Point Projective and Permutation Invariants

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Abstract. The paper deals with features of a point set which are invariant with respect to projective transform. First, projective invariants for five-point sets, which are simultaneously invariant to the permutation of the points, are derived. They are expressed as functions of five-point cross-ratios. Possibilities of the choice of their roots are referred and their normalization is discussed.
Then, the invariants for more than five points are derived. Stability and discriminability of the features is demonstrated by numerical experiments.

1 Introduction

One of the important tasks in image processing and computer vision is a recognition of objects on images captured under different viewing angles. If the scene is planar, then the distortion between two frames can be described by *projective transform*

$$\begin{aligned} x' &= (a_0 + a_1 x + a_2 y)/(1 + c_1 x + c_2 y) \\ y' &= (b_0 + b_1 x + b_2 y)/(1 + c_1 x + c_2 y) \ , \end{aligned} \tag{1}$$

where x, y are the coordinates in the first frame and x', y' are the coordinates in the second one.

This contribution deals with point-based features for recognition of projectively deformed point sets. The points must be labeled by some way, but the correspondence between the points can be unknown and then the correct labeling of the points is also unknown. Therefore the features are designed as invariant to the choice of labeling of the points, i.e. to the permutation of the points.

The plane transformed by the projective transform contains a straight line, which is not mapped into the second plane (more precisely it is mapped into the infinity) and which divides the plane into two parts. If the transform has form (1), then it is the line

$$1 + c_1 x + c_2 y = 0 \tag{2}$$

If all points (the whole point set) lay in one part of the plane, then some additional theorems dealing with topology of the set holds for the transform, e.g.

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the convex hull is preserved during the projective transform in that case. It can be used for construction of invariants with less computational complexity [1].

This paper deals with the general case of the projective transform, when the points can lay in both parts of the plane, the convex hull is not preserved and all possibilities of the positions of the points must be taken care of. The only constraint is that the points do not lay directly at the straight line (2).

A projective invariant can be defined for five points minimally. The simplest invariant is a five-point cross-ratio

$$\varrho(1,2,3,4,5) = \frac{P(1,2,3)P(1,4,5)}{P(1,2,4)P(1,3,5)}, \tag{3}$$

where $P(1,2,3)$ is the area of the triangle with vertices 1,2 and 3.

The work [2] deals with five-point projective and permutation invariants. It proposes to use median of all possible values. The more precise description of relations between various values of the cross-ratio during permutation of the five points can be found in [3]. Our first attempt to solve this problem was published in [4].

2 Five-point permutation invariants

The five-point cross-ratio (3) can be used for construction of the invariants to the choice of labeling.

2.1 The derivation of the invariants

Reiss in [2] uses the function $\varrho + \varrho^{-1}$, which is unstable near zero. If some triplet of five points in (3) is collinear, then the function $\varrho + \varrho^{-1}$ is infinite. Thus the more suitable function is $\psi = \frac{2}{\varrho + \varrho^{-1}} = \frac{2\varrho}{\varrho^2+1}$. If ϱ or ϱ^{-1} is zero, then the function ψ is zero.

If invariance to the choice of labeling of the points is required, some symmetric function must be used for considering all possibilities of labeling. We can use additions and multiplications

$$\begin{aligned} F'_{+}(1,2,3,4,5) &= \psi(1,2,3,4,5) + \psi(1,2,3,5,4) + \psi(1,2,4,5,3), \\ F'_{\cdot}(1,2,3,4,5) &= \psi(1,2,3,4,5) \cdot \psi(1,2,3,5,4) \cdot \psi(1,2,4,5,3). \end{aligned} \tag{4}$$

It is suitable to use e.g. the function $F_{+} = 16(\frac{1}{5} - \frac{1}{6-F'_{+}})$ instead of F'_{+} because of higher discriminability of the invariants and normalize $F'_{\cdot}$ by the constant $F_{\cdot} = -\frac{3}{8}F'_{\cdot}$ to the interval from 0 to 1. Functions F_{+} and $F_{\cdot}$ are invariant to the choice of labeling of last four points, but the point 1 must be common at all cross-ratios. To obtain full invariance to the choice of labeling, we must alternate all five points as common ones

$$\begin{aligned} I_{s_1,s_2}(1,2,3,4,5) = {} & F_{s_1}(1,2,3,4,5) s_2 F_{s_1}(2,3,4,5,1) s_2 F_{s_1}(3,4,5,1,2) s_2 \\ & s_2 F_{s_1}(4,5,1,2,3) s_2 F_{s_1}(5,1,2,3,4), \end{aligned} \tag{5}$$

where s_1 and s_2 are either sign $+$ or $\cdot$.

The set of n points has $2n$ degrees of freedom and projective transform has 8 parameters. Therefore the set can have only

$$m = 2n - 8 \tag{6}$$

independent invariants to projective transform. That is why only two from four invariants I_{++}, $I_{\cdot+}$, $I_{+\cdot}$ and $I_{\cdot\cdot}$ can be independent.

2.2 The roots of the invariants

Lenz and Meer [3] deal with the five-point projective and permutation invariants in detail. They discovered that if the common point stays the same and the other points are permuted, then the values of the cross-ratios are

$$\varrho_1 = \varrho,\ \varrho_2 = \frac{1}{\varrho},\ \varrho_3 = 1 - \varrho,\ \varrho_4 = \frac{1}{1-\varrho}, \varrho_5 = \frac{\varrho}{\varrho - 1},\ \varrho_6 = \frac{\varrho - 1}{\varrho}. \tag{7}$$

If we construct a function $F(\varrho)$, which has the same value for all these values of ϱ $(\varrho_1, \varrho_2, \cdots, \varrho_6)$, it is invariant to the permutation of four points. If we change the common point, we receive another value of the cross-ratio, let us say σ, and a function

$$K(\varrho, \sigma) = F(\varrho) + F(\sigma) + F(\frac{\varrho}{\sigma}) + F(\frac{\varrho - 1}{\sigma - 1}) + F(\frac{\varrho(\sigma - 1)}{\sigma(\varrho - 1)}) \tag{8}$$

is the five-point projective and permutation invariant.

As our research implies, if the function $F(\varrho)$ is constructed as the quotient of two polynomials and ϱ is its root, then each of the values $\varrho_1, \varrho_2, \cdots, \varrho_6$ must be its root. We can consider it in form

$$F(\varrho) = \frac{P_1(\varrho)}{P_2(\varrho)}, \tag{9}$$

$$P_1(\varrho) = (\varrho - \varrho_1)(\varrho - \varrho_2)(\varrho - \varrho_3)(\varrho - \varrho_4)(\varrho - \varrho_5)(\varrho - \varrho_6) \tag{10}$$

and similarly $P_2(\varrho)$ (when all roots differ from zero).

It is useful in practice, when the invariants are defined everywhere. It means the polynomial $P_2(\varrho)$ in the denominator has only imaginary roots, because $F(\varrho)$ is undefined at points of real roots of $P_2(\varrho)$. There are proposed two such invariants in [3]

$$F_{14} = \frac{2\varrho^6 - 6\varrho^5 + 9\varrho^4 - 8\varrho^3 + 9\varrho^2 - 6\varrho + 2}{\varrho^6 - 3\varrho^5 + 3\varrho^4 - \varrho^3 + 3\varrho^2 - 3\varrho + 1}, \tag{11}$$

$$F_{15} = \frac{(\varrho^2 - \varrho + 1)^3}{\varrho^6 - 3\varrho^5 + 5\varrho^3 - 3\varrho + 1}. \tag{12}$$

We can express $P_2(\varrho)$ in form

$$\begin{aligned} P_2(\varrho) = {} & (\varrho - a_1 - b_1\mathrm{i})(\varrho - a_1 + b_1\mathrm{i})(\varrho - a_2 - b_2\mathrm{i}) \\ & (\varrho - a_2 + b_2\mathrm{i})(\varrho - a_3 - b_3\mathrm{i})(\varrho - a_3 + b_3\mathrm{i}) \end{aligned} \tag{13}$$

and we can derive from (7) that the following conditions must be satisfied for roots of $P_2(\varrho)$

$$
\begin{aligned}
a_1^2 + b_1^2 &= 1, \\
(a_2-1)^2 + b_2^2 &= 1,\ a_2 = 1 - a_1,\ b_2 = b_1, \\
a_3 &= \tfrac{1}{2},\ b_3 = \tfrac{b_1}{(a_1-1)^2+b_1^2} = \tfrac{b_2}{a_2^2+b_2^2},
\end{aligned}
\tag{14}
$$

see Fig. 1 left. The order of the roots can be changed. Our invariants has form in this context

$$
\begin{aligned}
F'_+ &= \tfrac{2\varrho}{\varrho^2+1} + \tfrac{2(1-\varrho)}{(1-\varrho)^2+1} + \tfrac{2\frac{\varrho}{\varrho-1}}{\frac{\varrho^2}{(\varrho-1)^2}+1} = \\
&= 2\tfrac{\varrho^6-3\varrho^5+3\varrho^4-\varrho^3+3\varrho^2-3\varrho+1}{2\varrho^6-6\varrho^5+11\varrho^4-12\varrho^3+11\varrho^2-6\varrho+2}, \\
F_{\cdot} &= -\tfrac{3}{8}\tfrac{(-8)\varrho^2(1-\varrho)^2}{(1+\varrho^2)(2-2\varrho+\varrho^2)(1-2\varrho+2\varrho^2)} = \\
&= \tfrac{3\varrho^2(1-\varrho)^2}{2\varrho^6-6\varrho^5+11\varrho^4-12\varrho^3+11\varrho^2-6\varrho+2}, \\
F_+ &= 16\left(\tfrac{1}{5} - \tfrac{1}{6-F'_+}\right) = \tfrac{8}{5}\tfrac{\varrho^2(1-\varrho)^2}{(\varrho^2-\varrho+1)^3}.
\end{aligned}
\tag{15}
$$

The roots of the numerators of the invariants are 0 and 1, that of the denominator of $F_{\cdot}$ are $1\pm\mathrm{i}$, $\pm\mathrm{i}$ and $0.5\pm0.5\mathrm{i}$ and the roots of the denominator of F_+ are $0.5\pm\mathrm{i}\frac{\sqrt{3}}{2}$.

The function F_+ was chosen instead of F'_+ because of its different denominator than $F_{\cdot}$, what promises better discriminability of the invariants. The invariants F_+ and $F_{\cdot}$ were chosen because their roots are farther from the imaginary axis than that of F_{14} and F_{15} and therefore they promise to be more stable and also generalization for more than 5 points is easier.

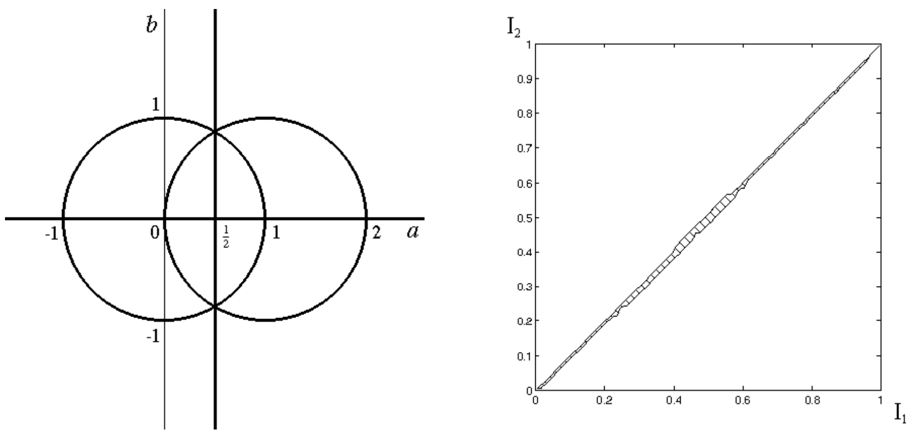

Fig. 1. Possible values of roots (left) and the invariants I_1, I_2 (right)

2.3 The normalization of the invariants

The invariants $I_1 = I_{++}$ and $I_2 = I_{.+}$ corresponding the functions F_+ and $F_.$ utilize the space ineffectively (see Fig. 1 right).

A better result can be reached as the sum and the difference (see Fig. 2 left)

$$I'_1 = (I_1 + I_2)/2, \qquad I'_2 = (I_1 - I_2 + 0.006) \cdot 53, \tag{16}$$

but good utilization of the given range is reached by subtraction and division

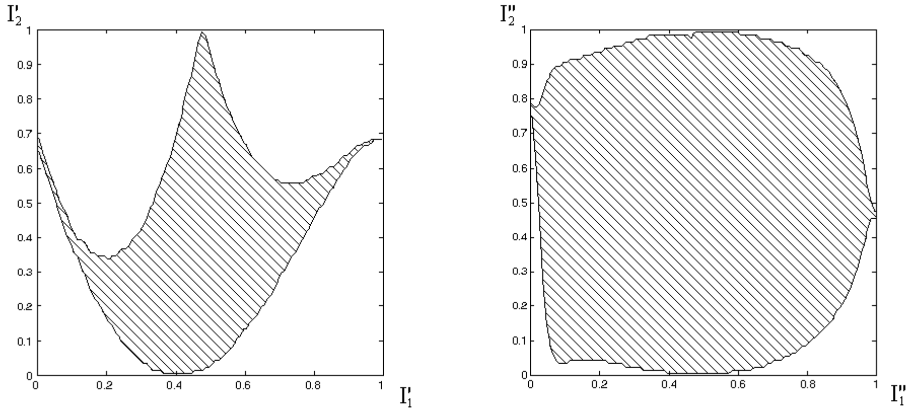

Fig. 2. Possible values of the invariants I'_1, I'_2 (left) and I''_1, I''_2 (right)

by the polynomials (see Fig. 2 right)

$$I''_1 = I'_1, \qquad I''_2 = \frac{1 - I'_2 + p(I'_1)}{d(I'_1)}, \tag{17}$$

where $p(I'_1) = 10.110488 \cdot {I'_1}^6 - 27.936483 \cdot {I'_1}^5 + 31.596612 \cdot {I'_1}^4 - 16.504259 \cdot {I'_1}^3 - 0.32251158 \cdot {I'_1}^2 + 3.0473587 \cdot I'_1 - 0.66901966$. If $I'_1 < 0.475$ then $d(I'_1) = 17.575974 \cdot {I'_1}^4 - 16.423212 \cdot {I'_1}^3 + 9.111527 \cdot {I'_1}^2 - 0.43942294 \cdot I'_1 + 0.016542258$ else $d(I'_1) = 3.9630392 \cdot {I'_1}^4 - 13.941518 \cdot {I'_1}^3 + 21.672754 \cdot {I'_1}^2 - 17.304971 \cdot I'_1 + 5.6198814$.

3 Invariants for more than five points

There is a number of solutions of the problem of the generalization of the invariants from the previous section to n points ($n > 5$). One of them with good experimental results can be addition of powers of the invariants I''_1, I''_2 over all possible combinations of 5 from n points C_5^n

$$I_{1,k} = \sum_{Q \in C_5^n} {I''_1}^k(Q), \qquad I_{2,k} = \sum_{Q \in C_5^n} {I''_2}^k(Q), \qquad k = 1, 2, \ldots, n-4. \tag{18}$$

The number of these invariants is chosen as $2n-8$ according to (6). The computing complexity is approximately $\binom{n}{5}T$, i.e. $O(n^5)$, where T is the computing complexity of one five-point invariant. This result is 5!=120 times faster than the solution in [4]. The number of added values is very high and that is why a normalization of the results is suitable. To preserve values inside acceptable intervals, we can use

$$I'_{s,k} = \sqrt[k]{\frac{1}{\binom{n}{5}} \sum_{Q \in C_5^n} I''^k_s(Q)}, \tag{19}$$

where $s = 1$ or $s = 2$.

Another, may be more sophisticated, normalization is following. We can suppose the values of the five-point invariants I''_1, I''_2 as random with approximately uniform distribution on the interval from 0 to 1. Then the distribution function $F_k(x)$ of the k-th power of the invariant is

$$\begin{array}{l} F_k(x) = 0 \quad \text{from } -\infty \text{ to } 0 \\ F_k(x) = x^{\frac{1}{k}} \text{ from } 0 \text{ to } 1 \\ F_k(x) = 1 \quad \text{from } 1 \text{ to } \infty. \end{array} \tag{20}$$

The mean value $\mu_k = \frac{1}{1+k}$ and the variance $\sigma_k^2 = \frac{k^2}{(1+k)^2(1+2k)}$. The number of summands in the sum (18) is relatively high and the central limit theorem imply the distribution of the sum is approximately Gauss, the mean value is the sum of the mean values μ_k and the variance is the sum of the variances σ_k^2. The given range is the best utilized in case of uniform distribution of the resulting invariants and therefore we can normalize the invariants with Gauss distribution function

$$G(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(\xi-\mu)^2}{2\sigma^2}} d\xi \tag{21}$$

with the mean value $\mu = \binom{n}{5}\frac{1}{1+k}$, the variance $\sigma^2 = \binom{n}{5}\frac{k^2}{(1+k)^2(1+2k)}$ and the standard deviation $\sigma = \frac{k}{1+k}\sqrt{\binom{n}{5}\frac{1}{1+2k}}$

$$I''_{s,k} = G(I_{s,k}; \mu, \sigma). \tag{22}$$

An approximate substitution of the function G is used in practice.

4 Numerical experiment

The following numerical experiment was carried out to observe the stability and discriminability of the invariants (19) and (22).

A set of 11 points was created by a pseudo-random generator. They are uniformly distributed from 0 to 511. Independent noise with Gauss distribution with zero mean value and gradually increasing standard deviation was added to the coordinates both of all points and of one point. In the first experiment the standard deviation σ_2 increased from 0 to 9.5 with the step 0.5, in the second one σ_1 increased from 0 to 190 with the step 10. The distances d between original and noisy set in the space of the invariants were computed. To obtain a curve of dependency of the distance on the noise standard deviation the averages of many experiments were used. The average of gradually 10, 20, 100 and 1000 experiments was used and only the curve from 1000 values was little enough dependent on the concrete realization of the noise. The result of the invariants (19) normalized by the average and by the root is in Fig. 3 left.

The scale of the horizontal axis is different for both cases so the curves were as similar as possible. The ratio of the scales is 7.36; it means if two sets differ by only one point, then the distance of the points must be at least approximately 7.36 times greater than the noise standard deviation to be possible to distinguish both sets. If the noise standard deviation is greater than approximately 9, i.e. about 2 % of the coordinate range, then the sets cannot be distinguished at all.

The results of the invariants (22) normalized by the Gauss distribution function is in Fig. 3 right.

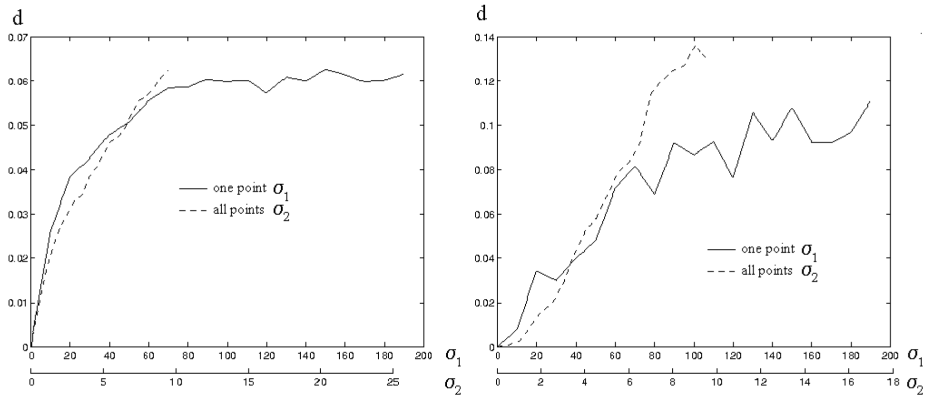

Fig. 3. The normalization by means of the average and the root (left) and by means of the Gauss distribution function (right)

The result is similar to previous case, the ratio of the scales is 11.18, that means little bit worse discriminability. The main difference is the scale on the vertical axis, which is about twice greater. It means these invariants utilize the given range better.

5 Summary

The features for recognition of the projectively deformed point sets were derived. They are invariant simultaneously with respect to projective transform and to the permutation of the points. They are expressed as functions of five-point cross-ratios. Possibilities of the choice of the roots of the polynomials in the denominators of the invariants and the normalization of these invariants are shown. The generalization for more than five points is made and their stability is studied. The performance of the features is demonstrated by numerical experiments.

6 References

1. N. S. V. Rao, W. Wu, C. W. Glover, "Algorithms for Recognizing Planar Polygonal Configurations Using Perspective Images", *IEEE Trans. on Robotics and Automation*, Vol. 8, No. 4, pp. 480-486, 1992.
2. T. H. Reiss, "Recognition Planar Objects Using Invariant Image Features", Lecture Notes in Computer Science 676, Springer, 1993.
3. R. Lenz, P. Meer, "Point configuration invariants under simultaneous projective and permutation transformations", *Pattern Recognition*, Vol. 27, No. 11, pp. 1523-1532, 1994.
4. T. Suk, J. Flusser, "The Features for Recognition of Projectively Deformed Images", *Proc. ICIP'95*, Washington, Vol. III, pp. 348-351, 1995.

7 Appendix

Sometimes a task how to save and load information about combinations to and from a memory can be need to solve. We have got the combinations k elements from n and we can save this information by following way:

$a = 0$
for $i_1 := 0; i_1 < n$
for $i_2 := i_1 + 1; i_2 < n$
$\vdots$
for $i_k := i_{k-1} + 1; i_k < n$
$\{\ m[a] :=$ information $(i_1, i_2, \ldots, i_k)$
$a := a + 1\ \}$

and then we need to load the information about k-tuple $(i_1, i_2, \ldots, i_k)$. That is why we need to compute the address a from the k-tuple. If we sort the indices by size so it holds

$$i_1 < i_2 < \ldots < i_k$$

then this address can be computed as

$$a = \binom{n}{k} - 1 + \sum_{j=1}^{k} \sum_{m=0}^{k-j+1} (-1)^{m+1} \binom{n}{k-j-m+1} \binom{i_j + m}{m}. \tag{23}$$