AN ADAPTIVE CHANGE DETECTION SCHEME
FOR A NONLINEAR BEAM MODEL[1]

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In this paper, we consider parameter estimation techniques for detecting changes of a nonlinear nature in an Euler–Bernoulli beam model. The nonlinear stiffness used provides a very simple model of damage, and the adaptive estimation algorithm is used to track the onset of the nonlinearity. Using Lyapunov redesign methods, extended and applied to infinite dimensional systems, a stable learning scheme is developed. The resulting parameter adaptation rule is able to "sense" the instance of the fault occurrence. In addition, it identifies the location and the shape of the fault where the beam is persistently excited. Simulation studies are used to illustrate the applicability of the theoretical results.

1. INTRODUCTION

An important aspect of many structural vibration problems is the online (non-destructive) detection of structural changes, particularly changes that indicate impending failure. In recent years there has been a great deal of research into adaptive parameter estimation techniques, which provide a means for online model calibration, e. g., [4, 28, 30, 33, 39].

More recently the detection and diagnosis of failures of dynamical systems attracted the attention of many researchers working on many engineering problems, [3, 27, 34, 35]. As was noted in [35] and the references therein, many fault detection schemes deal with either linear or nonlinear finite dimensional systems (lumped parameter systems). Many physical systems though, are described by partial differential, integrodifferential and functional differential equations. These systems are infinite dimensional and their parameters are often distributed in nature. For distributed parameter systems, such as flexible structures (beams and trusses), damage detection based on modal and frequency analysis has been studied by many researchers, see for example [2, 20, 21, 29, 32, 38] and the references therein. As was pointed out in [12] and argued in [11] and their references, modal based methods are highly unreliable when dealing with estimation of variable material parameters such as mass, stiffness and damping. As was already mentioned above, damages in

beams (often modeled by changes in stiffness, [12]) are distributed in nature and affect individual modes differently, [12]. The analysis of damages in parameterized partial differential equations with Galerkin approximation techniques validates non-destructive damage detection since it incorporates information on location/geometry of damages in structures.

We consider in this paper an extension of these ideas for a simple model of structural change. We use an Euler–Bernoulli model of a vibrating beam, with the change being an abrupt transition from linear to nonlinear stiffness. The idea is that the nonlinear stiffness provides a simple model of damage to the beam. Our goal is to investigate adaptive algorithms which can detect the occurrence of this nonlinearity.

This problem involves approximation at several levels. The plant is modeled by a partial differential equation, which must be solved numerically. The identification algorithm considered here requires "full state feedback" which means we must estimate the infinite dimensional state from finite dimensional observations. We give a complete description of our algorithm, along with a numerical example which illustrates its utility.

The paper is organized as follows. In Section 2 we give the basic problem statement. In Section 3, we outline the adaptive algorithm and give some well-posedness results. State and parameter convergence is the topic of discussion of Section 4, and in Section 5, we give our numerical results. Comments with future extensions follow in Section 6.

2. PROBLEM STATEMEMT

In mathematically modeling the flexible beam, we assume that it is of length l with uniform rectangular cross section of height h and width b. We let $w(t,x)$ denote the transverse displacement of the beam at position x along its span at each time t. This is measured relative to the x-axis in the coordinate frame determined by the longitudinal axis of the beam in its undeformed state with origin located at the beam's left fixed end, see Figure 2.1. We assume a cantilevered Euler Bernoulli beam with Kelvin–Voigt damping for the modeling of the dynamics and dissipation, see [6, 8, 15, 16, 22], and [42]. It is assumed that the beam undergoes only small deformations (i.e. $|w(t,x)| << l$, and $|(\partial w/\partial x)(t,x)| \ll 1$). The Euler–Bernoulli theory including Kelvin–Voigt viscoelastic damping yields the partial differential equation

$$\rho_b w_{tt} + \frac{\partial^2}{\partial x^2} M(t,x) = f(t,x), \quad 0 < x < l, \ t > 0, \tag{1}$$

with the boundary conditions (both ends kept fixed)

$$w(t,0) = w_x(t,0) = w(t,l) = w_x(t,l) = 0, \quad t > 0, \tag{2}$$

where ρ_b is the linear mass density, $M(t,x)$ is the internal moment and f is the external applied force. For an uncontrolled beam with Kelvin–Voigt damping, the moment is given by (see [16])

$$M(t,x) = EIw_{xx}(t,x) + c_D Iw_{txx}(t,x), \quad 0 < x < l, \ t > 0,$$

where E is Young's modulus, I is the cross sectional moment of inertia, and c_D is the damping modulus. For actuation, a piezoceramic patch is attached to the beam as shown in Figure 2.1. This patch is excited in such a way so as to produce a pure bending moment, see [16]. If H_0 is used to denote the Heaviside function with unit step at $x = 0$, the model for the beam is then given by

$$
\rho_b w_{tt}(t,x) + \left[EI w_{xx}(t,x) + c_D I w_{txx}(t,x) \right]_{xx}
$$
$$
= \left[EI \frac{K^B d_{31}}{\tau} u(t) \left[H_0(x - a_1) - H_0(x - a_2) \right] \right]_{xx}, \quad 0 < x < l, \ t > 0,
\tag{3}
$$

where $u(t)$ is the voltage applied to the patch at time t, K^B is a parameter which depends on the geometry and piezoceramic material properties, τ is the patch thickness, a_1 and a_2 denote the position of the patch and d_{31} is the piezoceramic strain constant (see [7, 16]). Equation (3) above models a linear beam with spatially invariant stiffness and damping coefficients. This would be used as a point of departure for providing a model for the nonlinear beam.

Fig. 2.1 Cantilevered beam with piezoceramic actuator patch.

We now consider the *nonlinear* Euler–Bernoulli beam with Kelvin–Voigt viscoelastic damping

$$
\rho_b w_{tt}(t,x) + \left[EI(t,x,w_{xx}(t,x)) + c_D I w_{txx}(t,x) \right]_{xx} = f(t,x)
\tag{4}
$$

with boundary conditions given by

$$
w(t,0) = w_x(t,0) = w(t,l) = w_x(t,l) = 0
$$

and initial conditions given by

$$
w(0,\cdot) = w_0(\cdot) \in H_0^2(0,l), \qquad w_t(0,\cdot) = w_1(\cdot) \in L^2(0,l).
$$

The nonlinear stiffness term $EI(t, x, w_{xx}(t, x))$ is given by

$$EI(t, x, w_{xx}(t, x)) = EI_0 w_{xx}(t, x) + H_0(t - t^*) \chi_{[x_1, x_2]}(x) g(w_{xx}(t, x))$$

$$= EI_0 w_{xx}(t, x) + H_0(t - t^*) \chi_{[x_1, x_2]}(x) \begin{cases} EI_1 w_{xx}(t, x) & \text{if } w_{xx}(t, x) > 0 \\ EI_2 w_{xx}(t, x) & \text{otherwise,} \end{cases}$$

where $\chi_{[x_1, x_2]}(x)$ denotes the characteristic function over the interval $[x_1, x_2]$, and which consists of the *nominal* (linear) term $EI_0 w_{xx}(t, x)$ (same as the one in (3)) and the nonlinear term $g(w_{xx})$, depicted in Figure 2.2 and given by the equation below

$$g(\phi) = EI_1 \phi H_0(\phi) + EI_2 \phi (1 - H_0(\phi)). \tag{5}$$

Fig. 2.2. Nonlinear stiffness term $g(w_{xx})$.

The latter is zero before failure and at the failure (i.e. at time $t \geq t^*$) depends on the curvature of the beam $w_{xx}(t, x)$ and acts on part of the beam (i.e. the characteristic function $\chi_{[x_1, x_2]}(x)$). Specifically, the beam stiffness parameter prior to the failure (i.e. for $t < t^*$) is given by

$$EI(t, x, w_{xx}(t, x)) = EI_0 w_{xx}(t, x), \quad 0 \leq x \leq l,$$

and after the failure (i.e. for $t \geq t^*$) by

$$EI(t, x, w_{xx}(t, x)) = \begin{cases} EI_0 w_{xx}(t, x) + \chi_{[x_1, x_2]}(x) EI_1 w_{xx}(t, x) & \text{if } w_{xx}(t, x) > 0 \\ EI_0 w_{xx}(t, x) + \chi_{[x_1, x_2]}(x) EI_2 w_{xx}(t, x) & \text{otherwise.} \end{cases}$$

It is assumed, for the sake of simplicity, that the nominal beam stiffness EI_0 is known. In addition, we assume that the *damping coefficient* $c_D I$ is also a known constant.

The goal here is to detect the failure modeled by the nonlinear term $g(w_{xx})$ and identify the beam stiffness parameters EI_1 and EI_2 adaptively. The failure time t^* is unknown and it is desired to detect (implicitly) this failure time by monitoring the system, via an appropriately chosen state observer. The proposed state estimator will detect changes in the system which would be an indication of the failure and thus t^* would be identified. It is assumed that the beam displacement $w(t, x)$ and velocity $w_t(t, x)$ are available for measurement at each time t. In the next section, we present our algorithm for detecting failures and estimating the beam parameters from state observations.

3. ABSTRACT FORMULATION

Before we proceed with the abstract formulation of the beam model, we need to provide some details on the abstract spaces involved. We consider the Hilbert space $L^2(0, l)$ as the state space. We also consider the Sobolev space $H_0^2(0, l)$ as the space of test functions, see [1]. Using the fact that the Sobolev space $H_0^2(0, l)$ is embedded densely and continuously in the Hilbert space $L^2(0, l)$, [41, 43], it follows that

$$H_0^2(0, l) \hookrightarrow L^2(0, l) \hookrightarrow H^{-2}(0, l), \tag{6}$$

where $H^{-2}(0, l)$ denotes the dual of $H_0^2(0, l)$, see [1, 18, 43]. In particular, we assume that there exists an *embedding* constant $K_{emb} > 0$ such that $|\varphi|_{L^2} \leq K_{emb} |\varphi|_{H_0^2}$, $\varphi \in H_0^2(0, l)$.

Here we use $\langle \cdot, \cdot \rangle$ to denote the usual duality product obtained as the extension by continuity of the $L^2(0, l)$- inner product from $L^2(0, l) \times H_0^2(0, l)$ to $H^{-2}(0, l) \times H_0^2(0, l)$, see [5, 43]. The $L^2(0, l)$-inner product (energy inner-product) is given by

$$\langle \psi, \phi \rangle_{L_2} = \int_0^l \rho_b \psi(x) \cdot \phi(x) \, \mathrm{d}x, \qquad \psi, \phi \in L_2(0, l),$$

whereas the $H_0^2(0, l)$-inner product is given by

$$\langle \psi, \phi \rangle_{H_0^2} = \int_0^l \psi_{xx}(x) \cdot \phi_{xx}(x) \, \mathrm{d}x, \qquad \psi, \phi \in H_0^2(0, l).$$

In order to simplify equation (4) above, we rewrite (5) as

$$g(w_{xx}(t, x)) = EI_1 w_{xx}(t, x) \, \alpha(t, x) + EI_2 w_{xx}(t, x) \, (1 - \alpha(t, x)), \tag{7}$$

where we define the indicator function $\alpha(t, x)$ (the Heaviside function) by

$$\alpha(t, x) = \begin{cases} 1 & \text{if } w_{xx}(t, x) > 0 \\ 0 & \text{otherwise} \end{cases}.$$

In the rest of this note we denote the characteristic function $\chi_{[x_1, x_2]}(x)$ by χ_I.

We now write the beam equation in weak or variational form

$$\langle w_{tt}(t), \varphi \rangle_{L^2} + \langle EI_0 w(t), \varphi \rangle_{H_0^2} + \langle c_D I w_t(t), \varphi \rangle_{H_0^2} \tag{8}$$
$$+ \; H_0(t - t^*) \langle EI_1 \chi_I \alpha(t) \, w_{xx}(t) + EI_2 \chi_I (1 - \alpha(t)) \, w_{xx}(t), \varphi_{xx} \rangle_{L^2} = \langle Bu(t), \varphi \rangle_{L^2},$$

where the test function $\varphi \in H_0^2(0, l)$, and the input operator $B \in \mathcal{L}(U, H^{-2}(0, l))$ is given by the right hand side of equation (3) and it is assumed to be known, see [13] for well-posedness of second order systems with unbounded input operators. The space U is called the input or control space. The initial conditions associated with (8) are

$$w(0, \cdot) = w_0(\cdot) \in H_0^2(0, l), \qquad w_t(0, \cdot) = w_1(\cdot) \in L^2(0, l). \tag{9}$$

The function $H_0(t - t^*)$ that represents the *time profile* of the failure is assumed to be the Heaviside function given by $H_0(t - t^*) = 0$ for $t < t^*$ and $H_0(t - t^*) = 1$ for $t \geq t^*$.

When the beam is actuated by a centered piezoceramic patch (Figure 2.1), we can rewrite the above equation (8) explicitly in terms of integrals over the spatial domain by

$$
\begin{aligned}
&\int_0^l \rho_b w_{tt}(t, x) \cdot \varphi(x)\, \mathrm{d}x + \int_0^l EI_0 w_{xx}(t, x) \cdot \varphi_{xx}(x)\, \mathrm{d}x \\
+ \quad &\int_0^l c_D I w_{txx}(t, x) \cdot \varphi_{xx}(x)\, \mathrm{d}x \\
+ \quad &H_0(t - t^*) \int_0^l \left[EI_1 \alpha(t, x) + EI_2(1 - \alpha(t, x)) \right] \chi_{[x_1, x_2]}(x) w_{xx}(t, x) \cdot \varphi_{xx}(x)\, \mathrm{d}x \\
= \quad &\left(\int_0^l \mathcal{K}^B \chi_{[a_1, a_2]}(x) \cdot \varphi_{xx}(x)\, \mathrm{d}x \right) u(t), \quad \varphi \in H_0^2(0, l),
\end{aligned}
\tag{10}
$$

where $u(t) \in U$ is the voltage applied to the patch, and $\mathcal{K}^B = EI \frac{K^B d_{31}}{\tau}$, see [13, 17, 18] for additional details on the modeling equations of a beam with piezoceramic actuators.

Since in this note we deal with two constant parameters, then the *parameter space* Q is identified with the Euclidean space $\mathbb{R}^2$, i.e. $Q = \mathbb{R}^2$. Before we proceed with the state and parameter estimator, we must impose a boundedness condition on the state of the beam.

Assumption 3.1. (Boundedness of plant.) A plant is a triple (EI_1, EI_2, w) with w a solution to the initial-value problem (8) with $w \in H_0^2(0, l)$, a. e. $t > 0$, for which there exists a constant $\mu > 0$ such that

$$
\left| \left\langle \left(EI_1 \alpha(t, x) + EI_2(1 - \alpha(t, x)) \right) \chi_I(x)\, w_{xx}(t, x), \phi_{xx} \right\rangle_{L^2} \right| \leq \mu \left| \begin{array}{c} EI_1 \\ EI_2 \end{array} \right|_{R^2} \cdot |\phi|_{H_0^2}
$$

for almost every $t > 0$ and all $\phi \in H_0^2(0, l)$.

Remark 3.2. The well posedness of the plant equation (10) has been treated in the paper by Banks, Gilliam and Shubov in [9, 10].

4. ESTIMATOR AND CONVERGENCE

We now proceed with the state estimator given in the form of an initial value problem

$$
\begin{aligned}
& \langle v_{tt}(t), \varphi \rangle_{L^2} + \langle EI^* v(t), \varphi \rangle_{H_0^2} + \langle c_D I^* v_t(t), \varphi \rangle_{H_0^2} + \langle EI_0 w(t), \varphi \rangle_{H_0^2} \\
+ \ & \langle c_D I w_t(t), \varphi \rangle_{H_0^2} - \langle EI^* w(t), \varphi \rangle_{H_0^2} - \langle c_D I^* w_t(t), \varphi \rangle_{H_0^2} \\
+ \ & \langle \widehat{EI}_1(t)\, \chi_I \alpha(t)\, w_{xx}(t) + \widehat{EI}_2(t)\, \chi_I (1 - \alpha(t))\, w_{xx}(t), \varphi_{xx} \rangle_{L^2} \\
= \ & \langle Bu(t), \varphi \rangle_{L^2},
\end{aligned}
\tag{11}
$$

where the parameters EI^* and $c_D I^*$ are some *tuning* parameters (see, for example, [26]); i.e. they are values of the stiffness and damping parameters chosen to affect the convergence of the estimator. The initial conditions for the state observer are taken to be the same as the ones for the plant, namely

$$
v(0, \cdot) = w(0, \cdot), \quad v_t(0, \cdot) = w_t(0, \cdot). \tag{12}
$$

The parameters $\widehat{EI}_1(t)$ and $\widehat{EI}_2(t)$ in (11) are the adaptive estimates of the unknown parameters EI_1 and EI_2. By denoting the state error $v - w$, by $e = v - w$ we then have that, using the Lyapunov redesign method, the unknown parameters can be adjusted via

$$
\frac{\mathrm{d}}{\mathrm{d}t} \widehat{EI}_1(t) = \lambda_1 \langle \chi_I \alpha(t)\, w_{xx}(t), \gamma e_{txx}(t) + e_{xx}(t) \rangle_{L^2} \tag{13}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t} \widehat{EI}_2(t) = \lambda_2 \langle \chi_I (1 - \alpha(t))\, w_{xx}(t), \gamma e_{txx}(t) + e_{xx}(t) \rangle_{L^2} \tag{14}
$$

for some $\gamma > 0$, $\lambda_1, \lambda_2 > 0$ and with initial conditions given by

$$
\widehat{EI}_1(0) = \widehat{EI}_2(0) = 0. \tag{15}
$$

The choice of these initial conditions will become clear below in the treatment of the convergence properties of the state and parameter estimator. This is similar to what was done in the finite dimensional case, see [35, 36].

We denote the parameter errors by $r_1(t) = \widehat{EI}_1(t) - EI_1$ and $r_2(t) = \widehat{EI}_2(t) - EI_2$, respectively. Using the definition of the state error, and using (8), (11), (13) and (14), we arrive at the *(state and parameter) error equations*

$$
\begin{aligned}
& \langle e_{tt}(t), \varphi \rangle_{L^2} + \langle EI^* e(t), \varphi \rangle_{H_0^2} + \langle c_D I^* e_t(t), \varphi \rangle_{H_0^2} \\
& + \langle [r_1(t)\, \alpha(t) + r_2(t)\, (1 - \alpha(t))]\, \chi_I w_{xx}(t), \varphi_{xx} \rangle_{L^2} = 0,
\end{aligned}
\tag{16}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t} r_1(t) = \lambda_1 \langle \chi_I \alpha(t)\, w_{xx}(t), \gamma e_{txx}(t) + e_{xx}(t) \rangle_{L^2}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t} r_2(t) = \lambda_2 \langle \chi_I (1 - \alpha(t))\, w_{xx}(t), \gamma e_{txx}(t) + e_{xx}(t) \rangle_{L^2}
$$

with initial conditions given by

$$
e(0) = e_t(0) = r_1(0) = r_2(0) = 0. \tag{17}
$$

Remark 4.1. The parameter errors $r_1(t)$ and $r_2(t)$ are given by

$$r_1(t) = \widehat{EI}_1(t) - 0 \quad \text{and} \quad r_2(t) = \widehat{EI}_2(t) - 0 \qquad \text{for } t < t^*,$$

$$r_1(t) = \widehat{EI}_1(t) - EI_1 \quad \text{and} \quad r_2(t) = \widehat{EI}_2(t) - EI_2 \quad \text{for } t \geq t^*,$$

since the unknown (failure) parameters EI_1, EI_2 are zero prior to the (unknown) failure time t^*.

Before we present any convergence results, we define the *energy* functional by

$$V(t) = \gamma \left\{ \langle EI^* e(t), e(t) \rangle_{H_0^2} + |e_t(t)|_{L^2}^2 \right\} + 2 \langle e(t), e_t(t) \rangle_{L^2} \tag{18}$$

$$+ \langle c_D I^* e(t), e(t) \rangle_{H_0^2} + r_1^2(t) + r_2^2(t),$$

where the constant $\gamma > 0$ will be defined below. With no additional assumptions we have the following convergence result.

Theorem 4.1. Assume that the plant satisfies the boundedness condition given by Assumption 3.1. If the constant γ satisfies

$$\gamma > \max \left\{ K_{emb}, \frac{K_{emb}}{EI^*}, \frac{K_{emb}^2}{c_D I^*} \right\}, \tag{19}$$

then, for $t < t^*$ we have

$$V(t) = |e(t)|_{H_0^2} = |e_t(t)|_{L^2} = r_1(t) = r_2(t) = 0,$$

and for $t \geq t^*$ we have

$$|e(t)|_{H_0^2}^2 + |e_t(t)|_{L^2}^2 + r_1^2(t) + r_2^2(t) + \rho \int_{t^*}^t \left\{ |e(\tau)|_{H_0^2}^2 + |e_\tau(\tau)|_{H_0^2}^2 \right\} \, d\tau$$

$$\leq \quad \sigma \left\{ |e(t^*)|_{H_0^2}^2 + |e_t(t^*)|_{L^2}^2 + r_1^2(t^*) + r_2^2(t^*) \right\},$$

for some $\rho, \sigma > 0$.

Proof. Using the fact that for $t < t^*$, the term $H_0(t - t^*)$ in (8) is zero, we have that $EI_1 = EI_2 = 0$ for that time interval and thus $r_1(0) = \widehat{EI}_1(0) - 0 = 0$, $r_2(0) = \widehat{EI}_2(0) - 0 = 0$. When the time derivative of equation (18) is calculated, it yields

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t) = -2\gamma c_D I^* |e_t(t)|_{H_0^2}^2 + 2|e_t(t)|_{L^2}^2 - 2EI^* |e(t)|_{H_0^2}^2$$

$$\leq -\left(2\gamma c_D I^* - 2K_{emb}^2\right) |e_t(t)|_{H_0^2}^2 - 2EI^* |e(t)|_{H_0^2}^2. \tag{20}$$

Using the fact that $V(t) \geq \sigma_0 \left\{ |e(t)|^2_{H^2_0} + |e_t(t)|^2_{L_2} + r^2_1(t) + r^2_2(t) \right\}$, for some $\sigma_0 > 0$ and by integrating the above equation from 0 to $t < t^*$ we obtain

$$|e(t)|^2_{H^2_0} + |e_t(t)|^2_{L^2} + r^2_1(t) + r^2_2(t) + \rho \int_0^t \left\{ |e(\tau)|^2_{H^2_0} + |e_\tau(\tau)|^2_{H^2_0} \right\} \, \mathrm{d}\tau$$
$$\leq \quad \sigma \left\{ |e(0)|^2_{H^2_0} + |e_t(0)|^2_{L^2} + r^2_1(0) + r^2_2(0) \right\} \equiv 0,$$

for $\rho, \sigma > 0$, which yields the desired result. After the failure, i.e. for $t \geq t^*$, we integrate equation (20) from t^* to some $t > t^*$ to obtain

$$|e(t)|^2_{H^2_0} + |e_t(t)|^2_{L^2} + r^2_1(t) + r^2_2(t) + \rho \int_{t^*}^t \left\{ |e(\tau)|^2_{H^2_0} + |e_\tau(\tau)|^2_{H^2_0} \right\} \, \mathrm{d}\tau$$
$$\leq \quad \sigma \left\{ |e(t^*)|^2_{H^2_0} + |e_t(t^*)|^2_{L^2} + r^2_1(t^*) + r^2_2(t^*) \right\},$$

where we used the fact that $V(t) \leq \sigma_1 \left\{ |e(t)|^2_{H^2_0} + |e_t(t)|^2_{L^2} + r^2_1(t) + r^2_2(t) \right\}$, for some $\sigma_1 > 0$. This then concludes the proof of the theorem. □

From the above theorem, only a boundedness condition can be established for $t \geq t^*$, i.e. we have that the state error $e(t)$ satisfies

$$e \in L^2(t^*, t; H^2_0(0, l)) \cap L^\infty(t^*, t; H^2_0(0, l)),$$

with its derivative satisfying

$$e_t \in L^2(t^*, t; H^2_0(0, l)) \cap L^\infty(t^*, t; L^2(0, l)),$$

while the parameter errors satisfy

$$r_1, r_2 \in L^\infty(t^*, t; \mathbb{R}).$$

Using arguments similar to those for establishing *Barbălat's* lemma, see [30, 33, 37] for finite dimensional systems, and [14] for infinite dimensional systems, we can establish the convergence of the state error to zero; see [25] for similar results for the adaptive parameter identification of second order distributed parameter systems.

Theorem 4.3. Assume that the boundedness condition given by Assumption 3.1 is satisfied and that γ satisfies condition (19). If the adaptation (13), (14) is used with the state estimator (11), then we have that for $t > t^*$

$$\lim_{t \to \infty} |e(t)|_{H^2_0} = \lim_{t \to \infty} |e_t(t)|_{L^2} = 0.$$

P r o o f . The proof is identical to the case of parameter identification for second-order distributed parameter systems in [25] and it is therefore omitted. □

Because of the structure of the failure assumed, the conditions required for parameter convergence, i.e. $\lim_{t \to \infty} r_1(t) = 0$ and $\lim_{t \to \infty} r_2(t) = 0$ are identical to those used for the parameter identification of second-order distributed parameter systems presented in [25], namely the condition of *persistence of excitation*. We present this condition as it applies to the specific problem under study.

Definition 4.4. [25, 26] The plant (8) is said to be *persistently excited* if there exists $T_0, \delta_0, \epsilon_0 > 0$, and a sequence of positive real numbers $\{t_k\}_{k=1}^\infty$ with $\lim_{t\to\infty} t_k = \infty$, such that for each $\boldsymbol{p} = (p_1, p_2) \in \mathbb{R}^2$ with $|\boldsymbol{p}|_{R^2} = 1$ and each positive integer k, there exists a $t_k^* \in [t_k, t_k + T_0]$ such that

$$\sup_{|\phi|_{H_0^2} \leq 1} \left| \int_{t_k^*}^{t_k^* + \delta_0} \left\langle \left(p_1 \alpha(\tau) + p_2(1 - \alpha(\tau)) \right) \chi_I w_{xx}(\tau), \phi_{xx} \right\rangle_{L^2} \mathrm{d}\tau \right| \geq \epsilon_0.$$

The above can be written explicitly as

$$\epsilon_0 \quad \leq \quad \sup_{|\phi|_{H_0^2} \leq 1} \left| \int_{t_k^*}^{t_k^* + \delta_0} \left(\int_0^l \left[p_1 \alpha(\tau, x) + p_2(1 - \alpha(\tau, x)) \right] \right. \right.$$
$$\left. \left. \chi_{[x_1, x_2]}(x) w_{xx}(\tau, x) \cdot \phi_{xx}(x) \, \mathrm{d}x \right) \mathrm{d}\tau \right|$$
$$= \quad \sup_{|\phi|_{H_0^2} \leq 1} \left| \int_{t_k^*}^{t_k^* + \delta_0} \left(\int_{x_1}^{x_2} \left[p_1 \alpha(\tau, x) + p_2(1 - \alpha(\tau, x)) \right] \right. \right.$$
$$\left. \left. w_{xx}(\tau, x) \cdot \phi_{xx}(x) \, \mathrm{d}x \right) \mathrm{d}\tau \right|.$$

We can now prove parameter convergence by imposing the persistence of excitation condition onto the plant information.

Theorem 4.5. If the plant (8) is *persistently excited* then we have

$$\lim_{t\to\infty} \left| \begin{pmatrix} r_1(t) \\ r_2(t) \end{pmatrix} \right|_{R^2} = 0.$$

P r o o f . The proof of this theorem is similar to the one given for the linear case in [25]. □

5. NUMERICAL RESULTS

In this section we describe the implementation scheme and present some of our numerical findings. Using the Galerkin scheme outlined in [23, 24, 25], we discretize the beam in terms of spline expansions (see [19, 31, 40]). Modified cubic splines on the interval $(0, l)$ with respect to the uniform mesh $\{0, \frac{1l}{n}, \frac{2l}{n}, \dots, l\}$ were used to approximate (11)−(14). We denote the 1-D cubic splines by $\{B_i^n\}_{i=1}^{n-1}$ and the approximating subspace $H^n = \text{span}\{B_i^n\}_{i=1}^{n-1}$. By letting

$$w_n(t, x) = \sum_{j=1}^{n-1} W_j^n(t) B_j^n(x),$$

where $W^n(t) = \left[W_1^n(t), \ldots, W_{n-1}^n(t)\right]^T \in \mathbb{R}^{n-1}$ is the coordinate vector for $w_n(t,x)$ with respect to the spline basis $\{B_j^n\}_{j=1}^{n-1}$, we have that

$$W^n(t) = [K^n]^{-1} \int_0^l w_{xx}(t,x) \cdot \left[\overrightarrow{B_{xx}^n}(x)\right] \, \mathrm{d}x,$$

where K^n is the $(n-1) \times (n-1)$ *stiffness* matrix and is given by

$$K^n = \int_0^l \left[\overrightarrow{B_{xx}^n}(x)\right] \cdot \left[\overrightarrow{B_{xx}^n}(x)\right]^T \, \mathrm{d}x,$$

with $\left[\overrightarrow{B_{xx}^n}(x)\right] = \left[[B_1^n(x)]_{xx}, \ldots, [B_{n-1}^n(x)]_{xx}\right]^T$. Now we let $V^n(t) = [V_1^n(t), \ldots$ $\ldots, V_{n-1}^n(t)]^T \in \mathbb{R}^{n-1}$ be the vector representation of the state estimator $v^n(t,x)$,

$$v^n(t,x) = \sum_{j=1}^{n-1} V_j^n(t) \, B_j^n(x).$$

Then the finite dimensional state estimator equation corresponding to (11) is given by

$$M^n \ddot{V}^n(t) + c_D I^* K^n \left(\dot{V}^n(t) - \dot{W}^n(t)\right) + EI^* K^n \left(V^n(t) - W^n(t)\right) + c_D I K^n \dot{V}^n(t)$$

$$+ EI_0 K^n V^n(t) + \widehat{EI}_1^n(t) \left[\int_0^l \alpha^n(t,x) \left[\overrightarrow{B_{xx}^n}(x)\right] \cdot \left[\overrightarrow{B_{xx}^n}(x)\right]^T \, \mathrm{d}x\right] W^n(t)$$

$$+ \widehat{EI}_2^n(t) \left[\int_0^l (1 - \alpha^n(t,x)) \left[\overrightarrow{B_{xx}^n}(x)\right] \cdot \left[\overrightarrow{B_{xx}^n}(x)\right]^T \, \mathrm{d}x\right] W^n(t) = \mathcal{K}^B F^n(t)$$

(21)

where the $(n-1) \times (n-1)$ *mass* matrix M^n is given by

$$M^n = \int_0^l \left[\overrightarrow{B^n}(x)\right] \cdot \left[\overrightarrow{B^n}(x)\right]^T \, \mathrm{d}x,$$

with $\left[\overrightarrow{B^n}(x)\right] = \left[[B_1^n(x)], \ldots, [B_{n-1}^n(x)]\right]^T$, and $F^n(t)$ is given by

$$\begin{aligned}
F^n(t) &= \left[\int_0^l \chi_{[a_1,a_2]}(x) \left[\overrightarrow{B_{xx}^n}(x)\right] \, \mathrm{d}x\right] u_{patch}(t) \\
&= \left[\int_{a_1}^{a_2} \left[\overrightarrow{B_{xx}^n}(x)\right] \, \mathrm{d}x\right] u_{patch}(t) \\
&= \left(\left[\overrightarrow{B_x^n}(a_2)\right] - \left[\overrightarrow{B_x^n}(a_1)\right]\right) u_{patch}(t).
\end{aligned}$$

The parameter estimator equations corresponding to (13), (14) are given by

$$\dot{\widehat{EI}}_1^{\,n}(t) \;=\; \lambda_1 [W^n(t)]^T \left[\int_{l_1}^{l_2} \alpha^n(t,x) \left[\overrightarrow{B^n_{xx}}(x) \right] \cdot \left[\overrightarrow{B^n_{xx}}(x) \right]^T \mathrm{d}x \right] G_n(t), \qquad (22)$$

$$\dot{\widehat{EI}}_2^{\,n}(t) \;=\; \lambda_2 [W^n(t)]^T \left[\int_{l_1}^{l_2} (1 - \alpha^n(t,x)) \left[\overrightarrow{B^n_{xx}}(x) \right] \cdot \left[\overrightarrow{B^n_{xx}}(x) \right]^T \mathrm{d}x \right] G_n(t), \quad (23)$$

where $G_n(t)$ is given by $G_n(t) = \left[E_n(t) + \gamma \dot{E}_n(t) \right]$ with $E_n(t) = W^n(t) - V^n(t)$ and λ_1, λ_2 are positive constants acting as *adaptive gains* (see [25]).

For our numerical simulations we assumed that the nonlinear stiffness term $(EI(t,x,w_{xx}(t,x)) = g(w_{xx}(t,x)))$ is given by

$$g(w_{xx}(t,x)) = \begin{cases} 0 w_{xx}(t,x) & \text{if } w_{xx}(t,x) > 0 \\ -5 w_{xx}(t,x) & \text{otherwise}, \end{cases}$$

for $0 \le x \le l$, $t > 0$, the nominal damping parameter is $c_D I(x) = 0.005 Nm^2 \sec$, $0 \le x \le l$, the nominal stiffness parameter is $EI_0(x) = 15 N\,\mathrm{m}^2$, and the linear mass density is $\rho_b = 1.35\,\mathrm{kg/m^3}$. The tuning parameters (see [26]) EI^* and $c_D I^*$ are chosen to be

$$EI^*(x) = 20, \;\; c_D I^*(x) = 0.01, \quad 0 \le x \le l.$$

The adaptive gains λ_1, λ_2 in (22), (23) are

$$\lambda_1 = \lambda_2 = 1 \times 10^5,$$

the parameter γ in (21) is $\gamma = 1 \times 10^3$, the initial guesses for the parameter estimates are

$$\widehat{EI}_1(0) = \widehat{EI}_2(0) = 0$$

and the plant and estimator states are

$$\begin{aligned} w(0,x) &= v(0,x) = 2 \times 10^{-3} x^2 (x-l)^2, \\ w_t(0,x) &= v_t(0,x) = 1 \times 10^{-2} \sin^2(2\pi x/l) \cos(2\pi x/l), \end{aligned}$$

for $0 < x < l$. The beam length is $l = 0.60 m$ and the (centered) patch covers a half of the beam length, i.e. $a_1 = 0.15 m$ and $a_2 = 0.45 m$. The piezoceramic constant is $\mathcal{K}^B = 0.002331655$ and the patch voltage is

$$u_{patch}(t) = 10 \left[\sin(150\pi t) + \sin(650\pi t) + \sin(400\pi t) + \sin(800\pi t) \right].$$

We now summarize the implemented stiffness for our numerical simulations. We simulated the plant (10) with

$$EI(t,x,w_{xx}(t,x)) = 15 w_{xx}(t,x) \qquad 0 \le t < 2,$$

$$\begin{aligned} &EI(t,x,w_{xx}(t,x)) \\ &= \; 15 w_{xx}(t,x) + \begin{cases} 0 w_{xx}(t,x)\, \chi_{[0.30,0.36]}(x) & \text{if } w_{xx}(t,x) > 0 \\ -5 w_{xx}(t,x)\, \chi_{[0.30,0.36]}(x) & \text{otherwise} \end{cases} \quad 2 \le t \le 5. \end{aligned}$$

The above integrals for the matrices and input vectors were computed numerically using a Gauss quadrature. Both the plant and the state estimator were approximated using a 16 cubic spline finite element method. They were also integrated using the ODE solver `rkf45`, a Fehlberg fourth-fifth order Runge–Kutta method solver.

We run two sets of simulations, namely one where the state initial conditions are assumed known and another one where the plant initial conditions were unknown. The latter, is often encountered in actual cases as it is seldom the case that initial conditions are known exactly. This in a way, tests the robustness of the adaptive estimator.

Case (i): Zero initial conditions of the state error. In this part, we simulated the plant and its estimator with the same initial conditions, namely

$$w(0, x) = v(0, x) = 2 \times 10^{-3} x^2 (x - l)^2$$
$$w(0, x) = v(0, x) = 1 \times 10^{-2} \sin^2(2\pi x/l) \cos(2\pi x/l).$$

This stiffness simulates a plant that initially ($0 \leq t < 2$) has a linear stiffness parameter that becomes nonlinear for $2 \leq t \leq 5$ and assumes different values depending on the sign of the curvature ($\alpha(t) = 1$ if $w_{xx} > 0$). In Figure 5.3 we plot the actual (dashed) values of the parameters ($EI_1 = EI_2 = 0$, for $t < 2$, $EI_1 = 0$, $EI_2 = -5$, for $t \geq 2$), and their estimates (solid). We observe that both parameters $\widehat{EI}_1$ and $\widehat{EI}_2$ are identified and that the time ($t = 2$) that the nonlinearity occurs is *sensed* by the estimator. In addition, the evolution of the state error is depicted in Figure 5.4. There, we observe that both $e_{xx}(t)$ and $e_t(t)$ assume a large value at $t = 2$ and then converge to zero around $t = 3$ seconds.

Fig. 5.3. Evolution of the parameter estimates $\widehat{EI}_1(t)$ and $\widehat{EI}_2(t)$.

Fig. 5.4. Evolution of state errors $|e(t,x)|_{H_0^2}$ and $|e_t(t,x)|_{L^2}$.

Case (ii): Non-zero initial conditions of the state error. In this case, we tested the robustness of the scheme by using non-zero initial conditions for the state error (i.e $e(0) = v(0) - w(0) = 0 - w(0) \neq 0$). This was done by using zero initial conditions of the state estimator while the plant had the same initial conditions as above. The convergence of the state error to zero prior to the failure is not guaranteed by Theorem 4.1. Similarly, the parameter errors for $t < t^*$ are not zero. Recalling Theorem 4.1 we have that for $t < t^*$ we have

$$|e(t)|_{H_0^2}^2 + |e_t(t)|_{L^2}^2 + r_1^2(t) + r_2^2(t) + \rho \int_0^t \left\{ |e(\tau)|_{H_0^2}^2 + |e_\tau(\tau)|_{H_0^2}^2 \right\} \mathrm{d}\tau$$

$$\leq \quad \sigma \left\{ |e(0)|_{H_0^2}^2 + |e_t(0)|_{L^2}^2 \right\} \neq 0$$

This can be observed in Figures 5.5 and 5.6, where we plot the time evolution of the two parameters r_1, r_2 and the state errors. In Figure 5.5 we observe that initially, both parameters start at non zero value, converge to zero around $t = 0.5$ sec and for $t \geq 2$ they start converging to the true values.

In Figure 5.6 we only included the same y-axis as in Figure 5.4 so that Figures 5.4 and 5.6 can be compared on the same axes. It is noted in Figure 5.6 that for $t < 0.5$ both $|e(t)|_{H_0^2}$ and $|e_t(t)|_{L^2}$ are non-zero, remain at zero for $0.5 \leq t < 2$, assume non-zero value at the failure time $t^* = 2$ and converge again to zero around $t = 3$ sec.

Fig. 5.5. Evolution of the parameter estimates $\widehat{EI}_1(t)$ and $\widehat{EI}_2(t)$.

Fig. 5.6. Evolution of state errors $|e(t,x)|_{H_0^2}$ and $|e_t(t,x)|_{L^2}$.

6. COMMENTS AND FUTURE EXTENSIONS

A simple model for a nonlinear stiffness in an Euler–Bernoulli beam with Kelvin–Voigt viscoelastic damping was utilized to test an online detection scheme. The failure was actually modeled as a nonlinear function of the beam's stiffness occur-

ring abruptly at some unknown failure time t^*. A state observer in the form of an adaptive estimator was used to first monitor any changes in the system's dynamics, and thus identifying implicitly the failure time t^*, and second, to identify the nonlinear stiffness parameters. The proposed state estimator assumed the same initial conditions as the plant (beam) and *sensed* the failure time t^* (time when stiffness abruptly changed from linear to nonlinear function). In addition it identified the nonlinear parameters. When the state estimator was simulated with different initial conditions, it still detected the failure time t^*. This in a way tested the robustness of the state estimator with respect to initial conditions. In this case, the state error already converged to zero much before the abrupt change in the system's dynamics occurred. Of course, if the change occurs before this pre-failure state error evolution has settled down, we *may not* see convergence of the state error with nonzero initial conditions. Further numerical results are needed in this regard.

One might choose the tuning parameters EI^* and $c_D I^*$ to affect the convergence properties of both $|e(t)|_{H_0^2}$ and $e_t(t)|_{L^2}$, see [26]. If for certain values of the tuning parameters, both errors $\big(|e(t)|_{H_0^2}$ and $|e_t(t)|_{L^2}\big)$ are not converged to zero prior to the failure time t^*, then the estimator might not be able to sense the time of failure t^*. This robustness property is currently investigated along with some more general cases of estimators which can not only identify the stiffness parameters EI_1 and EI_2, but also the location that the nonlinearities act on the beam (i. e. the length of the characteristic function) and even the distribution. In this case, the nonlinear part of the stiffness parameter might be given by

$$
\begin{aligned}
& EI(t, x, w_{xx}(t, x)) \\
= \quad & \chi_{[x_1, x_2]}(x) \cdot \left[EI_1(x) w_{xx}(t, x)\, H(w_{xx}) + EI_2(x) w_{xx}(t, x)\, (1 - H(w_{xx})) \right]
\end{aligned}
$$

where the parameters EI_1 and EI_2 are not constants, but functions of the spatial variable x that vanish outside the interval $[x_1, x_2]$. A further goal is to propose an estimator that will also identify the interval $[x_1, x_2]$ and $EI_1(x)$, $EI_2(x)$.

The persistence of excitation condition, needed for parameter convergence, might be hard to check in systems governed by nonlinear hyperbolic p.d.e.'s. For the linear case this was presented in [26], but even in linear systems with spatially varying parameters this condition might be difficult to prove. Perhaps, by taking advantage of the nonlinear nature of the system, the persistence of excitation condition given by Definition 4.4 might be something that can be concluded by imposing simple conditions on the input patch voltage. This warrants additional theoretical studies.

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