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EVALUATION OF THE REACHABILITY SUBSPACE OF GENERAL FORM POLYNOMIAL MATRIX DESCRIPTIONS (PMDs)

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We consider the concept of Reachability for systems described by PMDs, generalizing various known results from the theory of generalized state space systems using time domain analysis,which takes into account the finite and infinite pole-zero structure of the associated matrix. We extend also the theory of admissible initial conditions and we introduce the concept of Reachable subspace for PMDs providing a precise form for all future(reachable) states of our system.

1. INTRODUCTION

Let a multivariable system described by a Polynomial Matrix Description (PMDs) i. e. systems of the form $\sum$:

$$\begin{aligned} A(\rho)\beta(t) &= B(\rho)\,u(t) \\ y(t) &= C(\rho)\,\beta(t), \end{aligned} \tag{1}$$

where $\rho := \frac{\mathrm{d}}{\mathrm{d}t}$ is the differential operator, $A(\rho) = \sum\limits_{i=0}^{q_1} A_i\rho^i \in \Re^{r\times r}\,[\rho]$, $A_i \epsilon\, \Re^{r\times r}$, $i = 0,1,2,\ldots,\ q_1 \geq 1$ with $\mathrm{rank}_{\Re} A_{q_1} < r$, $B(\rho) = \sum\limits_{i=0}^{\sigma} B_i\rho^i \in \Re^{r\times m}[\rho]$, $B_j \in \Re^{r\times m}$, $j = 0,1,2,\ldots,\sigma \geq 0$, $C(\rho) = \sum\limits_{i=0}^{\sigma_1} C_i\rho^i \in \Re^{m_1\times r}[\rho]$, $C_j \in \Re^{m_1\times r}$, $j = 0,1,2,\ldots,\sigma_1 \geq 0$, $\beta(t) : (0^-,\infty) \to \Re^r$ the *pseudo-state* of the system $(\sum)$ and $u(t) : [0,\infty) \to \Re^m$ the control input to the system $(\sum)$. Polynomial Matrix Descriptions are governed by singular differential equations which endow the systems with many special features that are not found in regular state space systems. Among these are impulse terms and input derivatives in the free and forced pseudo-state responce, nonproperness of the transfer function matrix, noncausality between input and pseudo-state (or input and output), inconsistent and admissible initial conditions and many others which make the study of PMDs more complicated than the study of the classical regular systems. Starting from the fact that generalized state space systems i. e. systems of the form $\sum_1$: $E\rho x(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$, where $E \in \Re^{r\times r}$,

$\text{rank}_{\mathbf{R}} E < r$, $A \in \Re^{r\times r}$, $B \in \Re^{r\times m}, C \in \Re^{m_1\times r}$ represent a particular case of PMDs, we generalize various known results regarding the smooth and impulsive solutions of the homogeneous and the non-homogeneous system $(\sum_1)$ to the more general case of PMDs $(\sum)$. In recent papers (see [10, 9, 6]) various known results regarding the smooth and impulsive solutions of homogeneous generalized state space systems have been translated to the more general case of PMDs. Also relying heavily on the theory regarding the Smith–McMillan form of a rational matrix at infinity and applying it to the polynomial matrix $A(s) = L_{-}[A(\rho)]$ the theory of Weierstrass canonical form of a regular matrix pencil $Es - A$ under strict equivalence to the more general case of polynomial matrix $A(s)$ was generalized [9].

2. MAIN RESULTS

Theorem 1. [9] Let

$$A(s) = A_0 + A_1 s + \ldots + A_{q_1} s^{q_1} \in \Re^{r\times r}[s] \tag{2}$$

$\text{rank}_{\Re(s)} A(s) = r$, $q_1 \geq 1$ with Smith–McMillan form at $s = \infty$ given by $\mathbf{S}^{\infty}_{A(s)}(s) = \text{block diag}\left[s^{q_1}, \mathbf{s}^{q_2}, \ldots, \mathbf{s}^{q_k}, \frac{1}{s^{\widehat{q}_{k+1}}}, \ldots, \frac{1}{s^{\widehat{q}_r}}\right]$, where $1 \leq k \leq r$ and $\widehat{q} = -\widehat{q}_i$, $i = k+1, \ldots, r$ so that $q_1 \geq q_2 \geq \ldots \geq q_k \geq 0$ and $\widehat{q}_r \geq \widehat{q}_{r-1} \geq \ldots \geq \widehat{q}_{k+1} \geq 0$.We can write: $\mathbf{A}^{-1}(\mathbf{s}) = \mathbf{H}_{\text{pol}}(\mathbf{s}) + \mathbf{H}_{\text{spr}}(\mathbf{s})$, where $\text{H}_{\text{pol}}(s) \in \Re^{r\times r}[s]$ and $H_{\text{spr}}(s) \in \Re^{r\times r}_{pr}(s)$ is strictly proper. Let $n = \deg |A(s)|$. Then $n = \delta_M(H_{\text{spr}}(s))$. Let $\mu = \sum_{i=k+1}^{r}(\widehat{q}_i + 1)$. Then $\delta_M(H_{\text{pol}}(s)) = \mu$. Now let $C \in \Re^{r\times n}$, $J \in \Re^{n\times n}$, $B \in \Re^{n\times r}$ be a minimal realization of $H_{\text{spr}}(s)$ and $C_\infty \in \Re^{r\times\mu}, J_\infty \in \Re^{\mu\times\mu}, B_\infty \in \Re^{\mu\times r}$ be a minimal realization of $H_{\text{pol}}(s)$. Then C, J is a finite Jordan pair of A(s) and C_∞, J_∞ is an infinite Jordan pair of $A(s)$. Furthermore $A^{-1}(s)$ can be written:

$$\mathbf{A}^{-1}(\mathbf{s}) = \begin{bmatrix} C & C_\infty \end{bmatrix} \left[\begin{array}{c|c} sI_n - J & 0_{n,\mu} \\ \hline 0_{\mu,n} & I_\mu - sJ_\infty \end{array}\right]^{-1} \begin{bmatrix} B \\ B_\infty \end{bmatrix}. \tag{3}$$

The solution of the homogeneous matrix differential equation $A(\rho)\,\beta(t) = 0$ is found to be [9]:

$$\beta^{h}(t) = [C\ C_\infty] \begin{bmatrix} e^{Jt} x_s(0^-) \\ -\sum_{i=1}^{\widehat{q}_r} \delta^{(i-1)} J_\infty^i x_f(0^-) \end{bmatrix}, \tag{4}$$

where

$$x_f(0^-) := [B_\infty, J_\infty B_\infty, \ldots, J_\infty^{q_1-1} B_\infty] \begin{bmatrix} A_0 & A_1 & \cdots & A_{q_1-1} \\ 0 & A_0 & \cdots & A_{q_1-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix} \begin{bmatrix} \beta(0^-) \\ \beta^{(1)}(0^-) \\ \vdots \\ \beta^{(q_1-1)}(0^-) \end{bmatrix} \in \Re^\mu \tag{5}$$

and

$$x_s(0^-) := [J^{q_1-1}B, J^{q_1-2}B, \ldots, B] \begin{bmatrix} A_{q_1} & 0 & \cdots & 0 \\ A_{q_1-1} & A_{q_1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_1 & 0 & \cdots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0^-) \\ \beta^{(1)}(0^-) \\ \vdots \\ \beta^{(q_1-1)}(0^-) \end{bmatrix} \in \Re^n \tag{6}$$

$x_s(0^-)$ is the "slow state at $t = 0^-$" and $x_f(0^-)$ is the "fast state at $t = 0^-$ ([9]).

Consider the PMD (1). Now we shall present the solution of a non-homogeneous matrix differential equation:

$$A(\rho)\,\beta(t) = B(\rho)\,u(t). \tag{7}$$

Taking the L_- Laplace transform of (7) and assuming that the initial conditions are zero i.e. $\beta^{(i)}(0^-) \equiv 0$, $i = 0, 1, \ldots, q_1 - 1$, $u^{(i)}(0^-) \equiv 0$, $i = 0, 1, \ldots, \sigma - 1$, we obtain:

$$A(s)\,\hat{\beta}(s) = B(s)\,\hat{u}(s). \tag{8}$$

Hence in light of (3) we can write:

$$A^{-1}(s)\,B(s) = C_\infty [I_\mu - sJ_\infty]^{-1} B_\infty\, B(s) + C[sI_n - J]^{-1} BB(s) \tag{9}$$

which after some matrix manipulations [9] can be written:

$$A(s)^{-1}B(s) = [C \;\; C_\infty] \begin{bmatrix} J^{\sigma-1}B, J^{\sigma-2}B, \ldots, B & 0_{n,(\widehat{q}_r+1)r} \\ 0_{\mu,\sigma r} & B_\infty, J_\infty B_\infty, \ldots, J_\infty^{\widehat{q}_r} B_\infty \end{bmatrix} \tag{10}$$

$$\times \begin{bmatrix} B_\sigma & 0 & \cdots & 0 & 0 & & \cdots & 0 \\ B_{\sigma-1} & B_\sigma & \cdots & 0 & 0 & & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & & \vdots \\ B_1 & B_2 & \cdots & B_\sigma & 0 & & \cdots & 0 \\ B_0 & B_1 & \cdots & B_{\sigma-1} & B_\sigma & & \cdots & 0 \\ 0 & B_0 & \cdots & B_{\sigma-2} & B_{\sigma-1} & B_\sigma & \cdots & 0 \\ \vdots & \vdots & & & & & \ddots & \vdots \\ 0 \cdots & & B_0 & \cdots & & & & B_\sigma \end{bmatrix} \begin{bmatrix} I_m \\ sI_m \\ \vdots \\ s^{\widehat{q}_r+\sigma} I_m \end{bmatrix}$$

$$+C[sI_n - J]^{-1} [J^\sigma BB_\sigma + J^{\sigma-1}BB_{\sigma-1} + \cdots + BB_0].$$

Taking the inverse Laplace transform of (10) and in light of (8) we obtain the solution of (7) [7]:

$$\beta^n(t) = [C \;\; C_\infty] \begin{bmatrix} \displaystyle\int_0^t e^{Jt}\Omega u(\tau)\mathrm{d}\tau + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u^{(i)}(t) \\ \displaystyle\sum_{i=0}^{\widehat{q}_r} J_\infty^i \overline{\Omega} u^{(\sigma+i)}(t) + \sum_{i=0}^{\sigma-1} Z_i u^{(i)}(t) \end{bmatrix}, \tag{11}$$

where the superscript (i) means distributional derivative, σ is the maximum power of s in $B(s)$ and

$$\Omega = \sum_{i=0}^{\sigma} J^i B B_i = J^\sigma B B_\sigma + J^{\sigma-1} B B_{\sigma-1} + \cdots + B B_0 \tag{12}$$

$$\Phi_j = \sum_{i=0}^{\sigma-j} J^i B B_{i+j} \qquad j = 1, 2, \ldots, \sigma \tag{13}$$

$$\overline{\Omega} = \sum_{i=0}^{\sigma} J^i B_\infty B_{(\sigma-i)} = B_\infty B_\sigma + J_\infty B_\infty B_{\sigma-1} + \cdots + J_\infty^\sigma B_\infty B_0 \tag{14}$$

$$Z_{(\sigma-j)} = \sum_{i=0}^{\sigma} J_\infty^i B_\infty B_{(\sigma-j)-i} \qquad j = 1, 2, \ldots, \sigma \tag{15}$$

with $B_{(\sigma-j)-i} \equiv 0$ for $i, j : (\sigma - j) - i < 0$.

We obtain that the *complete solution* of (1) is given by:

$$\beta^c(t) = \beta^h(t) + \beta^n(t) = \tag{16}$$

$$[C\ C_\infty] \begin{bmatrix} e^{Jt} x_s(0^-) + \int_0^t e^{Jt} \Omega u(\tau) \mathrm{d}\tau + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u^{(i)}(t) \\ -\sum_{i=1}^{\widehat{q}_r} \delta^{(i-1)} J_\infty^i x_f(0^-) + \sum_{i=0}^{\widehat{q}_r} J_\infty^i \overline{\Omega} u^{(\sigma+i)}(t) + \sum_{i=0}^{\sigma-1} Z_i u^{(i)}(t) \end{bmatrix},$$

where the superscript (i) means distributional derivative. Let us now denote $u^{[i]}(t)$ the ith (ordinary) derivative of $u(t)$. Using the identity (see [1] p. 52)

$$u^{(i)}(t) = u^{[i]}(t) + \delta u^{[i-1]}(0) + \cdots + \delta^{[i-1]} u(0) \qquad i = 1, 2, \ldots \tag{17}$$

$\beta^c(t)$ can be written (see [7]) as $\beta^c(t) = \beta_1^c(t) + \beta_2^c(t)$ where:

$$\begin{aligned} \beta_1^c(t) = C \Bigg[& e^{Jt} x_s(0^-) + \int_0^t e^{Jt} \Omega u(\tau) \mathrm{d}\tau + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u^{[i]}(t) \\ & + \sum_{i=0}^{\sigma-2} \delta^{[i]} \left[\sum_{j=0}^{\sigma-2-i} \Phi_{j+2+i} u^{[j]}(0^-) \right] \Bigg] \end{aligned} \tag{18}$$

$$\begin{aligned}
\beta_2^c(t) = & -C_\infty \sum_{i=1}^{\widehat{q}_r} \delta^{(i-1)} J_\infty^i x_f(0^-) \\
& + C_\infty \left[\sum_{i=0}^{\sigma-1} Z_i u^{[i]}(t) + \sum_{i=0}^{\sigma-2} \delta^{[i]} \left[\sum_{j=0}^{\sigma-2-i} Z_{j+1+i} u^{[j]}(0^-) \right] \right] \\
& + C_\infty \left[\sum_{i=0}^{\widehat{q}_r} J_\infty^i \overline{\Omega} u^{[\sigma+i]}(t) + \sum_{i=0}^{\sigma-1} \delta^{(i)} \left[\sum_{j=0}^{\widehat{q}_r} J_\infty^j \overline{\Omega} u^{[\sigma+j-i-1]}(0^-) \right] \right. \\
& \left. + \sum_{i=\sigma}^{\sigma+\widehat{q}_r-1} \delta^{(i)} \left[\sum_{j=i-(\sigma-1)}^{\widehat{q}_r} J_\infty^j \overline{\Omega} u^{[j-1]}(0^-) \right] \right].
\end{aligned} \tag{19}$$

It is obvious that the complete solution of (1) may have impulsive components. Since discontinous (impulsive) behaviour is not desirable we have the following:

Definition 2. A point $\beta_0^c \equiv \beta^c(0^-) \in \Re^r$ is said to be an Admissible Initial Condition (A.I.C.) for the system (1) if the solution $\beta^c(t;\, 0^-, \beta_0^c,\, u(t))$ is continously differentiable on $[0, T]$ for some input $u(t)$ and for some $T > 0$, i. e. $\beta^c(t;\, 0^-, \beta_0^c,\, u(t))$ is impulse-free.

It follows from (18) and (19) that a point β_0^c is an A.I.C. if the following conditions hold:

$$C_\infty \sum_{i=1}^{\widehat{q}_r} \delta^{(i-1)} J_\infty^i x_f(0^-) = 0 \Rightarrow x_f(0^-) \in \mathrm{Ker}[J_\infty] \tag{20}$$

$$\sum_{j=0}^{\sigma-2-i} \Phi_{j+2+i} u^{[j]}(0^-) = 0 \qquad i = 0, 1, \ldots, \sigma-2 \tag{21}$$

$$\sum_{j=0}^{\sigma-2-i} Z_{j+1+i} u^{[j]}(0^-) = 0 \qquad i = 0, 1, \ldots, \sigma-2 \tag{22}$$

$$\sum_{j=0}^{\widehat{q}_r} J_\infty^j \overline{\Omega} u^{[\sigma+j-i-1]}(0^-) = 0 \qquad i = 0, 1, \ldots, \sigma-1 \tag{23}$$

$$\sum_{j=i-(\sigma-1)}^{\widehat{q}_r} J_\infty^j \overline{\Omega} u^{[j-1]}(0^-) = 0 \qquad i = \sigma, \ldots, \sigma+\widehat{q}_r-1. \tag{24}$$

The set of Admissible states for $t \geq 0^-$ is given by:

$$\beta^c(t) = [C\ C_\infty] \begin{bmatrix} e^{Jt}x_s(0^-) + \int_0^t e^{Jt}\Omega u(\tau)\mathrm{d}\tau + \sum_{i=0}^{\sigma-1} \Phi_{i+1}u^{[i]}(t) \\ \sum_{i=0}^{\widehat{q}_r} J_\infty^i \overline{\Omega} u^{[\sigma+i]}(t) + \sum_{i=0}^{\sigma-1} Z_i u^{[i]}(t) \end{bmatrix}. \tag{25}$$

From (25) for $t = 0^-$ the set of A.I.C. is:

$$H_{Iu} = \left\{ \beta^c(0^-) \in \Re^r / \beta^c(0^-) = [C\ C_\infty] \begin{bmatrix} x_s(0^-) + \sum_{i=0}^{\sigma-1} \Phi_{i+1}u^{[i]}(0^-) \\ \sum_{i=0}^{\widehat{q}_r} J_\infty^i \overline{\Omega} u^{[\sigma+i]}(0^-) + \sum_{i=0}^{\sigma-1} Z_i u^{[i]}(0^-) \end{bmatrix} \right\} \tag{26}$$

or equivalently:

$$H_{Iu} = \left\{ \beta^c(0^-) = [C\ C_\infty] \begin{bmatrix} x_s^c(0^-) \\ x_f^c(0^-) \end{bmatrix} / x_s^c(0^-) \in \Re^n, \right. \\ \left. \text{and } x_f^c(0^-) \in \sum_{i=0}^{\widehat{q}_r} J_\infty^i \operatorname{Im} \overline{\Omega} + \sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i + \operatorname{Ker} J_\infty \right\}. \tag{27}$$

Remark 3. Note that the zero vector 0 belongs to H_{Iu} because there exist $x_s(0^-) \equiv 0$ and input $u(t)$ such that $u^{[i]}(0^-) \equiv 0$ for $i = 0, 1, 2, \ldots, \widehat{q}_r$ or $i = 0, 1, 2, \ldots, \sigma - 2$ in the case $\sigma - 2 > \widehat{q}_r$.

Now we shall generalize the notions of Reachability given in [8, 11] in such a way to cover the general case of PMDs as in (1).

Definition 4. Given a point $\beta_0^c = \beta^c(0^-) \in H_{Iu}$, we say that another point $\beta_T \in \Re^r$ is *Reachable* from β_0^c if there exists an input $u(t)$ and $T > 0$ such that $\beta^c(t) = \beta^c(t; 0^-, \beta_0^c, u(t))$ is impulse-free on $[0^-, T]$ and holds:

$$\beta^c(T) = \beta_T. \tag{28}$$

Let $R(\beta_0^c)$ denote the set of Reachable states from $\beta_0^c \in H_{I_u}$. $R(\beta_0^c) \neq \emptyset$ means that there exists an input which will make the solution $\beta^c(t)$ impulse-free on $[0, T]$. We shall try to describe $R(\beta_0^c)$ in terms of its finite and infinite spectral data i. e. the finite Jordan triple (C, J, B) and the infinite Jordan triple $(C_\infty, J_\infty, B_\infty)$ of the matrix $A(s)$.

We firstly assume that $\beta_0^c = 0 \in H_{I_u}$ and describe the set $R(0)$ i. e. the set of Reachable states from $0 \in H_{I_u}$. We introduce the following notation (see also [11]):

$$\langle A \,/\, \operatorname{Im} B\rangle := \operatorname{Im} B + A \operatorname{Im} B + \cdots + A^{n-1} \operatorname{Im} B. \tag{29}$$

Following the lines of [11] we can prove that:

Theorem 5.

$$R(0) = [C\ C_\infty] \begin{bmatrix} \langle J \,/\, \operatorname{Im}\Omega\rangle + \sum_{i=0}^{\sigma-1} \operatorname{Im}\Phi_{i+1} \\ \langle J_\infty \,/\, \operatorname{Im}\overline{\Omega}\rangle + \sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i \end{bmatrix},$$

where Z_i, $i = 0, 1, \dots \sigma - 1$ is given by (15) and Φ_j, $j = 1, \dots, \sigma$ is given by (13).

In the above theorem we have examined the structure of $R(0)$. We shall now examine the structure of $R(\beta_1)$ with $\beta_1 \equiv \beta^c(0^-) \neq 0 \in \Re^r$. To this end consider the following two sets of *admissible initial conditions* (taken from (27)):

i) A.I.C. with $x_s^c(0^-) = 0 \in \Re^n$ and $x_f^c(0^-) \neq 0$ i. e.

$$H_2 = \left\{ \beta^c(0^-) = [C\ C_\infty] \begin{bmatrix} x_s^c(0^-) \\ x_f^c(0^-) \end{bmatrix} / x_s^c(0^-) = 0 \in \Re^n,\ x_f^c(0^-) \in \sum_{i=0}^{\widehat{q}_r} J_\infty^i \operatorname{Im}\overline{\Omega} \right.$$
$$\left. + \sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i + \operatorname{Ker} J_\infty \right\} \tag{30}$$

and

ii) A.I.C. with $x_s^c(0^-) \neq 0$ and $x_f^c(0^-) = 0$ i. e.

$$H_3 = \left\{ \beta^c(0^-) = [C\ C_\infty] \begin{bmatrix} x_s^c(0^-) \\ x_f^c(0^-) \end{bmatrix} / x_s^c(0^-) \neq 0 \in \Re^n \text{ and } x_f^c(0^-) = 0 \in \Re^\mu \right\}. \tag{31}$$

The complete set of A.I.C. can be written:

$$H_{Iu} = \left\{ \beta^c(0^-) \in \Re^r / \ \beta^c(0^-) \subset [C\ C_\infty] \begin{bmatrix} \Re^n \\ \langle J_\infty \,/\, \operatorname{Im}\overline{\Omega}\rangle + \sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i \end{bmatrix} \right\} \tag{32}$$

or equivalently from (30)–(31) and Remark 3:

$$H_{Iu} = H_2 \cup H_3 \cup \{0\}, \tag{33}$$

where $\{0\}$ denotes the zero vector corresponding to $x_s(0^-) \equiv 0$ and to an input $u(t)$ such that $u^{[i]}(0^-) \equiv 0,\ i = 1, 2, \dots, \widehat{q}_r + 1 + \sigma$. Now the complete set of Reachable states $\beta^c(T) \in \Re^r$ from $\beta \in H_{I_u}$ is:

$$\widetilde{R} = \bigcup_{\beta \in H_{Iu}} R(\beta) = R(0) \cup R(\beta_2) \cup R(\beta_3), \tag{34}$$

where $R(0)$ is the set of Reachable states from $0 \in H_{I_u}$, $R(\beta_2)$ is the set of Reachable states from $\beta_2 \in H_2$, i.e. from all A.I.C. which have $x_s^c(0^-) = 0$ and $x_f^c(0^-) \neq 0$ and

$$R(\beta_3) = \left\{ \beta_3(t) \in H_3 / \beta_3(t) = [C\ C_\infty] \begin{bmatrix} x_s^c(t) \\ x_f^c(t) \end{bmatrix} / x_s^c(t) = e^{Jt} x_s(0^-) \right. \\ \left. + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u^{[i]}(t) \in \Re^n,\ x_f(t) \equiv 0\ \in \Re^\mu\ \forall t > 0 \right\} \tag{35}$$

which represents the free-state reachable set from starting point(state) $\beta_3(0^-) = Cx_s(0^-) + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u^{[i]}(0^-)$. From Theorem 5 we have the form of $R(0)$. From the form of $R(\beta_3)$ in (35) we have:

$$R(\beta_3) \in [C\ C_\infty]\,[\Re^n \oplus \{0\}]. \tag{36}$$

Hence it remains only to find $R(\beta_2)$ where $\beta_2 \in H_2$.We can easily prove that:

Proposition 6. Let $\beta_2 \in H_2$ as in (30). Then:

$$R(\beta_2) = [C\ C_\infty] \begin{bmatrix} \Re^n \\ \left\langle J_\infty \,/\, \operatorname{Im} \overline{\Omega} \right\rangle + \sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i \end{bmatrix}. \tag{37}$$

Taking into account that $\langle J \,/\, \operatorname{Im} \Omega \rangle + \sum_{i=0}^{\sigma-1} \operatorname{Im} \Phi_{i+1} \subset \Re^n$ and $\{0\} \subset \left\langle J_\infty \,/\, \operatorname{Im} \overline{\Omega} \right\rangle + \sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i$ from (34) and Theorem 5, (36) and (37) we obtain that the complete set of Reachable states from any $\beta \in H_{Iu}$ is given by:

$$\widetilde{R} = \bigcup_{\beta \in H_{Iu}} R(\beta) = [C\ C_\infty] \begin{bmatrix} \Re^n \\ \left\langle J_\infty \,/\, \operatorname{Im} \overline{\Omega} \right\rangle + \sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i \end{bmatrix}. \tag{38}$$

Remark 7. Taking into account that $\langle J \,/\, \operatorname{Im} \Omega \rangle + \sum_{i=0}^{\sigma-1} \operatorname{Im} \Phi_{i+1} \subset \Re^n$ we obtain that every point y in R, where:

$$R := [C\ C_\infty] \begin{bmatrix} \langle J \,/\, \operatorname{Im} \Omega \rangle + \sum_{i=0}^{\sigma-1} \operatorname{Im} \Phi_{i+1} \\ \left\langle J_\infty \,/\, \operatorname{Im} \overline{\Omega} \right\rangle + \sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i \end{bmatrix} \tag{39}$$

is Reachable (according to Definition 4) from every point x in R.

We have the following definition:

Definition 8. The system (1) is called Reachable if every point $\beta_T \in \Re^r$ is Reachable from every point $\beta_0 \in H_{Iu}$.

Proposition 9. The system (1) is Reachable iff: $R = \Re^r$.

Definition 10. The set R as in (39) is called the *Reachable subspace* of the system (1).

Now we shall give some useful Reachability tests for Polynomial Matrix Descriptions which are natural extensions of the corresponding tests for generalized state space systems. Let the subspace $R_s := \langle J \,/\, \mathrm{Im}\,\Omega\rangle + \sum_{i=0}^{\sigma-1} \mathrm{Im}\,\Phi_{i+1} \subset \Re^n$. R_s is spanned by the linearly independent columns of the matrix:

$$Q_s = [\Omega,\ J\Omega, \dots, J^{n-1}\Omega,\ \Phi_1,\ \Phi_2, \dots, \Phi_\sigma] \in \Re^{n\times(n+\sigma)m}. \tag{40}$$

Let also the subspace $R_f := \left\langle J_\infty \,/\, \mathrm{Im}\overline{\Omega}\right\rangle + \sum_{i=0}^{\sigma-1} \mathrm{Im}\, Z_i \subset \Re^\mu$. R_f is spanned by the linearly independent columns of the matrix:

$$Q_f = [\overline{\Omega},\ J_\infty\overline{\Omega}, \dots, J_\infty^{\widehat{q}_r}\overline{\Omega},\ Z_0,\ Z_1, \dots, Z_{\sigma-1}] \in \Re^{\mu\times(\widehat{q}_r+1+\sigma)m}. \tag{41}$$

From the form of R in (39) and (40)−(41) it follows:

Definition 11. The Reachable subspace R is spanned by the linearly independent columns of the matrix

$$Q = [C\ C_\infty] \begin{bmatrix} Q_s & 0 \\ 0 & Q_f \end{bmatrix} \in \Re^{r\times(n+\widehat{q}_r+1+2\sigma)m} \tag{42}$$

which is called *pseudo-state Reachability matrix* of (1).

Combining (42) with Proposition 9 we can state the obvious:

Theorem 12. Every $\beta_T \in \Re^r$ is Reachable iff:

$$R \equiv \Re^r \Rightarrow \mathrm{rank}[Q] = r. \tag{43}$$

Remark 13. We have the following:

$$[C\ C_\infty] \in \Re^{r\times(n+\mu)} \quad \text{and} \quad \mathrm{rank}\,[C\ C_\infty] = r \tag{44}$$

$$n + \mu = r + \sum_{i=1}^{k}(q_i - 1). \tag{45}$$

Hence generally it holds:

$$n + \mu > r. \tag{46}$$

From Theorem 12 and Remark 13 we can state the following:

Corollary 14. The system (1) is Reachable iff:

$$\operatorname{rank}[C\ C_\infty] = r \tag{47}$$

and

$$\operatorname{rank}\begin{bmatrix} Q_s & 0 \\ 0 & Q_f \end{bmatrix} \geq r. \tag{48}$$

3. ILLUSTRATIVE EXAMPLE

Let $A(s) = \begin{bmatrix} s+1 & s^2 \\ 0 & 1 \end{bmatrix}$ be a polynomial matrix with Smith–McMillan form at $s = \infty : S^{\infty}_{A(s)}(s) = \begin{bmatrix} s^2 & 0 \\ 0 & \frac{1}{s} \end{bmatrix}$ and $r = 2$, $n = 1$, $\mu = 2$; hence $n + \mu = 1 + 2 = 3 > 2 = r$. Let also $C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $J = [-1]$, $B = [1 -1]$ a minimal realization of the strictly proper part of $A^{-1}(s)$ and $C_\infty = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, $J_\infty = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B_\infty = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ a minimal realization of the polynomial part of $A^{-1}(s)$. Then

$$\operatorname{rank}[C\ C_\infty] = \operatorname{rank}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 2 = r.$$

Hence the first condition (47) of Corollary 14 holds true.

CASE A. Let $B(s) = B_0 + B_1 s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s = \begin{bmatrix} s+1 & 0 \\ 0 & s+1 \end{bmatrix}$ i. e. $\sigma = 1$. Then:

$$\Omega = JBB_1 + BB_0 = [0,0], \qquad \Phi_1 = BB_1 = [1,-1]$$

$$\overline{\Omega} = B_\infty B_1 + J_\infty B_\infty B_0 = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}, \qquad Z_0 = B_\infty B_0 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

i) $\operatorname{rank}[Q_s] = \operatorname{rank}[\Omega, \Phi_1] = \operatorname{rank}[0,0,1,-1] = 1$

ii) $\operatorname{rank}[Q_f] = \operatorname{rank}\left[\overline{\Omega}, J_\infty\overline{\Omega}, Z_0\right] = \operatorname{rank}\begin{bmatrix} 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix} = 2$

iii) $\operatorname{rank}\begin{bmatrix} Q_s & 0 \\ 0 & Q_f \end{bmatrix} = \operatorname{rank}\left[\begin{array}{cccc|cccccc} 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \end{array}\right] = 3 > r$

i. e. the system is *Reachable* according to Corollary 14.

CASE B. Let $B(s) = B_0 + B_1 s = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} s = \begin{bmatrix} s \\ 0 \end{bmatrix}$ i. e. $\sigma = 1$. Then:

$$\Omega = JBB_1 + BB_0 = [-1], \quad \Phi_1 = BB_1 = [1]$$

$$\overline{\Omega} = B_\infty B_1 + J_\infty B_\infty B_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad Z_0 = B_\infty B_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i) $\text{rank}[Q_s] = \text{rank}\,[\Omega, \Phi_1] = \text{rank}\,[-1, 1] = 1$

ii) $\text{rank}\,[Q_f] = \text{rank}\,[\overline{\Omega}, J_\infty \overline{\Omega}, Z_0] = \text{rank} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

iii) $\text{rank} \begin{bmatrix} Q_s & 0 \\ 0 & Q_f \end{bmatrix} = \text{rank} \left[\begin{array}{cc|ccc} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] = 1 < r$

i. e. the system is *Not Reachable* because the condition (48) does not hold.

4. CONCLUSIONS

The concept of Reachability for Polynomial Matrix Descriptions (PMDs) is considered. After generalizing various known results regarding the smooth and impulsive solutions of generalized state space systems (which represent a particular case of PMDs) we developed a theory regarding Reachability properties of PMDs using time-domain analysis. This analysis extends in a general way a number of results previously known only for regular and generalized state space systems. Finally we have to point out that our definition of Reachability is equivalent and natural generalization of the notions of Controllability [2], C-Controllability [11] and Reachability [8]. However, the way that our theory is related to further aspects such as the notions of Strong Controllability, Observability and duality for the case of PMDs are topics for further research.

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