq -a\,. \tag{46}$$

Under validity of the null hypothesis $\tilde{T}_u/T_u^* \to 1$ a. e. , and according to [9], p. 404 the distributions $\mathcal{L}(\log\tilde{T}_u) \to \chi_v^2$. Thus if costants $\{t_u\}$ are such that the tests (34) based on $T_u = \log T_u^*$ satisfy (11), then (35) is fulfilled. Further, it is obvious that there exist an index u_1 and a positive real number M such that for all $u \geq u_1$ on the set C_u the inequality $\tilde{T}_u/T_u^* \leq M$ holds. Hence according to Theorem 2

$$\limsup_{u\to\infty}\frac{1}{n_u}\log P_\theta\big[C_u \cap \{\log T_u^* \leq t_u\}\big] \leq$$

$$\leq \limsup_{u\to\infty}\frac{1}{n_u}\log P_\theta\big[\log\tilde{T}_u \leq t_u + \log M\big] = -K(\Omega_0,\theta,p)\,.$$

Combining this with (46) we get

$$\limsup_{u\to\infty}\frac{1}{n_u}\log P_\theta\big[\log T_u^* \leq t_u\big] \leq -K(\Omega_0,\theta,p)\,,$$

which together with (12) means that the tests based on T_u^* are H–L optimal for testing the hypothesis (42) under the normality assumptions.

Let $X = \{1, ..., k\}$ be a finite set,

$$\Xi = \left\{ (p_1, ..., p_{k-1})' \in R^{k-1};\ \min_i p_i > 0,\ \sum_{i=1}^{k-1} p_i < 1 \right\} \tag{47}$$

and

$$f(x,p) = p_x\,, \qquad p_k = 1 - \sum_{j=1}^{k-1} p_j \tag{48}$$

denotes a density with respect to the counting measure μ on $(X, 2^X)$. In this notation the following theorem holds.

Theorem 3. If Ω_0 is a non-empty subset of $\Theta = \Xi^q$ and T_u are the statistics (23), then the tests (34) are H–L optimal for testing Ω_0 against $\Theta - \Omega_0$.

Proof. Since (40) holds with $m = k-1$, the tests (34) have the property (35). To prove *(A I)–(A V)*, let us denote

$$\Xi_1 = \overline{\Xi} \tag{49}$$

the closure of (47), and let $f(x,p)$ be the densities (48). Since

$$K(p, p^*) = \sum_{j=1}^{k} p_j \log \frac{p_j}{p_j^*}$$

where $0 \log \frac{0}{x} = 0$, the assumption *(A III)(c)* can be proved similarly as the lemma 4.4(a) in [5], and all the other statements in *(A I)–(A III)* can be easily proved by means of the compactness of Ξ_1.

The assumption *(A IV)* is fulfilled with $A_n = X^n$ and

$$\hat{\theta}_n = (\hat{p}_1, \dots, \hat{p}_{k-1})'\,, \qquad \hat{p}_x = \frac{n_x}{n}$$

where n_x denotes the number of occurrences of x in $x_1, \dots, x_n$.

Making use of the first equality in (48), the relation (2.4) in [5] and proceeding as in the proof of the inequality (2.10) in [5], we obtain, that *(A V)* holds.

Thus the assumptions of Theorem 1 are fulfilled and the proof is completed. □

Let $X = \{0, 1, 2, ...\}$,

$$\Xi = (0, +\infty) \tag{50}$$

and

$$f(x, \lambda) = \frac{e^{-\lambda} \lambda^x}{x!} \tag{51}$$

be density of the Poisson distribution $\overline{P}_\lambda$ with respect to the counting measure μ on $(X, 2^X)$. In this notation the following theorem holds.

Theorem 4. If Ω_0 is a non-empty subset of $\Theta = \Xi^q$ and T_u are the statistics (23), then the tests (34) are H–L optimal for testing Ω_0 against $\Theta - \Omega_0$.

Proof. Since (40) holds with $m = 1$, the tests (34) have the property (35). To prove *(A I)* – *(A V)*, we denote

$$\Xi_1 = \langle 0\,, +\infty)\,.$$

Since for λ, $\lambda^* \in \Xi_1$

$$K(\lambda, \lambda^*) = \begin{cases} \lambda^* - \lambda + \lambda \log \dfrac{\lambda}{\lambda^*} & \lambda^* > 0 \\ +\infty & \lambda^* = 0,\ \lambda > 0 \end{cases}$$

where $0 \log \frac{0}{x} = 0$, the assumptions *(A I)* – *(A III)* hold. The assumption *(A IV)* holds with $A_n = X^n$ and $\hat{\theta}_n = n^{-1} \sum_{j=1}^{n} x_j$. Since the axiom *(A V)* can be proved either by means of the relation (6.22) in [6] or by means of the Lemma 4.3 in [7], the assumptions of Theorem 1 are fulfilled. □

(Received December 22, 1992.)

REFERENCES

[1] R. R. Bahadur: Some Limit Theorems in Statistics. SIAM, Philadelphia, 1971.

[2] L. D. Brown: Non-local asymptotic optimality of appropriate likelihood ratio tests. Ann. Math. Statist. *42* (1971), 1206–1240.

[3] D. G. Herr: Asymptotically optimal tests for multivariate normal distributions. Ann. Math. Statist. *38* (1967), 1829–1844.

[4] J. L. Hodges and E. L. Lehmann: The efficiency of some nonparametric competitors of the t-test. Ann. Math. Statist. *27* (1956), 324–335.

[5] W. Hoeffding: Asymptotically optimal tests for multinomial distributions. Ann. Math. Statist. *36* (1965), 369–408.

[6] F. Rublík: On optimality of the LR tests in the sense of exact slopes. II. Application to individual distributions. Kybernetika *25* (1989), 117–135.

[7] S. Kourouklis: Hodges-Lehmann optimality of the likelihood ratio test in regular exponential families of distributions. TR. No 73, Dept. of Statistics, The Pennsylvania State University, 1987.

[8] S. Kourouklis: Hodges-Lehmann efficacies of certain tests in multivariate analysis and regression analysis. Canadian J. Statist. *16* (1988), 87–95.

[9] A. M. Kshirsagar: Multivariate Analysis. Marcel Dekker, New York, 1972.

[10] M. S. Srivastava and C. G. Khatri: An Introduction to Multivariate Statistics. North-Holland, New York 1979.

[11] I. Vajda: Generalization of discrimination-rate theorems of Chernoff and Stein. Kybernetika *26* (1990), 273–288.

RNDr. František Rublík, CSc., Institute of Measurement of SAS, Dúbravská cesta 9, 842 19 Bratislava. Slovak Republic.