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CONTINUOUS-TIME DEADBEAT OBSERVATION PROBLEM WITH APPLICATION TO PREDICTIVE CONTROL OF SYSTEMS WITH DELAY

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A continuous–time deadbeat observation paradigm is discussed. Two observers are shown to estimate the state of a linear dynamic system deadbeatly in continuous time with respectively finite and infinite memory. Among other properties, BIBO-stability is proved for both structures. Based on the theory devised, deadbeat and asymptotic predictors for plants with delayed control are developed and shown to give rise to predictive feedback controllers assigning finite spectrum to the closed–loop system.

1. INTRODUCTION

It is well known that deadbeat performance can be achieved in discrete–time systems by placing all roots of the characteristic polynomial of an observer or controller at the origin. Therefore, any pole–placement design technique provides the desired transient response property. The notion of Finite-Input Finite-Output stability introduced by Kučera and Kraus [4] for discrete–time systems can be understood as a generalization of the deadbeat strategy.

In contrast, the continuous–time deadbeat problem does not naturally arise from pole placement and has drawn serious research attention only recently. The presence of time delays in the control law or observer structure is inevitable in order to drive the control or observation error to zero deadbeatly. It appears that this phenomenon is well known and referred to as Pointwise Degeneracy in the theory of differential equations with time delays.

Being a new research area, the continuous–time deadbeat problem is treated only in a few papers. In [5], the finite–memory deadbeat observation problem has been solved by a direct state–space approach, related to deterministic least squares.

In [2] a general solution to the deadbeat tracking and stabilization problems is obtained via finite Laplace transform , a common technique in differential–difference equations theory.

The paper is composed as follows. Firstly, we investigate the properties of a Finite–Memory Deadbeat Observer (FMDO), with particular emphasis on its stability. Then, a combination of a conventional Luenberger observer and FMDO is exploited to achieve desired performance in an Infinite–Memory Deadbeat Observer (IMDO). It is shown that the ideas of deadbeat estimation can be used for predicting the state of the plants with time delays along the forward signal path either with deadbeat or asymptotic settling of the prediction error. Using these predictors for feedback control results in structures which generalize the Smith Predictor Method, in the sense that the time delay is excluded from the characteristic polynomial of the closed–loop system.

2. PROBLEM STATEMENT

Consider the Linear Time Invariant (LTI) system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control vector, and $y(t) \in R^\ell$ is the observation vector. A, B, C are real matrices of appropriate dimensions.

The Deadbeat Observation Problem is formulated so as to find a dynamic system (observer) which estimates the state vector of (1) from continuous measurements of y and u so that the estimation error

$$e(t) = x(t) - \hat{x}(t)$$

vanishes outside some predefined time interval $[t_0, t^*)$, i. e. $e(t) \equiv 0; t \geq t^*$.

3. FINITE–MEMORY OBSERVER

A dynamic system whose output at any time instant t does not depend on the system input outside of the time interval $[t, t-\tau]$ for some positive real τ is said to possess a *finite memory*. A dynamic system that does not possess finite memory is termed an *infinite memory* system.

Introduce the operator $\Psi^{\tau_i} : L_2 \to L_2$ defined by

$$(\Psi^{\tau_i} v)(t) = \int_{t-\tau_i}^{t} \exp(A(t-\xi-\tau_i))\, Bv(\xi)\, \mathrm{d}\xi.$$

Theorem 1. Provided that the matrix

$$\mathcal{W}_k = \sum_{i=0}^{k} \exp(-A^T\tau_i)\, C^T C \exp(-A\tau_i)$$

is positive definite, then the observer

$$\begin{aligned} \hat{x}_k(t) &= \mathcal{W}_k^{-1} \sum_{i=0}^{k} \exp(-A^T\tau_i)\, C^T Y_i(t) \\ Y_i(t) &= y(t-\tau_i) + C(\Psi^{\tau_i} u)\,(t) \end{aligned} \tag{2}$$

has the following properties: (i) finite memory, limited by the largest time–delay $\tau_k = \max(\tau_i)$; (ii) dead–beat performance, i.e. $e(t) = x(t) - \hat{x}_k(t) = 0;\ t > \tau_k$ for any initial function $\phi_0 = y(t),\ t \in [-\tau_k, 0]$; (iii) bounded–input bounded-output (BIBO) stability.

Proof. For a proof of the first two properties see [7]. BIBO stability of (2) follows immediately from the finite–memory property of the FMDO, as all the matrices involved are bounded. □

By extending the applicability area of the FMDO beyond the class finite–dimensional systems, the following result is of practical importance.

Corollary 1. The observer (2) provides deadbeat state vector estimation for the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t - \tau). \end{aligned} \tag{3}$$

Proof. Follows immediately assuming $\tau = \tau_0$. □

Discrete deadbeat systems are known to generate control signals of high magnitude when the sampling period is chosen to be too small. A similar behavior can be anticipated in continuous deadbeat systems. This necessitates an evaluation of the transient response of the observer during the phase preceding deadbeat performance.

Let $|\cdot|$ be any vector norm in R^n inducing the matrix norm $\|\cdot\|$ and the matrix measure $\mu(\cdot)$ [1].

Theorem 2. For all u and y satisfying

$$|y| \le m_1; |u| \le m_2$$

an upper bound for the estimate $\hat{x}_k$ is given by

$$|\hat{x}_k(t)| \le M_1 \sum_{i=0}^{k} e^{-\mu(A)\tau_i}(m_1 + m_2 M_{2,i}), \tag{4}$$

where $M_1 = \|\mathcal{W}_k^{-1}\|\,\|C\|$

$$M_{2,i} = \begin{cases} \|C\|\,\|B\|\frac{1-e^{-\mu(A)\tau_i}}{\mu(A)} & \text{if } \mu(A) \neq 0 \\ \|C\|\,\|B\|\tau_i & \text{otherwise.} \end{cases}$$

Proof. See the Appendix.

The estimate (4) relates the time delays of the FMDO to the maximal amplitude attainable by the the estimate $\hat{x}_k(t),\ t < \tau_k$ and can be used for design purposes.

Noteworthy, in Theorem 1 the plant is not required to be stable to guarantee stability of the FMDO, whereas other structures, e. g. the Smith Predictor, cannot be used for unstable systems [2]. Unfortunately, this property is not inherited by realizations of Ψ^{τ_i}. Indeed, taking the derivative of $(\Psi^{\tau_i}u)(t)$ with respect to t, it is straightforward to show that assuming zero initial conditions, the following differential–difference systems possess the same input–output mapping $u \mapsto z$ as the operator Ψ^{τ_i},

$$\dot{z}(t) = Az(t) + \exp(-A\tau_i)Bu(t) - Bu(t-\tau_i) \tag{5}$$

and

$$\begin{aligned} \dot{x}_m(t) &= Ax_m(t) + Bu(t) \\ z(t) &= \exp(-A\tau_i)\,x_m(t) - x_m(t-\tau_i). \end{aligned} \tag{6}$$

If (1) is unstable then, naturally, both (5) and (6) are unstable as well. However, in practice, one seldom deals with unstable plants allowed to function in open loop. More likely, an unstable plant is stabilized by a feedback controller which prevents the signals in the closed–loop system from an unlimited rise. Then, the FMDO can be used for implementation of the feedback control law and the instability of A is not an issue any more, since the closed–loop stability safeguards boundedness of all signals. Of course, all the three realizations of Ψ^{τ_i} yield the same transfer function from u to z. With respect to the observer complexity, (6) is preferable over (5), since all $\Psi^{\tau_i},\ i = 0,\ldots,k$ in (2) can be implemented using only one model of the plant.

There is a striking analogy between the observer (2) and an observer based on multiple derivatives of the plant input and output, a so–called ideal observer [3]. Provided all derivatives up to kth order of the input and output are available, the plant (1) can be parameterized as follows,

$$Y = \mathcal{O}x(t) + \mathcal{B}U, \tag{7}$$

where

$$\begin{aligned} Y &= \begin{pmatrix} y^T & \frac{\mathrm{d}y}{\mathrm{d}t}^T & \ldots & \frac{\mathrm{d}^k y}{\mathrm{d}t^k}^T \end{pmatrix}^T \\ U &= \begin{pmatrix} u^T & \frac{\mathrm{d}u}{\mathrm{d}t}^T & \ldots & \frac{\mathrm{d}^k u}{\mathrm{d}t^k}^T \end{pmatrix}^T \\ \mathcal{O} &= \begin{pmatrix} C^T & A^TC^T & \ldots & A^{k^T}C^T \end{pmatrix}^T \end{aligned}$$

and $\mathcal{B}$ is the lower block triangular Toeplitz matrix with the structure

$$\mathcal{B} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ CB & 0 & \ldots & 0 \\ CAB & CB & \ldots & 0 \\ \vdots & \vdots & \vdots & 0 \\ CA^{k-2}B & CA^{k-3}B & \ldots & 0 \end{pmatrix}.$$

For an observable pair (A, C) and $k = n-1$, the state vector $x(t)$ is given by the solution of (7),

$$\hat{x}(t) = (\mathcal{O}^T\mathcal{O})^{-1}\mathcal{O}^T(Y - \mathcal{B}U)$$

or, after block multiplications,

$$\begin{aligned} \hat{x}(t) &= \mathcal{V}^{-1}\sum_{i=0}^{n-1} A^{i^T} C^T \bar{Y}_i \\ \bar{Y}_i &= \frac{\mathrm{d}^i y}{\mathrm{d}t^i} - \sum_{j=0}^{i-1} CA^{i-j-1}B\frac{\mathrm{d}^j u}{\mathrm{d}t^j} \\ \mathcal{V} &= \sum_{i=0}^{n-1} A^{i^T} C^T C A^i. \end{aligned} \tag{8}$$

Clearly, (8) represents a deadbeat observer which guarantees zero estimation error for any time instant. Comparing (2) with (8) shows that they are closely related and are, in fact, two different representations of (1). Both of them directly exploit system observability obtaining the state estimate as the solution to a system of algebraic equations, with the difference that the delay operator is used in (2) instead of the differential operator in (8). Most likely, many other pseudodifferential operators can be used for the same purpose.

A geometric interpretation of (2) can also be suggested. Introduce the following notation

$$W = \begin{pmatrix} C\exp(-A\tau_0) \\ \vdots \\ C\exp(-A\tau_k) \end{pmatrix}; \quad Y = \begin{pmatrix} Y_0 \\ \vdots \\ Y_k \end{pmatrix}.$$

Let the linear independent vectors $w_i \in R^{\ell(k+1)}, i = 1, \dots, n$ be the columns of W

$$W = \begin{pmatrix} w_1 & \dots & w_n \end{pmatrix}.$$

It is easy to see that $\mathcal{W}_k$ is the matrix of scalar products of the vector set w_i, $i = 1, \dots, n$

$$\mathcal{W}_k = \begin{pmatrix} w_1^T w_1 & \dots & w_1^T w_n \\ \vdots & \ddots & \vdots \\ w_n^T w_1 & \dots & w_n^T w_n \end{pmatrix}.$$

As is shown in [7] an orthonormal set of w_i may be obtained by applying a nonsingular transformation T to the state vector of (1) so that $x_t = Tx$. Let the transformation be taken as

$$T = \Sigma^{1/2}U^T,$$

where the matrices Σ and U are those of the singular value decomposition of the symmetric positive definite matrix $\mathcal{W}_k$

$$\begin{aligned} \mathcal{W}_k &= U\Sigma U^T; \quad UU^T = I \\ \Sigma &= \mathrm{diag}(\sigma_1, \dots, \sigma_n), \quad \sigma_i > 0. \end{aligned}$$

Then, the gramian matrix of the transformed system is the unit matrix and the corresponding set of vectors w_i associated with the new state vector x_t is orthonormal. The system (1) becomes balanced wih respect to $\mathcal{W}_k$ and the deadbeat state estimate takes the form

$$\hat{x}_t = \begin{pmatrix} w_1^T Y \\ \vdots \\ w_n^T Y \end{pmatrix}.$$

The existence condition of the FMDO (2) formulated in Theorem 1 involves not only the parameters of the plant (1) but as well the design parameters of the observer itself. This obscures the answer to the question how general the observer is, i. e. whether or not it is possible to design an FMDO for any observable system (1). The following theorem shows that the existence of FMDO is guaranteed by observability of the plant.

Theorem 3. If the pair (A, C) is observable then any nonzero interval $I \in [0, \infty)$ contains a set of time delays

$$\tau_i, i = 0, 1, \ldots, k; \quad k \geq n - 1$$

such that $\mathrm{rank}(\mathcal{W}_k) = n$.

P r o o f. See the Appendix.

4. INFINITE MEMORY DEADBEAT OBSERVER

The continuous–time deadbeat phenomenon is not only restricted to finite–memory structures, but can also be accomplished in infinite–memory structures.

Theorem 4. Provided that the matrix $A_c = (A - KC)$ is Hurwitz and the matrix

$$\mathcal{U}_k = \sum_{i=0}^{k} \exp(-A_c^T \tau_i) C^T C \exp(-A_c \tau_i)$$

is positive definite, then the observer

$$\begin{aligned} \hat{x}(t) &= \bar{x}(t) + e_d(t) \\ \dot{\bar{x}}(t) &= A_c \bar{x}(t) + Bu(t) + Ky(t) \\ e_d(t) &= \mathcal{U}_k^{-1} \sum_{i=0}^{k} \exp(-A_c^T \tau_i)\, C^T y_\ell(t - \tau_i) \\ y_\ell(t) &= y(t) - C\bar{x}(t) \end{aligned} \tag{9}$$

has the following properties: (i) infinite memory; (ii) deadbeat performance in the sense that the estimation error $e(t) = x(t) - \hat{x}(t)$ is zero for all $t \geq \tau_k$; (iii) BIBO stability.

P r o o f . The first property follows immediately from the fact that the estimate $\hat{x}$ includes the Luenberger observer estimate $\bar{x}$ as an additive term.

The Luenberger observer's estimation error $\bar{e}(t) = x(t) - \bar{x}(t)$ satisfies the differential equation

$$\dot{\bar{e}}(t) = A_c \bar{e}(t) \tag{10}$$

and, apparently, can be deadbeatly reconstructed from the innovations signal $y_\ell(t)$. The FMDO for $\bar{e}(t)$ is given by the expression for $e_d(t)$ in (9). According to Theorem 1, $e_d(t) = \bar{e}(t)$ for all $t \geq \tau_k$, and the following relationship for the state estimation holds

$$\hat{x}(t) = \bar{x}(t) + e_d(t) = x(t); \quad t \geq \tau_k$$

that is the estimation error $e(t)$ vanishes deadbeatly, which proves (ii).

To verify the third property of the IMDO, note that it is BIBO stable if both the Luenberger observer for $\bar{x}$ and the FMDO estimating $\bar{e}$ are BIBO-stable. Since the matrix A_c is Hurwitz, the former is BIBO-stable by design. BIBO stability of the latter follows from the boundedness of $\bar{e}$ and the assertion (iii) of Theorem 1. □

Using the same notation as in Theorem 2, an upper bound for $\hat{x}$ can be obtained as follows.

Theorem 5. Suppose the initial estimation error is such that

$$|\bar{e}(0)| \leq m_3.$$

Then the following inequality holds for all t,

$$|\hat{x}(t)| \leq -\frac{M_3}{\mu(A_c)} + M_4 \sum_{i=0}^{k} e^{\mu(A_c)(t-\tau_i)}, \tag{11}$$

where

$$\begin{aligned} M_3 &= \|B\| m_2 + \|K\| m_1 \\ M_4 &= \|\mathcal{U}_k^{-1}\| \|C\| m_3. \end{aligned}$$

Strictly speaking, it is not necessary to impose the stability assumption on the matrix A_c since the FMDO is able to reconstruct the state vector of an unstable system, as well. However, it follows from (11) that an unstable A_c leads to violation of BIBO stability of the IMDO.

The following statement shows that the difference in the existence conditions of the FMDO and IMDO is only superficial.

Theorem 6. If the pair (A, C) is observable then any nonzero interval $I \in [0, \infty)$ contains a set of time delays

$$\tau_i, i = 0, 1, \dots, k; \quad k \geq n - 1$$

such that $\text{rank}(\mathcal{U}_k) = n$.

Proof. See Appendix.

Combining the results of Theorem 3 and Theorem 6, it becomes evident that both the FMDO and IMDO exist for any observable system (1) and their applicability area coincide with that of the Luenberger observer. In discrete time, a natural analogy of the FMDO is the FIR filter, while the IMDO is a counterpart of the discrete Luenberger observer with deadbeat performance. These associations help to achieve quite nice symmetry in continuous vs. discrete theory.

5. CONTINUOUS DEADBEAT OBSERVERS IN FEEDBACK CONTROL

In the previous sections we have shown that the FMDO and IMDO can be designed for any observable LTI system. In fact, as for example Corollary 1 indicates, the same approach is applicable to a more general class of LTI systems with time delay along the forward signal path. Bearing in mind the vast area of the Luenberger observer in control applications, it is an intriguing question whether the continuous deadbeat observers possess the same kind of potential when it comes to feedback control of LTI systems with delays.

Dealing with the control of time-delay systems, a natural design objective is to find a controller which in some sense excludes from the closed–loop system the impact that the delay has on the system behavior. Having a time delay in the input or output signal of the plant makes it necessary to use a predictor in order to enhance system performance. As early as in the fifties, the idea of excluding the time delay from the characteristic polynomial of the closed–loop system was implemented in the Smith Predictor.

Insofar as the Smith Predictor is structurally unstable whenever the plant is unstable, a predictor without this weak point has been introduced by Furukawa and Shimemura [2]. When the state vector cannot be measured directly, a Luenberger observer is used to obtain a state estimate, which is then fed into the predictor. Naturally, it takes two models of the system to implement this scheme – one for the observer and one for the predictor. Moreover, being predicted, the observer estimation error might cause undesirable transients in the closed–loop system.

Generally, all prediction schemes are implicitly or explicitly based on the Finite Spectrum Assignment Method by Olbrot [9], the purpose of which is to place an infinite number of eigenvalues of the plant at a finite number of prescribed points of the complex plane.

Consider the LTI system with delay in control

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t - \tau_0) \\ y(t) &= Cx(t). \end{aligned} \tag{12}$$

Theorem 7. Consider the system (12) and assume that the pair (A, C) is observable. Then there is a set of time delays $\tau_i > \tau_0; i = 1, \dots, k$ such that the observer

$$\begin{aligned} \dot{\bar{x}}(t+\tau_0) &= A\bar{x}(t+\tau_0) + Bu(t) \\ Z_i(t-\tau_0) &= y(t-\tau_i) - C\bar{x}(t-\tau_i) \\ \hat{e}_k(t) &= \mathcal{W}_k^{-1} \sum_{i=0}^{k} \exp(-A^T\tau_i) C^T Z_i(t) \\ \hat{x}(t+\tau_0) &= \bar{x}(t+\tau_0) + \hat{e}_k(t) \end{aligned} \tag{13}$$

is a Deadbeat Predictor (DP) of $x(t)$ in the sense that the prediction error $e(t) = x(t+\tau_0) - \hat{x}(t+\tau_0)$ is equal to zero for all $t \geq \max(\tau_i)$.

Proof. Combining (12) and (13) gives the differential equation governing the prediction error $\bar{e}(t) = x(t+\tau_0) - \bar{x}(t+\tau_0)$,

$$\begin{aligned} \dot{\bar{e}}(t) &= A\bar{e}(t) \\ Z_i(t) &= C\bar{e}(t-\tau_i). \end{aligned} \tag{14}$$

The predictor residual Z_0 is measurable and therefore the prediction error can be deadbeatly reconstructed by applying the result of Corollary 1. The resulting observer produces the estimate $\hat{e}_k$ in (13). Now, since $\hat{e}_k = e(t)$ for all $t \geq \tau_k = \max(\tau_i)$, it follows that

$$\begin{aligned} \hat{x}(t+\tau_0) &= \bar{x}(t+\tau_0) + \hat{e}_k(t) \\ &= \bar{x}(t+\tau_0) + e(t) = x(t+\tau_0). \end{aligned}$$

Taking advantage of the relaxation of the FMDO existence condition stated in Theorem 3 completes the proof. □

A natural application of the predictor above is the feedback control of systems with delay in control.

Theorem 8. If the pair (A, B) is controllable, then the closed–loop system comprising the plant (12) controlled by the feedback law

$$u(t) = r(t) + G\hat{x}(t+\tau_0)$$

possesses the following properties: (i) the transfer function from the reference input to the output is

$$\begin{aligned} y(s) &= C(sI - A_p)^{-1} B e^{-s\tau_0} r(s) \\ A_p &= A + BG \end{aligned} \tag{15}$$

(ii) the characteristic polynomial is

$$\det(sI - A_p)\det(sI - A) = 0 \tag{16}$$

(iii) the prediction error e has no effect on the plant state x for all $t \geq \max(\tau_i)$.

P r o o f. The closed–loop system equations are

$$\begin{aligned}\dot{x}(t) &= A_p x(t) + Br(t-\tau_0) + BG\delta(t)\\ \delta(t) &= \hat{e}_k(t-\tau_0) - e(t-\tau_0)\\ \hat{e}_k(t) &= \mathcal{W}_k^{-1}\sum_{i=0}^{k}\exp(-A^T\tau_i)\,C^T Ce(t-\tau_i)\\ \dot{e}(t) &= Ae(t).\end{aligned}$$

Consider the difference δ. Due to the result of Corollary 1 this difference vanishes for all $t \geq \max(\tau_i)$ which proves (iii).

Assuming zero initial conditions and taking the Laplace transform of the closed–loop system equations results in

$$\begin{pmatrix}x(s)\\ e(s)\end{pmatrix} = \begin{pmatrix}D_{11}(s) & D_{12}(s)\\ 0 & D_{22}(s)\end{pmatrix}^{-1}\begin{pmatrix}B\\ 0\end{pmatrix} r(s),$$

where

$$\begin{aligned}D_{11}(s) &= sI - A_c\\ D_{12}(s) &= -BG\big(S(s)-I\big)e^{-s\tau_0}\\ S(s) &= \mathcal{W}_k^{-1}\sum_{i=0}^{k}\exp(-A^T\tau_i)\,C^T Ce^{-s\tau_i}\\ D_{22}(s) &= sI - A.\end{aligned}$$

Taking into account that

$$y(s) = \begin{pmatrix}C & 0\end{pmatrix}\begin{pmatrix}x(s)\\ e(s)\end{pmatrix}$$

both (ii) and (i) follow immediately. □

As can easily be seen, the control law stated in Theorem 8 is a modification of the Smith Predictor, though with deadbeat performance in the predictor part. Indeed, the Smith Predictor for the plant(12) is given by

$$\dot{\hat{x}}_p(t+\tau) = A_p\hat{x}_p(t+\tau) + B_p u(t) \tag{17}$$

and its prediction error $\epsilon(t) = x_p(t+\tau) - \hat{x}_p(t+\tau)$ is governed by the differential equation

$$\dot{\epsilon}(t) = A\epsilon(t).$$

The feedback controller using both the output and output prediction is described in a state–space representation as

$$\begin{aligned}\dot{z}(t) &= Mz(t) + Gy_f(t)\\ y_f(t) &= r(t) - y(t) + C\hat{x}_p(t) - C\hat{x}_p(t+\tau)\\ u(t) &= Dz(t).\end{aligned} \tag{18}$$

where $z \in R^p$ is the controller state vector and M, G, D are real matrices. The closed–loop system (12), (18) has exactly the same transfer function as the system in Theorem 8 (i). However, the deadbeat performance of the predictor (13) makes a good deal of difference, since the prediction error influences the plant under a limited period of time, whereas in the case of the Smith Predictor the contribution of the prediction error subsides asymptotically. Furthermore, when the deadbeat predictor is used and an exact model of the plant is available, it appears that the prediction error is completely decoupled from the plant. Because of the finite–memory property of the FMDO estimating the prediction error, the transient response of the plant caused by $e(t),\ t \in [0, \max(\tau_i))$ can be attributed to initial conditions of the plant. The same decoupling property can be proved by demonstrating that

$$D_{12}e(s) = 0.$$

However, if the plant (12) is subject to unmodeled disturbances and is unstable, then, as can be easily concluded from the closed–loop equations, the variable $\bar{x}$ might be unbounded. To cure this problem, two approaches can be suggested. The first one is to exploit a dynamic model of the disturbance. Exactly in the same way as for the prediction error, the disturbance signal can be estimated from the observer residual and, after that, used in a control law to compensate for the disturbance contribution to the plant output. In more detail, though for systems without delay, this method is described in [6].

Another possible approach is to stabilize the model which simulates plant dynamics by means of a feedback from the residual signal. Note that a direct update of the estimate $\bar{x}$ by feeding back the weighted residual $K(y - C\hat{x})$ is feasible but complicated since it results in the error equation

$$\dot{e}(t) = Ae(t) - KCe(t - \tau_0).$$

Here K should be chosen so that the system is asymptotically stable. A possibility to develop an effective technique enabling such design is rather vague not least due to the fact that A could be unstable and KC is singular.

Theorem 9. Consider the system (12), and assume that the pair (A, C) is observable. Then there is a set of time delays $\tau_i > \tau_0; i = 1, \ldots, k$ such that the observer

$$\begin{aligned} \dot{\tilde{x}}(t + \tau_0) &= A\tilde{x}(t + \tau_0) + Bu(t) + K\tilde{e}_k(t) \\ Z_i(t - \tau_0) &= y(t - \tau_i) - C\tilde{x}(t - \tau_i) \\ \tilde{e}_k(t) &= \mathcal{W}_k^{-1} \sum_{i=0}^{k} \exp(-A_o^T\tau_i)\, C^T Z_i(t), \end{aligned} \tag{19}$$

where

$$\begin{aligned} A_o &= A - K \\ \mathcal{W}_k &= \sum_{i=0}^{k} \exp(-A_o^T\tau_i)\, C^T C \exp(-A_o\tau_i) \end{aligned}$$

is an Asymptotic Predictor (AP) of $x(t)$ in the sense that the prediction error $e(t) = x(t+\tau_0) - \tilde{x}(t+\tau_0)$ tends to zero as $t \to \infty$ at the same decay rate as the system $\dot{e}(t) = A_o e(t)$ does.

Proof. The prediction error for the observer (19) is governed by the differential equation

$$\begin{aligned} \dot{e}(t) &= Ae(t) - K\tilde{e}_k(t) \\ \tilde{e}_k(t) &= \mathcal{W}_k^{-1} \sum_{i=0}^{k} \exp(-A_o^T \tau_i)\, C^T C e(t-\tau_i). \end{aligned} \tag{20}$$

Compare now the equation above to the autonomous system

$$\dot{e}(t) = (A-K)e(t).$$

Assuming that the initial function $\phi_e = [e(t), -\max(\tau_i) \leq t \leq 0]$ belongs to a trajectory of the latter systems which is always true in this case, brings us to the conclusion that the two system are equivalent. □

The closed–loop properties of the AP are summarized in the following assertion.

Theorem 10. If the pair (A, B) is controllable, then the closed–loop system comprising the plant (12) and the controller

$$u(t) = r(t) + G\tilde{x}(t+\tau_0)$$

possesses the following properties: (i) the transfer function from the reference input to the output is

$$y(s) = C(sI - A_p)^{-1} B e^{-s\tau_0} r(s)$$

(ii) the characteristic polynomial is

$$\det(sI - A_p) \det(sI - A_o) = 0.$$

Proof. The closed–loop system equation is

$$\dot{x}(t) = A_p x(t) + Br(t-\tau_0) - BGe(t-\tau_0), \tag{21}$$

where e is given by (20). Since the differential equation describing the prediction error is autonomous, the term related to e has no influence on the reference signal transfer function. Thus, taking the Laplace transform of the closed–loop system equation yields (i). By the same reason, the characteristic polynomial of the closed–loop system is the product of the characteristic polynomial of the observer (19) and the characteristic polynomial of the controlled plant (21), as stated in (ii). □

In comparison with the DP, the AP does not include the original modes of the plant and provides an arbitrarily fast convergence rate of its prediction error. The

AP-based controller is very much akin to conventional Luenberger observer–based controllers, especially when it comes to design issues.

It is worth noting, that both the DP and AP render an arbitrary predefined finite spectrum to the infinite-dimensional plant (12) and reduce the controller design problem to that of an ordinary LTI system. Naturally, letting $\tau_0 = 0$ does not violate any assumption made in Theorem 7 or Theorem 9 which means that they are valid for LTI systems without delay as well. In this case, the predictors take the form of respectively a deadbeat and an asymptotic observer and they are still feasible for conventional feedback control.

6. NUMERICAL EXAMPLE

To exemplify the application of Continuous Deadbeat Observers to feedback control, we consider a simple numerical example. The state differential equation of the system to be simulated is

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ b \end{pmatrix} u(t-\tau_0) \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t), \end{aligned} \tag{22}$$

where the following numerical values are used

$$a = 4.6; \quad b = 0.787; \quad \tau_0 = 0.1.$$

First we apply the observer structure (13) to the state vector prediction problem for the plant (22). To minimize transient response time, the time delays for the predictor are chosen as $\tau_0 = 0.1; \tau_1 = 0.15$. This means that the prediction error residual is to be fed into the FMDO undelayed and delayed by $\tau_1 - \tau_0$. Thus, the prediction error vanishes for all $t > 0.15$, as can be seen in Figure 1.

Fig. 1. Continuous Deadbeat Predictor. The predicted values $\hat{x}(t+\tau_0)$ are delayed for τ_0 to facilitate comparison with the corresponding state variables of the plant.

Furthermore, the closed–loop performance of the DP is investigated. The plant (22) is controlled by the feedback law

$$\begin{aligned} u(t) &= r(t) + G\hat{x}(t+\tau_0) \\ r(t) &= 5\sin(5t). \end{aligned}$$

The gain matrix $G = \begin{pmatrix} -11.4358 & -1.7789 \end{pmatrix}$ places two closed–loop system poles at $s_{12} = -3$. Figure 2 shows the responses of the closed–loop system reference signal and the system without delay in the control loop. The model is given by

$$\dot{x}_m(t) = (A + BG)\, x_m(t) + Br(t).$$

Inspection of Figure 2 shows no difference in the dynamics of the model and the closed–loop system after the deadbeat time has expired, short of the constant time delay τ_0, which perfectly agrees with the theory presented in the previous section. Note also that the closed–loop system does not inherit the deadbeat performance of the predictor, and its transient response settling time is also defined by the eigenvalues of $(A + BG)$.

Fig. 2. Continuous Deadbeat Predictor. Closed–loop system reference signal response.

Consider now the implementation of predictive control through the AP (19). Assume also that the poles of the prediction error equation are to be placed at $s_1 = -1$, $s_2 = -2$. This can be achieved by the observer gain matrix

$$K = \begin{pmatrix} -1.6 & 0 \\ 9.36 & 0 \end{pmatrix}.$$

The observer prediction error is shown in Figure 3. Clearly, the transient response due to initial conditions is identical with that of the model in the form of ordinary differential equations.

Fig. 3. Asymptotic Predictor. The prediction error e is given in comparison with the corresponding state variables of the model $\dot{e}_m(t) = (A - K)e_m(t)$.

In the same manner as the DP's, the AP's estimate is used in the controller

$$u(t) = r(t) + G\tilde{x}(t + \tau_0).$$

Fig. 4. Asymptotic Predictor. Closed–loop system reference signal response.

Results of a simulation run are presented in Figure 4, and they are quite similar to those obtained for the closed–loop system based on the Deadbeat Predictor. Actually, this can be expected, taking into account that the closed–loop transfer functions are equal in the both cases. The differences at the initial stage of the transient response are explained by dissimilar processing of the initial conditions in the corresponding structures.

7. CONCLUSIONS

A deadbeat observation problem is posed for continuous linear multivariable systems. Two BIBO-stable deadbeat observers possessing respectively finite and infinite process memory are discussed.

As an application of the introduced continuous deadbeat paradigm, a predictive control problem for systems with time delay along the forward path is solved generalizing the Smith Predictor and ordinary Luenberger observer–based feedback control.

8. APPENDIX

Proof of Theorem 2. Assume that both u and y, the inputs to the observer, are bounded

$$|y| \leq m_1; |u| \leq m_2$$

then for each t

$$\begin{aligned} |Y_i(t)| &\leq |y(t-\tau_i)| + \|C\| \, |(\Psi^{\tau_i} u)(t)| \\ &\leq m_1 + \|C\| \, |(\Psi^{\tau_i} u)(t)|. \end{aligned}$$

To evaluate the second term on the right–hand side of the above inequality, consider the norm of the integrand in Ψ^{τ_i}. After the change of variables $\theta = t - \xi$ we get

$$(\Psi^{\tau_i} u)(t) = \int_0^{\tau_i} \exp(A(\theta - \tau_i)) \, Bu(t-\theta) \, \mathrm{d}\theta.$$

Using the properties of matrix measure

$$|\exp(A(\theta - \tau_i)) \, Bu(t-\theta)| \leq \|B\| \, m_2 e^{\mu(A)(\theta - \tau_i)},$$

where $\|\cdot\|$ and $\mu(\cdot)$ are respectively the matrix norm and matrix measure induced by the vector norm $|\cdot|$. Suppose now that $\mu(A) \neq 0$, then integrating both sides of the inequality over the interval $[0, \tau_i]$ yields

$$|(\Psi^{\tau_i} u)(t)| \leq \frac{\|B\| m_2}{\mu(A)} \left(1 - e^{-\mu(A)\tau_i}\right). \tag{23}$$

For the case $\mu(A) = 0$, the upper bound is given by

$$|(\Psi^{\tau_i} u)(t)| \leq \lim_{\mu(A) \to 0} \frac{\|B\| m_2}{\mu(A)} \left(1 - e^{-\mu(A)\tau_i}\right).$$

Application of l'Hospital's rule immediately brings us to the result

$$|(\Psi^{\tau_i} u)(t)| \leq \|B\| m_2 \tau_i. \tag{24}$$

Observing that

$$|\exp(-A^T\tau_i)\,C^T Y_i| \le \|C\||Y_i|e^{-\mu(A)\tau_i}$$

and taking into account (23) and (24) we arrive to the upper bound (4). □

P r o o f of Theorem 3. First note that the gramian matrix $\mathcal{W}_k$ can be factorized as

$$\mathcal{W}_k = W_k^T W_k,$$

where W_k is a block matrix of the form

$$W_k = \begin{pmatrix} C\exp(-A\tau_0) \\ \vdots \\ C\exp(-A\tau_k) \end{pmatrix}.$$

Therefore the condition $\det(\mathcal{W}_k) \neq 0$ is equivalent to $\operatorname{rank}(W_k) = n$.

Following [10], let the characteristic polynomial of A be

$$D(s) = s^n - (p_1 + p_2 s + \cdots + p_n s^{n-1}).$$

Define the auxiliary polynomials

$$D^{(j)}(s) = s^{n-j} - (p_{j+1} + p_{j+2}s + \cdots + p_n s^{n-j-1}).$$

Then if $\mathcal{G}$ is a closed contour enclosing all eigenvalues of A, we have

$$\begin{aligned} \exp(At) &= \sum_{j=1}^{n} \phi_j(t)\, A^{j-1} \\ \phi_j(t) &= \frac{1}{2\pi i}\oint_{\mathcal{G}} \frac{D^{(j)}(\theta)}{D(\theta)} e^{t\theta}\,\mathrm{d}\theta. \end{aligned} \tag{25}$$

Using the definitions above, we write the matrix W_k as

$$W_k = (\Phi_{k,n} \otimes I_\ell)P_o(A, C), \tag{26}$$

where $P_o(A, C)$ is the observability matrix of (A, C),

$$P_o^T(A, C) = \begin{pmatrix} C^T & A^T C^T & \ldots & A^{n-1\,T} C^T \end{pmatrix},$$

I_ℓ is the unit matrix of dimension ℓ, $\otimes$ denotes the Kronecker product, and

$$\Phi_{k,n} = \begin{pmatrix} \phi_1(-\tau_0) & \ldots & \phi_n(-\tau_0) \\ \vdots & \ddots & \vdots \\ \phi_1(-\tau_k) & \ldots & \phi_n(-\tau_k) \end{pmatrix}.$$

Consider the case when $k + 1 = n$ and the matrix $\Phi_{k,n}$ is square. We now show that there is always a set of τ_i, $i = 0, \ldots, k$ on I such that the matrix $\Phi_{k,n}$ is nonsingular. Assume the contrary, i. e. that the matrix $\Phi_{k,n}$ has linearly dependent rows for all possible values of τ_i in I. Letting the rows i and j be linearly dependent, it then follows from (25) that

$$\exp(-A\tau_i) = \rho \exp(-A\tau_j)$$

for all τ_i, τ_j and some real constant ρ, which brings us to the conclusion that $\rho = 1$, $A = 0$. This contradicts the assumptions made on (1) and therefore the matrix $\Phi_{k,n}$ is proved to be nonsingular. Due to the observability condition $\operatorname{rank}(P_o) = n$. Since

$$\det(\Phi_{k,n} \otimes I_\ell) = \det(\Phi_{k,n})^\ell$$

and

$$\det(\Phi_{k,n}) \neq 0$$

equation (26) implies that

$$\operatorname{rank}(W_k) = n.$$

In the case $k + 1 > n$, by the same reason as above, the matrix $\Phi_{n,k}$ can always be partitioned as

$$\Phi_{n,k} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix},$$

where $\Phi_1 \in R^{n\times n}$ and $\det(\Phi_1) \neq 0$. Substituting the partitioned $\Phi_{n,k}$ in (26) yields the desired result

$$\operatorname{rank}(W_k) = n$$

and the proof is completed. □

P r o o f of Theorem 5. To begin with, one can observe that

$$|\bar{x}(t)| \leq \int_0^\infty \|\exp(A_c\theta)\| \,\mathrm{d}\theta M_3$$

or using the properties of matrix measure,

$$|\bar{x}(t)| \leq \int_0^\infty e^{\mu(A_c)\theta} \,\mathrm{d}\theta M_3.$$

For any Hurwitz A_c the improper integral on the right–hand side of the inequality converges, giving

$$|\bar{x}(t)| \leq -\frac{1}{\mu(A_c)} M_3.$$

Further, the upper bound on the solution of (10) is

$$|\bar{e}(t)| \leq m_3 e^{\mu(A_c)t}.$$

Taking into account the finite memory used in the estimate e_d and (4) yields

$$|e_d(t)| \leq M_4 \sum_{i=0}^{k} e^{\mu(A_c)(t-\tau_i)}.$$

Summarizing the partial results above provides (11). □

Proof of Theorem 6. Along the lines of Example 3.3-5 in [3] but reasoning for the dual case, the relationship between the observability matrix of the pair (A, C) and the observability matrix of the pair (A_c, C) is given as follows

$$P_o(A,C) = \mathcal{D}P_o(A_c,C), \tag{27}$$

where

$$\mathcal{D} = \begin{pmatrix} I & 0 & 0 & \dots & 0 \\ CK & I & 0 & \dots & 0 \\ CAK & CK & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{n-2}K & CA^{n-3}K & CA^{n-4}K & \dots & I \end{pmatrix}.$$

This result can be easily checked out by substituting $A = A_c + KC$ in $P_o(A,C)$. Because of the unit matrices on the main diagonal it is clear that $\det \mathcal{D} \neq 0$ and

$$\text{rank} P_o(A,C) = \text{rank} P_o(A_c,C)$$

In other words, the estimation error of the Luenberger observer $\bar{e}$ is observable from the residual y_ℓ iff the pair (A, C) is observable. Carrying out the same argument as in the proof of Theorem 3 results in the conclusion that for any observable pair (A_c, C) there is a set of time delays $\tau_i,\ i = 0,\dots,k;\ k \geq n-1$ such that $\mathcal{U}_k$ is nonsingular. Noting that application of (27) equates observability of (A, C) with that of (A_c, C), completes the proof. □

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