KYBERNETIKA — VOLUME *30* (1994), NUMBER 3, PAGES 307 – 318

A GENERAL BOUNDED CONTINUOUS MOMENT PROBLEM AND ITS SETS OF UNIQUENESS

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Consider a compact metrizable space X and a countable set $B \subset C(X)$. Write $P = Q[\mod B]$ for a pair of Borel probability measures P and Q defined on X if $P(f) = Q(f)$ for each $f \in B$. A moment (countable bounded continuous) problem promoted as the decomposition of $\mathbb{P}(X)$ (the set of all Radon probability measures on X) by the above equivalence will be treated here. A representation of such a decomposition by a compact convex set is to be constructed with the aim to establish a mathematical setting which would be operational when trying to identify a pair of moment problems, to construct "a big moment problem" as an inverse limit of "small moment problems" and finally to characterize its compact sets of uniqueness.

Some of the ideas employed here come back to [5, 1, 2].

The results we present here are available also for the "bounded countable" moment problems defined in a similar way by a countable set B of Borel measurable functions defined on a Souslin space X. These results will be published elsewhere as the proofs require a space consuming effort.

1. CONVEX COMPACT REPRESENTATIONS

Having X a topological space we shall denote by $\mathbb{P}(X)$ and $C(X)$ the set of all Radon probability measures and the set of all continuous real functions defined on X, respectively. A triple (X, T, E) will be called a *generator* of a moment problem if

$$\begin{aligned} &X \text{ is a nonempty compact metrizable topological space,} \\ &E \text{ is a complete locally convex space} \end{aligned} \tag{1}$$

and

$$T : X \to E \text{ is a continuous map.} \tag{2}$$

Denote by $(bT) : \mathbb{P}(X) \to E$ the map which assignes to each $P \in \mathbb{P}(X)$ the barycentrum $b(TP) \in E$ of the image measure $TP \in \mathbb{P}(TX)$. Moreover let

$$S(X, T, E) = (bT)(\mathbb{P}(X)) = \{s \in E : s = (bT)\,(P) \text{ for some } P \in \mathbb{P}(X)\}$$

for a generator (X, T, E).

Having X a compact metrizable space, a decomposition M of $\mathbb{P}(X)$ will be called a (bounded continuous) *moment problem* if

$$M = \{(bT)^{-1}(s), \quad s \in S(X, T, E)\}$$

for some generator (X, T, E). In such a case we shall write $M = M(X, T, E)$ and call the set $S(X, T, E)$ a *compact convex representation of the moment problem M*. Finally, we denote by $\mathbb{M}$ the class of all bounded continuous moment problems and by $\mathbb{M}(X)$ the moment problems in $\mathbb{M}$ that are supported by a compact metrizable set X. (We will present an identification of moment problems in $\mathbb{M}$ with those defined in our abstract at the end of this Section. All concepts are illustrated in Section 3 of the present paper).

Recall that a point $bP \in E$ is called the *barycenter* of a measure $P \in \mathbb{P}(E)$ if

$$E' \subset L_1(P) \quad \text{and} \quad x'(bP) = \int_E x' \mathrm{d}P$$

for all $x' \in E'$, where E' denotes the topological dual to E. Now, the correctness of our definition of the map $(bT) : \mathbb{P}(X) \to E$ will follow easily by the following arguments:

$$TX \text{ is a compact metrizable subset of } E. \tag{3}$$

(Proposition 7.6.3, p. 126 in [3].)

$$\begin{aligned} &T : \mathbb{P}(X) \to \mathbb{P}(TX) \text{ is a continuous surjection w.r.t.} \\ &\text{the corresponding weak topologies in } \mathbb{P}(X) \text{ and } \mathbb{P}(TX), \text{ respectively} \end{aligned} \tag{4}$$

(Theorem 12, p. 39 in [4]), if we agree to keep further on the symbol T to denote also the image measure map $P \to TP$ from $\mathbb{P}(X)$ into the set of all Borel probability measures defined on X.

$$b : \mathbb{P}(TX) \to E \text{ is a correctly defined continuous affine map} \tag{5}$$

(Proposition 1.1.3, p.16 in [7]), thus

$$\begin{aligned} &(bT) = b \circ T \text{ is a continuous affine surjection of } \mathbb{P}(X) \\ &\text{onto } S(X, T, E) \text{ for each generator } (X, T, E). \end{aligned} \tag{6}$$

Theorem 1. Let $M(X, T, E) \in \mathbb{M}$. Then $S = S(X, T, E)$ is a compact convex metrizable set in E such that $S = \overline{\mathrm{co}}(TX)$ and $\mathrm{ex}S \subset TX$.

P r o o f . The set S is a continuous affine image of the compact metrizable convex set $\mathbb{P}(X)$ by (6), hence a compact convex metrizable set by Proposition 7.6.3, p. 126 in [3]. A standard argument using the theorem on the separation of a pair of compact convex sets by a hyperplane shows that $S \subset \overline{\text{co}}(TX)$. The rest of Theorem 1 follows easily by Krein–Milman theorem. □

Compact convex representations of moment problems in $\mathbb{M}(X)$ (X a fixed space) may serve when trying to establish relations as

$$M_1 = M_2 \text{ or } M_1 \geq M_2, \text{i. e. } M_1 \text{ is a finer decomposition than } M_2, \quad M_1, M_2 \in \mathbb{M}(X).$$

Theorem 2. Let $M_1 = M(X, T_1, E_1)$ and $M_2 = M(X, T_2, E_2)$ are moment problems in $\mathbb{M}(X)$. Then the following statements are equivalent:

(a) $M_1 \geq M_2$.

(b) There exists a continuous affine surjection $a : S(X, T_1, E_1) \to S(X, T_2, E_2)$ such that $a \circ (bT_1) = (bT_2)$ on $\mathbb{P}(X)$.

(c) There exists a continuous map $a : S(X, T_1, E_1) \to S(X, T_2, E_2)$ such that $a \circ T_1 = T_2$ on X and

$$b(aP) = a(bP) \text{ holds for each measure } P \in \mathbb{P}(T_1 X). \tag{7}$$

Remark that the requirement $a \circ T_1 = T_2$ in (c) is exactly as to say that the decomposition of X into the stalks $\{T_1^{-1}(s), s \in E_1\}$ is finer than the decomposition $\{T_2^{-1}(s), s \in E_2\}$. The condition (7) is a legitimate one by Corrolary 1.2.3 in [7], p. 23, which implies that both barycentra exist and are contained in $S_2(X, T_2, E_2)$ and $S(X, T_1, E_1)$, respectively.

P r o o f. Denote $S_i = S(X, T_i, E_i)$ for $i = 1, 2$.

(a) ⇒ (b): As $M_1 \geq M_2$, there exists a uniquelly determined map $a : S_1 \to S_2$ such that $a \circ (bT_1) = (bT_2)$ holds on $\mathbb{P}(X)$. The map is obviously both surjective and affine because the maps (bT_i) enjoy the properties according to (6). Further, (bT_1) is a quotient map as a continuous surjection of the compact set $\mathbb{P}(X)$ onto the compact set S_1 by (6) and 7.5.1 in [3], p. 122. Thus a is continuous by the definition of the quotient map, again by (6) applied to the map (bT_2).

(b) ⇒(c): Consider a map $a : S_1 \to S_2$ satisfying (b). As $(bT_i)(\varepsilon_x) = T_i(x)$ holds for each $x \in X$ (where ε_x denotes the point measure supported by x) the equation $a \circ T_1 = T_2$ follows easily. To verify (7) for a $P \in \mathbb{P}(T_1 X)$ we have to establish that

$$\int_S (x' \circ a)(s) P(\mathrm{d}s) = (x' \circ a)\,(bP)$$

holds for all $x' \in E_1'$. But, this follows immediately by Proposition 23.1.6 in [3], p. 402, as $x' \circ a$ is a continuous affine real function defined on the compact convex set S_1.

(c) $\Rightarrow$(a): Consider a map $a : S_1 \to S_2$ satisfying the requirements of (c). Take $P, Q \in I\!P(X)$ such that $(bT_1)(P) = (bT_1)(Q)$. As the image measures T_1P and T_1Q are in $I\!P(T_1X)$ it follows from (7) that

$$(bT_2)(P) = b((a \circ T_1)(P)) = a(b(T_1P)) = a(b(T_1Q)) = b((a \circ T_1)(Q)) = (bT_2)(Q).$$

Hence $M_1 \geq M_2$ and the proof is completed. □

Corollary 1. Generators (X, T_1, E_1) and (X, T_2, E_2) provide the same moment problem $M = M(X, T_1, E_1) = M(X, T_2, E_2)$ in $I\!M(X)$ if and only if

(a) $\{T_1^{-1}(s), s \in E_1\} = \{T_2^{-1}(s), s \in E_2\}$ (T_1 and T_2 define the same decomposition of X) and

(b) the map $a : T_1X \to T_2X$ uniquelly defined by $a \circ T_1 = T_2$ can be extended to a continuous affine bijection between $S(X, T_1, E_1)$ and $S(X, T_2, E_2)$.

Thus all compact convex representations of a given moment problem are isomorphic in the category of compact convex metrizable sets.

A compact set $D \subset X$ will be called a *set of uniqueness for a moment problem* $M \in I\!M(X)$ if each member of the decomposition M contains at most one measure $P \in I\!P(X)$ supported by the set D (i.e. $P \in I\!P(D)$).

Choquet theory provides a completely algebraic characterization of sets of uniqueness. Recall that a compact metrizable convex set S in a locally convex space E is called a *simplex*, or Choquet simplex, if the cone $C = R^+(S \times \{1\}) \subset E \times R$ is a lattice w.r.t. the ordering $\succeq$ defined on C by

$$c_1 \succeq c_2 \ \ iff \ \ c_1 - c_2 \in C,$$

(see [7], p. 47 or [3], p. 417).

Theorem 3. Consider $M = M(X, T, E) \in I\!M(X)$ and $D \subset X$ a compact set. Then D is a set of uniqueness for the moment problem M if and only if

(a) The map T restricted to D is an injection X into E

and

(b) $S(D) = S(D,\ T\,|\,D,\ E) = \overline{co}(TD)$ is a simplex with $exS(D) = TD$.

P r o o f. Note that

$$D \text{ is a set of uniqueness iff } (bT) : I\!P(D) \to S(D) \text{ is a bijective map} \tag{8}$$

and since $S(D)$ is a compact convex metrizable set by Theorem 1, we get using Choquet uniqueness theorem (23.6.5 in [3], p. 420) that

$$b : I\!P(\text{ex}S(D)) \to S(D) \text{ is a bijection iff } S(D) \text{ is a simplex.} \tag{9}$$

Since $(bT)\,|\,P(D) = b\circ(T\,|\,D)$ we verify that (a), (b) imply D to be a set of uniqueness simply using (8), (9). On the other hand, if D is such a set we get the validity of (a) as a consequence of (8) $[(bT)(\varepsilon_x) = T(x),\ x \in D]$. Using (8) once more we can see that $(bT) : I\!\!P(D) \to S(D)$ is an *afinne* bijection, hence $\mathrm{ex}S(D) = (bT)(I\!\!P(D)) = TD$. Now, a simple combination of (8), (a) and (9) may be used to prove the rest of (b).
□

Corollary 2. Let M be a moment problem generated by (X, T, E) where the locally convex space E has a finite dimension n. Then $D \subset X$ is a set of uniqueness for M iff T restricted to D is an injection and TD is a set of affinelly independent points in E. Hence if D is a set of uniqueness then $\mathrm{card}D \le n+1$.

To derive our Corollary from Theorem 3 note that the algebraic definition of a Choquet simplex we have referred before to generalizes that of a finite dimensional simplex (the convex hull of a set of affinely independent points in E, (see [7], p. 52-3)).

An obvious way how to construct a moment problem in $M(X)$ is suggested by our abstract:

Consider a subset $B \subset C(X)$ (not necessarily countable) and define a generator

$$(X, T^B, R^B) \text{ by } T^B(x) = (g(x), g \in B) \text{ for } x \in X, \tag{10}$$

where $E = R^B$ is topologized by its (locally convex) product topology. It is easy to see that

$$M(X, T^B, R^B) \text{ is a moment problem in } I\!\!M(X) \text{ for each } B \subset C(X). \tag{11}$$

It follows from the definition of the barycenter observing that the topological dual of R^B is linearly generated by the projections of R^B onto R that

$$\begin{aligned}&\text{the decomposition } M(X, T^B, R^B) \text{ is defined by the equivalence relation}\\ &P = Q[\mod B](\text{ iff } P(g) = Q(g) \text{ for each } g \in B,\ P, Q \in I\!\!P(X)).\end{aligned} \tag{12}$$

On the other hand we have

Theorem 4. For any $M \in I\!\!M(X)$ there is a *countable* $B \subset C(X)$ such that $M = M(X, T^B, R^B)$.

Hence, our proclamation made in abstract that we will treat countable bounded continuous moment problems is thus justified.

P r o o f . Choose an arbitrary generator of M, say $M = M(X, T, E)$. As $S(X, T, E)$ is a compact metrizable convex set (by Theorem 1) it follows from Propositions 3 and 4 in [4], p.104-5, that there is a sequence $\{x_i'\} \subset E'$ which separates points in $S(X, T, E)$. Hence $P = Q[\mod B]$ iff $TP(x_i') = TQ(x_i'),\ i \in N$, iff $x_i'(b(TP)) =$

$x_i'(b(TQ))$, $i \in N$, iff $(bT)(P) = (bT)(Q)$ holds for all $P, Q \in \mathbb{P}(X)$, where $B = \{x_i'(T), i \in N\} \subset C(X)$. Thus $M = M(X, T^B, R^B)$ by (12) and the proof is finished. □

In this setting, Theorem 3 may be complemented as follows:

Corollary 3. Let $B \subset C(X)$ and $D \subset X$ is a compact set. Then

(a) D is a set of uniqueness for $M(X, T^B, R^B)$,

(b) $C(D) = \{a[(g, g \in B)], a : \overline{co}(T^B D) \to R$ a continuous affine real function$\}$,

(c) $\mathcal{L}((B \mid D) \cup \{1\})$ is a dense set in $C(D)$,

are equivalent statements. ($\mathcal{L}$ denotes the linear hull operator).

P r o o f. (The equivalence of (a) and (b) was proved in [5].)

Put $T_1 = T^{(B|D)}$, $T_2 = T^{C(D)}$, $E_1 = R^{(B|D)}$, $E_2 = R^{C(D)}$, denote $S = \overline{co}(T^B D)$ and observe that $S = S(D, T_1, E_1)$ by Theorem 1. Now, if D is a set of uniqueness for $M(X, T^B, R^B)$ then both $M(D, T_1, E_1)$ and $M(D, T_2, E_2)$ are identical decompositions of $P(D)$ (into the singletons). Hence, by Theorem 2 (b), there exists an affine continuous map $A : S \to E_2$ that maps $T_1(D) = \{(g(x), g \in B), x \in D\}$ onto $T_2(D) = \{(f(x), f \in C(D)), x \in D\}$. Thus, given an $f \in C(D)$, $a = \mathrm{pr}_f \circ A$ is the real continuous affine function defined on S such that $a[(g, g \in B)] = f$. Hence (a) $\Rightarrow$ (b).

The implication (b) $\Rightarrow$ (c) follows easily as $\mathcal{L}(E_1' \cup \{1\})$ is a uniformly dense set in the set of all affine continuous functions defined on S (Proposition 23.1.6 in [3], p. 402), and $E_1' = \mathcal{L}(\mathrm{pr}_g, g \in B)$. The implication (c) $\Rightarrow$ (a) is obvious. □

2. INVERSE LIMITS OF MOMENT PROBLEMS

Let us consider the class $\mathbb{M}$ of moment problems defined by (1) and (2) as a *category* where q is a *morphism* from $M_1 \in \mathbb{M}(X_1)$ to $M_2 \in \mathbb{M}(X_2)$ iff

$$q : X_1 \to X_2 \text{ is a continuous } \textit{surjection} \text{ and } q^{-1}M_2 \le M_1.$$

(the symbol q denotes both the map from X_1 onto X_2 and the image measure map $P \to qP$ from $P(X_1)$ onto $P(X_2)$, thus $q^{-1}M_2$ is a decomposition of $P(X_1)$).

We write $q : M_1 \to M_2$ if q is a morphism from M_1 to M_2 and observe that $q_1 : M_1 \to M_2$, $q_2 : M_2 \to M_3 \Rightarrow q_2 \circ q_1 : M_1 \to M_3$, hence the usual composition of maps defines the composition law for the category $\mathbb{M}$, (see [3], p. 160). It is easy to see that

$$\begin{aligned}
&\text{a continuous surjection } q : X_1 \to X_2 \text{ is a morphism}\\
&\text{from } M_1 = M(X_1, T_1, E_1) \text{ to } M_2 = M(X_2, T_2, E_2)\\
&\text{iff } M(X_1, T_1, E_1) \ge M(X_1, T_2 \circ q, E_2),\\
&\text{i.e. iff } (bT_1)(P) = (bT_1)(Q) \Rightarrow (bT_2)(qP) = (bT_2)(qQ) \text{ for } P, Q \in \mathbb{P}(X_1).
\end{aligned} \tag{13}$$

Also observe that

$$\begin{aligned}&\text{a map } q: X_1 \to X_2 \text{ is an isomorphism of } M_1 \in I\!M(X_1) \text{ and}\\ &M_2 \in I\!M(X_2) \text{ iff } q \text{ is a homeomorphic bijection such that } qM_1 = M_2,\\ &\text{the last identity being equivalent to } M(X_1, T_1, E_1) = M(X_1, T_2 \circ q, E_2)\\ &\text{if } (X_i, T_i, E_i) \text{ is a generator of } M_i \text{ for } i = 1, 2.\end{aligned} \tag{14}$$

Recall that $[M_i, i \in I,\ q_{ij}, i \geq j, i, j \in I]$ is called an *inverse system* in $I\!M$ if $(M_i)_{i \in I} \subset I\!M$ is a net and $q_{ij} : M_i \to M_j,\ i \geq j,\ i, j \in I$ are morphisms such that

$$q_{ik} = q_{jk} \circ q_{ij} \text{ if } i \geq j \geq k, \quad i, j, k \in I. \tag{15}$$

Also recall that $[M_\infty, p_i,\ i \in I]$ is an *inverse limit* of an inverse system $[M_i,\ q_{ij}]$ if

$$M_\infty \in I\!M \text{ (i.e. } M \in I\!M(X) \text{ for some nonempty compact metrizable } X), \tag{16}$$

$$p_i \text{ are morphisms consistent with the family } (q_{ij}), \text{ i.e. } p_i = q_{ij} \circ p_i \text{ holds for } i \geq j \text{ and} \tag{17}$$

$$M_0 \in I\!M,\ u_i : M_0 \to M_i,\ u_i \text{ consistent with } (q_{ij}) \Rightarrow \Rightarrow \text{there exists a } \textit{unique} \text{ morphism } u : M_0 \to M_\infty \text{ such that } p_i \circ u = u_i \text{ for } i \geq j. \tag{18}$$

It is a well known fact (Theorem 11.6.2 in [3], p. 204) that if an inverse limit exists, it is unique up to a unique commuting isomorphism (see (14)). We write $[M_\infty, p_i] \in \lim_{\hookleftarrow}[M_i, q_{ij}]$ if (16), (17), (18) hold.

Theorem 4 yields a very helpful functor from the category $I\!M$ to the category $\mathcal{C}$ of compact metrizable convex sets with continuous affine maps as morphisms having made before a particular choice of generators (X, T, E) for each $M \in I\!M$: If this is the case then the functor

maps $M(X, T, E) \in I\!M$ to $S(X, T, E) \in \mathcal{C}$ (we write $M(X, T, E) \to S(X, T, E)$) and

maps $q : M(X_1, T_1, E_1) \to M(X_2, T_2, E_2)$ to $a : S(X_1, T_1, E_1) \to S(X_2, T_2, E_2)$ (we write $q \to a$), where the map a is uniquely determined by

$$a \circ (bT_1)(P) = (b(T_2 \circ q))(P), \quad P \in I\!P(X). \tag{19}$$

This perhaps needs some explanation: because $M(X_1, T_1, E_1) \geq M(X_2, T_2 \circ q, E_2)$ and $S(X_1, T_2 \circ q, E_2) = S(X_2, T_2, E_2)$ (q is a surjection X_1 onto X_2!) we may use the equivalent definition (b) in Theorem 2 to construct a map a satisfying (19). The uniqueness is due to the fact that both (bT_1) and $(b(T_2 \circ q))$ are surjections. It is easy to see that

$$q_1 \to a_1,\ q_2 \to a_2 \ \Rightarrow\ q_2 \circ q_1 \to a_1 \circ a_2$$

if the morphism $q_1 \circ q_2$ is defined. Thus each inverse system $[M(X_i, T_i, E_i), q_{ij}]$ in $I\!M$ promotes an inverse system $[S(X_i, T_i, E_i), a_{ij}]$ in $\mathcal{C}$ such that the morphisms are in correspondence $q_{ij} \to a_{ij}$ established by (19).

An obvious candidate for an inverse limit to an inverse system $[M(X_i, T_i, E_i), q_{ij}]$ is $[M(X_\infty, T_\infty, E_\infty), p_i]$, where $E_\infty = \prod E_i$ (the product topology), $p_i : \prod X_j \to X_i$ is the ith coordinate projection,

$$\begin{aligned} &X_\infty = \{(x_i)_{i\in I} \in \textstyle\prod X_i :\ q_{ij}(x_i) = x_j,\ i \geq j\} \\ &\text{(the set of all } (q_{ij})\text{-threads) and } T_\infty : X_\infty \to E_\infty \\ &\text{is the map defined by } T_\infty[(x_i)_{i\in I}] = (T_i(x_i))_{i\in I} \text{ for } (x_i)_I \in x_\infty. \end{aligned} \tag{20}$$

Observe that $[X_\infty, p_i]$ is the topological inverse limit of the inverse system $[X_i, q_{ij}]$ in the category of compact sets, hence X_∞ is a *nonempty compact set* and each $p_i : X_\infty \to X_i$ is a *surjection* by Proposition 11.8.5 in [3], p. 212. Assuming moreover that the net I is countably generated (i. e. there is a countable cofinal subset $J \subset I$) we get X_∞ to be a *metrizable compact set* as a continuous image of $X'_\infty = \{(x_j)_J \in \prod_J X_j : q_{ik}(x_i) = x_k, i \geq k,\ i, k \in J\}$, which is a compact subset in R^N. To see that $[M(X_\infty, T_\infty, E_\infty), p_i]$ is a legitimate candidate for an inverse limit it remains to show that $M[X_\infty, T_i \circ p_i, E_i) \preceq M(X_\infty, T_\infty, E_\infty)$ for each $i \in I$. But it presents no problem as it is easy to verify directly from the definition of barycenter in E_∞ that

$$\pi_i[(bT_\infty)\,(P)] = (b(T_i \circ p_i))\,(P) \tag{21}$$

holds for each $i \in I$, $P \in I\!P(X_\infty)$ and $\pi_i : E_\infty \to E_i$ is the ith coordinate projection.

Theorem 5. Let $[M(X_i, T_i, E_i), q_{ij}]$ be an inverse system in $I\!M$ such that the net I is *countably generated*. Then $(X_\infty, T_\infty, E_\infty)$ defined by (20) generates a moment problem in $I\!M$ such that

(a) $[M(X_\infty, T_\infty, E_\infty), p_i] \in \lim_{\hookleftarrow}[M(X_i, T_i, E_i), q_{ij}]$

and

(b) if $[M(X, T, E), q_i] \in \lim_{\hookleftarrow}[M(X_i, T_i, E_i), q_{ij}]$,
then $[S(X, T, E), a_i] \in \lim_{\hookleftarrow}[S(X_i, T_i, E_i), a_{ij}]$,

where $q_{ij} \to a_{ij},\ q_i \to a_i$ in the sense of (19).

P r o o f. First, we shall prove that

$$S(X_\infty, T_\infty, E_\infty) = \Big\{(s_i)_{i\in I} \in \prod S_i : a_{ij}(s_i) = s_j,\ i \geq j\Big\}$$

which implies that

$$\begin{aligned} &[S(X_\infty, T_\infty, E_\infty), \pi_i] \in \lim_{\hookleftarrow}[S(X_i, T_i, E_i), a_{ij}], \\ &\qquad \text{where } \pi_i : \textstyle\prod S_j \to S_i \text{ is the } i\text{th coordinate projection.} \end{aligned} \tag{22}$$

If $s = (s_i) \in S(X_\infty, T_\infty, E_\infty)$ then $s = (bT_\infty)(P)$ for some P in $I\!P(X_\infty)$. Because $\pi_i \circ T_\infty = T_i \circ p_i$ it follows by (21) that $s_i = (bT_i)(p_i P)$ for $i \in I$. Hence,

$$a_{ij}(s_i) = (b(T_j \circ q_{ij}))(p_i P) = (bT_j)(p_j P) = s_j$$

for $i \geq j$ according to (19). If $s = (s_i)$ is a thread in $\prod S_i$, i.e. $a_{ij}(s_i) = s_j$ for $i \geq j$, we have to exhibit a measure P in $I\!P(X_\infty)$ such that $s = (bT_\infty)(P)$ to show that $s \in S(X_\infty, T_\infty, E_\infty)$: Denote by $L_i : I\!P(X_\infty) \to S_i$ the map defined by $L_i(P) = (b(T_i \circ p_i))(P)$ for $P \in I\!P(X_\infty)$ and put $K_i = L_i^{-1}(s_i)$. Considering that L_i's are continuous surjections defined on the compact set $I\!P(X_\infty)$ it follows easily from the fact that $s = (s_i)_{i \in I}$ is a thread that $\{K_i\}$ is a centered system of nonempty compact sets. Choosing a P in $\cap K_i$ we get a measure with $(bT_\infty)(P) = s$ and (22) is proved.

To prove (a) we must only verify (18) with $M_\infty = M(X_\infty, T_\infty, E_\infty)$ and $M_0 = M(X_0, T_0, E_0)$. As $[X_\infty, p_i]$ is an inverse limit of $[X_i, q_{ij}]$ in the category of compact metrizable sets there is a *unique continuous map* $u : X_0 \to X_\infty$ such that $p_i \circ u = u_i$. The map u is a *surjection* because the morphisms u_i, q_{ij} are surjective and X_0 is a compact set. Thus, it remains to verify that $M(X_0, T_0, E_0) \geq M(X_0, T_\infty \circ u, E_\infty)$ to show that $u : M_0 \to M_\infty$ is a unique morphism satisfying $p_i \circ u = u_i$ for $i \in I$ and thus to complete the proof of the implication (18):

Letting $q_{ij} \to a_{ij}$ and $u_i \to a_i$ in the correspondence (19) we get an inverse system $[S(X_i, T_i, E_i), a_{ij}]$ in the category $\mathcal{C}$ and morphisms $a_i : S(X_0, T_0, E_0) \to S(X_\infty, T_\infty, E_\infty)$ that are consistent with the family (a_{ij}), such that (see (19))

$$a_i[(bT_0)(P)] = (b(T_i \circ u_i))(P) \text{ for each } i \in I \text{ and } P \in I\!P(X_0). \tag{23}$$

On the other hand it follows from (22) that there exists a unique continuous affine map $a : S(X_0, T_0, E_0) \to S(X_\infty, T_\infty, E_\infty)$ such that $\pi_i \circ a = a_i$ for $i \in I$. It follows from (23) that $a[(bT_0)(P)] = (b(T_\infty \circ u))(P)$ for $P \in I\!P(X_0)$. As $S(X_\infty, T_\infty, E_\infty) = S(X_0, T_\infty \circ u, E_\infty)$ and a is also a surjection we get $M(X_0, T_0, E_0) \geq M(X_0, T_\infty \circ u, E_\infty)$ by Theorem 2 (b).

Finally, the assertion (b) easily follows from (a) and (22) as inverse limits both in $I\!M$ and $\mathcal{C}$ are determined uniquelly up to a commuting isomorphisms. □

The following statement is an obvious consequence of Theorem 5 (a):

Corollary 4. Let X be a compact metrizable space and $B = \cup B_i$ for some countably generated nondecreasing net $B_i \subset C(X)$. Then

$$M(X, T^B, E^B) \in \lim_{\hookleftarrow} M(X, T^{B_i}, E^{B_i})$$

where all missing morphisms are equal to the identity map on X.

The following very simple statement suggests a possibility how to use Theorem 5 for "a limit construction" of sets of uniqueness.

Remark. Consider a countably generated net I and suppose that

(a) $[M(X,T,E),q_i] \in \lim_{\hookleftarrow}[M(X_i,T_i,E_i),q_{ij}]$.
(b) $K_i \subset X_i$ are sets of uniqueness for $M(X_i,T_i,E_i)$.
(c) $[K_i,q_{ij}]$ is an inverse system in the category of compact metrizable sets.

Then $K = \cap_i q_i^{-1} K_i$ is a set of uniqueness for $M(X,T,E)$.

Remark that $[K,q_i] \in \lim_{\hookleftarrow}[K_i,q_{ij}]$. The statement follows directly from the fact that each measure $P \in \mathbb{P}(K)$ is determined by its projections $(q_i P) \in \mathbb{P}(X_i)$.

3. MARGINAL AND TRANSSHIPMENT PROBLEM

Having $X = Y^2$, assuming Y to be a compact metrizable space recall that $M \in \mathbb{M}(X)$ is called a *marginal problem* if a pair of equivalent measures P, Q is defined by $P_1 = Q_1,\ P_2 = Q_2$ where $P_i' s (Q_i' s)$ are the marginals to P (Q). Also recall that $\mathcal{N} \in \mathbb{M}(X)$ is called a *transshipment problem*, see [1], if $P_1 - P_2 = Q_1 - Q_2$ defines the classes of equivalence in $\mathbb{M}$.

Assume first that $Y = \{1,2,\dots,n\}$ is a finite set and observe that

$$M = M(X,T,R^{2n}),\ \text{where}\ T(i,j) = (\delta_{i1},\dots,\delta_{in},\delta_{j1},\dots\delta_{jn})$$

$\delta_{\ell k} = I_{[\ell=k]}$, because $(bT)\,(P) = (P_1(1),\dots,P_1(n),P_2(1),\dots,P_2(n))$ completely defines the marginals P_1 and P_2 of each measure P in $\mathbb{P}(X)$. As T is an injection it follows from Corollary 2 that $D \subset X$ is a set of uniqueness for the marginal problem M iff TD is a set of affinely independent points in R^{2n} (i. e. $\operatorname{card}(D) \le 2n+1$). Similarily $\mathcal{N} = M(X,U,R^n)$, where the map U is defined by $U(i,j) = (\delta_{1i}-\delta_{1j},\delta_{2i}-\delta_{2j},\dots,\delta_{ni}-\delta_{nj})$, because $(bU)\,(P) = (P_1(1)-P_2(1),P_1(2)-P_2(2),\dots,P_1(n)-P_2(n))$ for each $P \in \mathbb{P}(X)$. Hence $D \subset X$ must avoid the diagonal in X to have a chance to be a set of uniqueness for $\mathcal{N}$ (Corollary 2) and in this case D is a set of uniqueness if and only if $U(D)$ is a set of affinely independent points in R^n (i. e. $\operatorname{card}(D) \le n+1$).

Consider now both problems in a continuous version with $Y = [0,1]$. Then

$$M = M(X,T,E),\ \text{where}\ \ E = (C[0,1])^2\ \text{and}\ T : (x,y) \to (e^{tx}, e^{sy})$$

and

$$\mathcal{N} = M(X,U,E),\ \text{where}\ E = C[0,1]\ \text{and}\ \ U : (x,y) \to e^{tx} - e^{ty}.$$

Both T and U are continuous when the space $C[0,1]$ is considered with topology of uniform convergence and generate properly M and $\mathcal{N}$, respectively, as each finite measure on $[0,1]$ is uniquelly determined by its Laplace transform restricted to the interval $[0,1]$ and

$$(bT)\,(P)\,(t,s) = \left(\int_0^1 e^{tx} P_1(\mathrm{d}x),\ \int_0^1 e^{sy} P_2(\mathrm{d}y)\right),$$
$$(bU)\,(P)\,(t) = \int_0^1 e^{tx}(P_1 - P_2)(\mathrm{d}x)$$

hold for $t, s \in [0,1]$ and $P \in I\!P(X)$. Thus, for a compact $D \subset X$ we have

$$\begin{aligned} S(D,U,C^2[0,1]) &= \{(L(P_1), L(P_2)), P_i \in I\!P[0,1] \\ &\quad \text{such that } P_i\text{'s are marginals of some } P \in I\!P(D)\} \text{ and} \\ S(D,U,C[0,1]) &= \{L(P_1 - P_2),\ P_i \in I\!P[0,1] \\ &\quad \text{such that } P_i\text{'s are the marginals of some } P \in I\!P(D)\}, \end{aligned}$$

where $L(P)(t) = \int_0^1 e^{tx} P(\mathrm{d}t),\ t \in [0,1]$.

Hence, Corrolary 2 could be used to get necessary and sufficient conditions for D to be a set of marginal uniqueness either for M and $\mathcal{N}$. (See also [1] and [6] for a more detailed study of these sets.)

Obviously, $\mathcal{N} \preceq M$ holds and therefore Theorem 2 ensures the existence of affine continuous surjection $a : S(X,T,C^2) \to S(X,U,C)$ such that $a \circ (bT) = (bU)$. The map is defined explicitly by

$$a(f,g)(t) = f(t) - g(t) \text{ for } t \in [0,1] \text{ and } (f,g) \in S(X,T,C^2)$$

Finally, putting

$$T_n(x,y) = \left(\sum_{k=0}^{n} \frac{(xt)^k}{k!}, \sum_{k=0}^{n} \frac{(ys)^k}{k!} \right)$$

for $(x,y) \in X$ and $(t,s) \in X$, we get a continuous map $T_n : X \to C^2[0,1]$ and hence a finite dimensional moment problem $M(X,T_n,C^2)$, ($P, Q \in I\!P(X)$ belong to the same class of equivalence iff $P_i(x^k) = Q_i(x^k)$ for $1 \le k \le n$ and $i = 1,2$). If $q_{ij} : X \to X$ are homeomorphisms such that $M(X, T_j \circ q_{ij}, C^2) \le M(X, T_i, C^2)$ hold for $i \ge j$ and the condition (15) is satisfied, then $[M(X,T_i,C^2), q_{ij}]$ is an inverse system in $I\!M(X)$. It follows directly from Theorem 5 (a) that there are uniquelly determined homeomorphisms $q_i : X \to X$ consistent with the family (q_{ij}) such that

$$[M(X,T,C^2), q_i] \in \lim_{\hookleftarrow} [M(X_i,T_i,C^2), q_{ij}]$$

Thus, the *marginal problem* is an inverse limit of an inverse system of standard *finite moment problems.*

(Received March 3, 1994.)

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