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On the inverse problem of rotation moment invariants

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1. Introduction

Numerous papers have been devoted to the moment invariants (MI). They usually describe their derivation, various invariant properties, numerical behavior, and their power to serve as the features in many pattern recognition tasks. Only few papers have dealt with the independence and completeness of the sets of MI's. However, both these properties are of fundamental importance from theoretical as well as practical point of view. In our recent paper [1] we presented a general approach to deriving rotation moment invariants. MI's were constructed as products of appropriate powers of complex moments. We also proposed how to construct the basis (independent and complete set) of the invariants of this kind.

In this paper we prove stronger theorem. We show the basis described in Ref. [1] is a basis of *all possible* rotation moment invariants (not only of those constructed as products of moment powers). In other words, knowing the invariants from this basis we can calculate for instance traditional Hu moment invariants [2], Zernike moment invariants [3], and Fourier–Mellin invariants [4], to name a few. This theorem can be equivalently formulated as the solution of the inverse problem. We show in this paper that all moments can be recovered from the invariant basis.

2. Recalling the construction of the basis

In this section we briefly recall the construction of rotation moment invariants based on complex moments and the definition of their basis as it was introduced in Ref. [1].

Complex moment c_{pq} of order $(p+q)$ of image $f(x,y)$ is defined as

$$c_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+\mathrm{i}y)^p (x-\mathrm{i}y)^q f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \qquad (1)$$

where i denotes the imaginary unit. Each complex moment can be expressed in terms of geometric moments m_{pq} as

$$c_{pq} = \sum_{k=0}^{p}\sum_{j=0}^{q} \binom{p}{k}\binom{q}{j}(-1)^{q-j} \times \mathrm{i}^{p+q-k-j} m_{k+j,\,p+q-k-j}. \qquad (2)$$

In polar coordinates, Eq. (1) becomes the form

$$c_{pq} = \int_{0}^{\infty}\int_{0}^{2\pi} r^{p+q+1} \mathrm{e}^{\mathrm{i}(p-q)\theta} f(r,\theta)\,\mathrm{d}r\,\mathrm{d}\theta. \qquad (3)$$

It follows immediately from Eq. (3) that $c_{pq} = c_{qp}^*$ (the asterisk denotes the complex conjugation). When image f is rotated, the magnitude of each complex moment is preserved while its phase is shifted by $(p-q)\alpha$, where α is the angle of rotation:

$$c'_{pq} = \mathrm{e}^{-\mathrm{i}(p-q)\alpha} c_{pq}. \qquad (4)$$

Using the above rotation property of the complex moments, rotation invariants can be easily constructed as the products of appropriate moment powers:

$$I = \prod_{i=1}^{n} c_{p_i q_i}^{k_i}, \qquad (5)$$

where $n \geqslant 1$ and k_i, p_i and $q_i; i = 1,\ldots,n$, are non-negative integers such that

$$\sum_{i=1}^{n} k_i(p_i - q_i) = 0.$$

The invariants (5) are in general complex valued. If we want to have real-valued features, we only take real and imaginary parts of each of them (or, equivalently, the magnitude and phase).

Eq. (5) allows us to construct an infinite number of the invariants for any order of moments, but most of them are dependent, i.e. they are functions of the others. By the term *basis* we intuitively understand the smallest set by means of which all other invariants can be expressed. The knowledge

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of the basis is a crucial point in all pattern recognition problems because it provides the same discriminative power as the set of all invariants at minimum computational costs.

In Ref. [1], the following basis $\mathscr{B}$ of the rotation invariants created from the complex moments up to the order r according to Eq. (5) was proposed:

$$(\forall p,q \mid p \geqslant q \wedge p+q \leqslant r)(\Phi(p,q) \equiv c_{pq}c_{q_0p_0}^{p-q} \in \mathscr{B}),$$

where p_0 and q_0 are arbitrary indices such that $p_0+q_0 \leqslant r$ and $p_0-q_0=1$. (Note that the basis is generally not unique, it depends on the choice of p_0 and q_0.)

The basic invariants $\Phi(p,q)$'s were proven to be mutually independent and, on the other hand, any invariant of the type (5) can be expressed as a certain function of the basic invariants.

3. The inverse problem

The above-mentioned theorem proves the completeness of the basis $\mathscr{B}$ only within a certain class of invariants. It has not been proven whether or not *any* moment-based rotation invariant, regardless of its particular type, is a function of $\mathscr{B}$. For answering this question it is sufficient to resolve so-called *inverse problem*, which means to recover all geometric moments up to the order r when knowing the basis $\mathscr{B}$. It can be done in two steps—we recover complex moments from the invariants and then geometric moments are calculated from the complex moments.

In the first step, the following non-linear system of equations must be resolved for the c_{pq}'s:

$$\begin{aligned}
\Phi(p_0,q_0) &= c_{p_0q_0}c_{q_0p_0},\\
\Phi(0,0) &= c_{00},\\
\Phi(1,0) &= c_{10}c_{q_0p_0},\\
\Phi(2,0) &= c_{20}c_{q_0p_0}^2,\\
\Phi(1,1) &= c_{11},\\
\Phi(3,0) &= c_{30}c_{q_0p_0}^3,\\
&\vdots\\
\Phi(r,0) &= c_{r0}c_{q_0p_0}^r,\\
\Phi(r-1,1) &= c_{r-1,1}c_{q_0p_0}^{r-2},\\
&\vdots
\end{aligned} \tag{6}$$

Since $\mathscr{B}$ is a set of rotation invariants, it does not reflect the orientation of the object. Thus, there is one degree of freedom when recovering the object moments which corresponds to the choice of the object orientation. Without loss of generality, we can choose such orientation in which $c_{p_0q_0}$ is real and positive. (An alternative, perhaps more intuitive, option is to set c_{20} real and positive. It leads to the traditional "standard position" of the object in which $m_{11}=0$ and $m_{20}>m_{02}$.) The first equation of Eq. (6) can be then immediately resolved for $c_{p_0q_0}$:

$$c_{p_0q_0} = \sqrt{\Phi(p_0,q_0)}.$$

Consequently, using the relationship $c_{q_0p_0} = c_{p_0q_0}$, we get the solutions

$$c_{pq} = \frac{\Phi(p,q)}{c_{q_0p_0}^{p-q}}$$

and

$$c_{pp} = \Phi(p,p)$$

for any p and q.

Recovering the geometric moments is based on Eq. (2), which forms in fact a linear system of $(r+1)(r+2)/2$ equations. This system can be decomposed into $r+1$ mutually independent subsystems, where the kth subsystem contains the moments of the kth order only. The number of equations in each subsystem is further reduced to one half due to the relation $c_{pq}=c_{qp}^*$. The simplified subsystems can be easily resolved for the geometric moments.

As an example we show the solution of the inverse problem up to the third order. When choosing $p_0=2$ and $q_0=1$, Eq. (6) becomes the form

$$\begin{aligned}
\Phi(2,1) &= c_{21}c_{12},\\
\Phi(0,0) &= c_{00},\\
\Phi(1,0) &= c_{10}c_{12},\\
\Phi(2,0) &= c_{20}c_{12}^2,\\
\Phi(1,1) &= c_{11},\\
\Phi(3,0) &= c_{30}c_{12}^3.
\end{aligned} \tag{7}$$

According to the procedure given above we get the complex moments

$$\begin{aligned}
c_{21} &= \sqrt{\Phi(2,1)} = c_{12},\\
c_{00} &= \Phi(0,0),\\
c_{10} &= \frac{\Phi(1,0)}{c_{12}} = c_{01}^*,\\
c_{20} &= \frac{\Phi(2,0)}{c_{12}^2} = c_{02}^*,\\
c_{11} &= \Phi(1,1),\\
c_{30} &= \frac{\Phi(3,0)}{c_{12}^3} = c_{03}^*.
\end{aligned} \tag{8}$$

Recovering the geometric moments is trivial for the zero and first orders:

$$m_{00} = c_{00},$$

$$m_{10} = \text{Re}(c_{10}),$$

$$m_{01} = \text{Im}(c_{10}). \quad (9)$$

For the second-order moments we get

$$m_{20} = \frac{1}{2}\,\text{Re}(c_{20} + c_{11}),$$

$$m_{11} = \frac{1}{2}\,\text{Im}(c_{20}),$$

$$m_{02} = \frac{1}{2}\,\text{Re}(c_{11} - c_{20}) \quad (10)$$

and, finally, for the third-order moments we obtain (when incorporating the chosen normalization constraint $\text{Im}(c_{21}) = 0$)

$$m_{30} = \frac{1}{4}\text{Re}(c_{30} + 3c_{21}),$$

$$m_{21} = \frac{1}{4}\text{Im}(c_{30}),$$

$$m_{12} = \frac{1}{4}\text{Re}(c_{21} - c_{30}),$$

$$m_{03} = -\frac{1}{4}\text{Im}(c_{30}). \quad (11)$$

It is worth noting that $\Phi(0,0)$ and $\Phi(1,0)$ are useless in practice because c_{00} is often used for normalization to scaling and $c_{10} = m_{10} + \text{i}m_{01}$ is used to achieve translation invariance.

4. Conclusion

In this paper, the full completeness of the basis of rotation moment invariants proposed earlier has been proven. We have shown that all complex moments (and, consequently, all geometric moments) up to the given order can be recovered from the basis. Thus, any moment-based rotation invariant can be expressed by means of the basic rotation invariants $\Phi(p,q)$. This result is a generalization of the theorem from our recent paper [1].

It should be pointed out that the basis is by no means unique. All considerations about the basis construction, its independence and completeness, which were here based on complex moments, can be analogously done when replacing complex moments by Zernike moments, Fourier–Mellin moments, or by any other type of moments having similar behavior under rotation as the complex moments. Such replacing will lead to different bases with similar structures and properties.

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References

[1] J. Flusser, On the independence of rotation moment invariants, Pattern Recognition 33 (2000) 1405–1410.

[2] M.K. Hu, Visual pattern recognition by moment invariants, IRE Trans. Inform. Theory 8 (1962) 179–187.

[3] M.R. Teague, Image analysis via the general theory of moments, J. Opt. Soc. Am. 70 (1980) 920–930.

[4] Y. Sheng, J. Duvernoy, Circular-Fourier—Radial-Mellin descriptors (FMD's) for pattern recognition, J. Opt. Soc. Am A 3 (1986) 885–888.