Cusp Catastrophe Theory: Application to U.S. Stock Markets

Jozef Baruník[1,2] , Miloslav Vošvrda[1,2]
[1]Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Prague
[2]Institute of Economic Studies, Faculty of Social Sciences, Charles University, Prague

Extended abstract

We show that the cusp catastrophe model explains the crash of stock exchanges much better than alternative linear and logistic models. On the data of U.S. stock markets we demonstrate that the crash of October 19, 1987 may be better explained by cusp catastrophe theory, which is not true for the crash of Sept. 11, 2001. With the help of sentiment measures, such as index put/call options ratio and trading volume (the former models the chartists, while the latter the fundamentalists), we have found that the 1987 returns are clearly bimodal and contain bifurcation flags. The cusp catastrophe model fits these data better than alternative models. Therefore we may say that the crash may have been led by internal forces. However, the causes for the crash of 2001 are external, which is also evident in much weaker presence of bifurcations in the data. Thus alternative models may be used for its explanation.

Keywords

cusp catastrophe, bifurcations, singularity, nonlinear dynamics, stock market crash

JEL C01, C53

1. Introduction

Unexpected stock market crashes has been a nightmare for the financial world ever since the capital market existed. The catastrophe theory attempts to unfold a part of information we might need to understand the crash phenomenon. It describes how small, continuous changes in control parameters, or independent variables influencing the state of the system, can have sudden, discontinuous effects on dependent variables. In the paper, we apply the theory to sudden stock market changes that are known as crashes. Zeeman [7] was the first to qualitatively describe the *"unstable behavior of stock exchanges"* by Thom [5] catastrophe theory. We extend his ideas by incorporating quantitative analysis.

The article is rather empirical as it puts the theory to test on financial data. As only a few papers deal with an empirical analysis of catastrophe theory, this paper may contribute to this research. We build on the Zeeman's qualitative description, and primary aim of the research is to answer the question of whether catastrophe models are capable of indicating the stock market crashes.

What we regard as the most significant aspect is testing on the real-world financial data. Our key assertion is that the cusp catastrophe model is able to fit the data more properly than an alternative linear regression model, and/or nonlinear (logistic) model. We fit the catastrophe model to the data of October 19, 1987 crash, known as Black Monday which was the greatest single-day loss (31%) that Wall Street has ever suffered in continuous trading. As for comparison, we use another large crash, that of September 11, 2001. The final part is devoted to the assumption that while in 1987 the crash was caused by internal forces, in 2001 it was external forces, namely 9/11 terrorist attack. Thus the catastrophe model should fit the data of 1987 well, as the bifurcations leading to instability are present. However, it does not seem to perform better than linear regression on the 2001 data. As the control variables we use the measures of sentiment, precisely OEX[1] Put/Call ratio which appears to be very good measure of the speculative money in the capital market, against trading volume as good proxy for large, fundamental investors.

2. The Cusp Catastrophe Model

Let us assume one dependent variable Y , and a set of n independent variables $\{X_1, X_2, \ldots, X_n\}$. Then y represents realization of a random variable Y, and x_i represents realizations of X_i . To

---

[1]OEX are options with the Standard & Poor's 100 Index underlying

obtain greater flexibility than using linear regression technique, $2n+2$ additional degrees of freedom are introduced. This could be done by defining control factors $\alpha_x = \alpha_0 + \alpha_1 x_1 + ... + \alpha_n x_n$ and $\beta_x = \beta_0 + \beta_1 x_1 + ... + \beta_n x_n$. These factors determine the predicted values of y given realizations of $\{x_1, x_2, \ldots, x_n\}$, meaning that for each value x there might be three predicted values of the state variable. The predictions will be roots of the following canonical form

$$0 = \alpha_x + \beta_x (y-\lambda)/\sigma - \left((y-\lambda)/\sigma\right)^3, \tag{1}$$

which describes the cusp catastrophe response surface containing a smooth pleat. λ and σ are location and scale parameters. In the literature on catastrophe theory, α_x and β_x are so called *normal* and *splitting* factors, however, we prefer the notions a*symmetry* and *bifurcation* factors, respectively. Hence the statistical estimation problem is to find the estimates for the $2n+4$ parameters: $\{\lambda, \sigma, \alpha_0, ..., \alpha_n, \beta_0, ..., \beta_n\}$ from n observations of the $n+1$ variables $\{Y, X_1, \ldots, X_n\}$.

2.1. Stochastic Dynamics and Probability Density Function (PDF)

Let y_t be the function of time t for $t \in \langle 0, T \rangle$. From a dynamic system's point of view, Equation (1) can be considered as the surface of the equilibrium points of a dynamic system of the state variable y_t which follows the ordinary differential equation $dy_t = g(x, y_t)dt$, where $g(x, y_t)$ is the right hand side of Equation (1). For real world applications, it is necessary to add a non-deterministic behavior into the system, as the system usually does not determine its next states entirely. We may obtain a stochastic form by an adding of the Gaussian white noise term[2]. The system is then described by a stochastic differential equation of the form

$$dy_t = \left(\alpha_x + \beta_x \frac{(y_t-\lambda)}{\sigma_{y_t}} - \left(\frac{(y_t-\lambda)}{\sigma_{y_t}} \right)^3 \right) dt + \sigma_{y_t} dW_t, \tag{2}$$

and $\sigma^2_{y_t}$ is an instantaneous variance of the process y_t. The W_t is a standard Wiener process and $dW_t \sim N(0, dt)$. Hartelman [4] has established a link between a deterministic function of catastrophe system and a pdf of the corresponding stochastic process. He showed that the pdf $f(y_t)$ will converge in time to a pdf $f_S(y_\infty)$ corresponding to a limiting stationary stochastic process. This has led to a definition of stochastic equilibrium state and bifurcation which is compatible with their deterministic counterpart. Instead of fitting the deterministic process where the equilibrium points of the system are of a main interest, the attention is drawn to relative extremes of the conditional density function of y. Following Wagenmarkers [6], the pdf of y is:

$$f_S(y_\infty \mid x) = \xi \exp\left[\alpha_x y_\infty + \frac{\beta_x}{2} \left(\frac{y_\infty - \lambda}{\sigma_{y_\infty}} \right)^2 - \frac{1}{4} \left(\frac{y_\infty - \lambda}{\sigma_{y_\infty}} \right)^4 \right] \tag{3}$$

The constant ξ normalizes the pdf so it has unit integral over its range. The modes and antimodes of the cusp catastrophe pdf can be obtained by solving the equation $df_S(.|.)/dy. = 0$, which will yield exactly implicit cusp surface equation - Equation (1). The parameters will be estimated by method of estimation developed by Hartelman [4], Wagenmarkers [6].

---

[2]Cobb[1], Cobb, Watson[2], Cobb, Zacks[3]

As β_x changes from negative to positive, the pdf $f_S(y_\infty|x)$ changes its shape from unimodal to bimodal. It is also the reason why the β_x factor is called *bifurcation* factor. For $\alpha_x = 0$, the pdf is symmetrical, other values control asymmetry, thus α_x is *asymmetry* factor. Thoughtful reader has certainly noted that catastrophe theory models are an extension to traditional models, therefore they have to satisfy the requirement of the empirical testability. It should be remembered, that there is no single statistical test for acceptability of the catastrophe model. Due to the multimodality of cusp catastrophe, traditional measure for goodness of fit cannot be used. Considered residuals can be determined only if the probability density function at time t is one-peaked, and as the model generally offers more than one predicted value, it is difficult to find a tractable definition for a prediction error. In testing we follow Hartelman [4] approach. A comparison of the cusp and a linear regression model is made by means of a likelihood ratio test, which is asymptotically chi-squared distributed with degrees of freedom being equal to the difference in degrees of freedom for two compared models. As it may not be sufficient to reliably distinguish between catastrophe and non-catastrophe models, Hartelman [4] compares catastrophe model also to a nonlinear logistic model. As the cusp catastrophe model and the logistic model are not nested, Akaike information criterion (AIC), and Bayesian information criterion (BIC) statistics are used in a testing routine to compare the models.

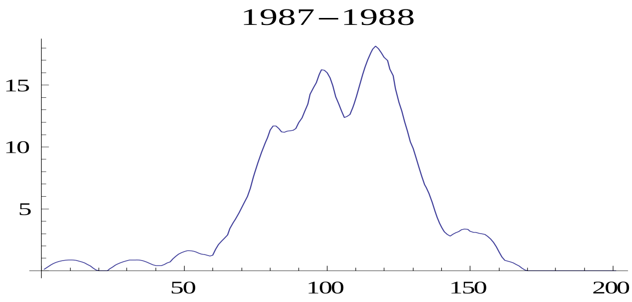
3. Empirical Testing

3.1. Data Description

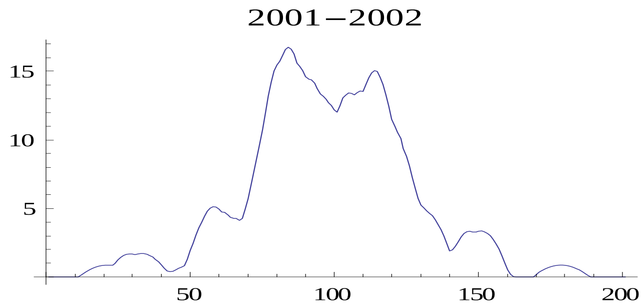
We primarily test the model on the set of daily data which contains most discussed stock market crash of October 19, 1987, known as Black Monday. The crash was the greatest single-day loss that Wall Street had ever suffered in continuous trading, 31%. The reasons for Black Monday have been widely discussed among professional investors and academics. However, not until today is there a consensus on the real cause. For comparison, we use another large crash, that of September 11, 2001. Our assumption is that while in 1987 the crash was caused by internal forces, the 2001 crash happened due to external force, namely the terrorist attack on the twin towers. Therefore the catastrophe model should fit the data of 1987 well as bifurcations leading to instability are present.
The data represents the daily returns of S&P 500 in the years 1987-1988 and 2001-2002 as the crashes took place inside these intervals. For the asymmetry side, we have chosen the daily change of down volume representing the volume of all declining stocks in the market. The trading volume represents good measure of the fundament, as it correlates with the volatility, and more importantly, good measure of what the large funds, representing fundamental investors, are doing. For bifurcation side OEX Put/Call ratio represents very good measure of speculative money. It is a ratio of daily put volume divided by daily call volume of the options with underlying Standard and Poor's 100 index. As financial options are the most popular vehicle for speculation, it represents the data of speculative money, while extraordinary biased volume or premium suggests excessive fear or greed in the stock market. These should be internal forces which causes the bifurcation.

3.2. Results

All the data are differenced once in order to gain stationarity. It can be seen that the data are leptokurtic, and much more interestingly, multimodal. For illustration of bimodality, we use kernel density estimation - see Graphs 1 and 2 (we use Epanechnikov kernel which is of following form: $K(u) = \frac{3}{4}(1-u^2)(|u| \leq 1)$ with smoother bandwidth so the bimodality can be seen):



Graph 1



Graph 2

Kernel density of the 2 year returns of 1987 and 1988 shows clear bimodality, and so does the kernel density of the second set of the data, i.e. years 2001 and 2002. The first test we consider is Hartelman's test for multimodality. It is evident from the previous figures that the returns are far from being unimodal. However as noted in Wangenmakers [6], there may occur inconsistencies between the pdf and the invariant function with respect to the number of stable states: examples of which can be found in Wangenmakers [6]. Thus, we make use of the proposed Hartelman's kernel program to test for the multimodality and we have found that there is 75% probability that the 1987-1988 data contains at least one bifurcation point, and 26% probability that the the 2001-2002 data contains at least one bifurcation point. These results are also consistent with our assumption, that the first crisis was drawn by internal market forces (c.f. the presence of the bifurcations in the data), whereas the 2001 crash was caused mainly due to external forces, 9/11 attack.

Encouraged by the knowledge that bifurcations are present in our datasets we can now move to cusp fitting. As has been mentioned before, we use Hartelman's cuspfit software[3] for this purpose. The methodology is simple. First, the linear, nonlinear (logistic) and the cusp catastrophe models have been fitted to the data. Then we have tested whether the cusp catastrophe model fits the data better than the other two models by the procedure described at the beginning of the empirical part of this paper. We have obtained the following results:

In Table 1 there are the results of the cusp fit to the data of 1987-1988 which contains the crash of October 19, 1987. We can see that log likelihood is largest for the cusp catastrophe model. Chi-squared test, Akaike and Schwarz-Bayesian information criteria also favor the catastrophe model, and R^2 is again much better for the cusp catastrophe. Thus we can conclude that the cusp catastrophe model offers a more suitable explanation for the 1987 stock market crash. We believe that the quality of the fit arises from the choice of the variables. We have also tried other possible variables in order to explain the bifurcations, but none has proved as successful. The choice of the variables is logical as the tests for the bifurcations in the data confirmed their presence.

Table 1: Results of the fits to the 1987-1988 data

model	linear	logistic	**cusp**
R^2	0.1452	0.2558	**0.4025**
log likelihood	-6.09×10^3	-5.17×10^2	$\mathbf{-4.95\times10^2}$
AIC	1.23×10^3	1.05×10^3	$\mathbf{1.00\times10^3}$
BIC	1.24×10^3	1.07×10^3	$\mathbf{1.03\times10^3}$
parameters	4	5	**6**

Table 2: Results of the fits to the 2001-2002 data

model	linear	**logistic**	cusp
R^2	0.1128	**0.4682**	0.2023
log likelihood	-0.61×10^3	$\mathbf{0.45\times10^3}$	-0.55×10^3
AIC	0.12×10^4	$\mathbf{0.91\times10^3}$	0.11×10^4
BIC	0.12×10^4	$\mathbf{0.93\times10^3}$	0.11×10^4
parameters	4	**5**	6

Let us have a look at the second set of the data that of years 2001-2002. The results are in Table 2, and

[3]Applications are available at Han van der Maas's Website (http://users.fmg.uva.nl/hvandermaas/)

we can see, that the catastrophe model in this case is rather superfluous. The log likelihood is greater than in the linear model, but lower than in the logistic model. Also other information criteria favor the logistic model. These results are in fact expected as of our earlier assumption, (i.e. that the 1987 crash was driven by internal forces, and the 2001 crash by external). While the 2001 data does have some bifurcations, the cusp catastrophe model clearly cannot fit the data significantly better than other models. This seems to be true, and for these data the catastrophe model did not perform better. However, for the 1987 crash the model seems to fit the data much better, and that is the sign, that the crash has occurred due to internal market forces.

4. Conclusions

Uncertain behavior of stock markets has always been on the leading edge of the research. Using the Cobb [1], Hartelman [4] and Wagenmarkers [6] results we have managed to test cusp catastrophe theory on the financial data, and we have arrived at very interesting results which may help to move the frontier of understanding the stock market crashes further on. We may thus confirm, that the catastrophe models explains the stock market crash much better then alternative linear regression models, or nonlinear logistic model. We have fitted the data of the two stock market crashes, the first being the crash of October 19, 1987, and the second September 11, 2001. We have used the sentiment measures to model the proportion of technical and fundamental players in the market. OEX put/call ratio is a very good measure of the technical players and represents the speculative money in our model and the trading volume is the measure of fundamental players and represents the excess demand.

We have clearly identified the bimodality of the returns using the test for multimodality which confirms that there is 75% probability that there is at least one bifurcation point in the data. Finally, the cusp catastrophe model fits these data much better than other models that have been used. Hence we conclude that the internal processes of the first dataset led to the crash in 1987. On the other hand, the crash of the September 11, 2001 can be better explained by the alternative logistic model. We have also found only 26% probability that there is at least one bifurcation point in these data, which is also in line with our second assumption: that due to the fact that this crash was caused by external forces the presence of the bifurcations in the data is much weaker.

Our findings may contribute to the frontier of the research, as it is the first attempt to quantitatively explain stock market crashes by cusp catastrophe theory. The testing has been conducted only on the restricted datasets. Thus further work is to test on different data which describes the situations when the changes in speculative money in the stock market lead to a crash. The main significant question, that of whether cusp catastrophe theory may help with an early indication of the stock market crashes still remains to be answered.

Acknowledgement

The authors would like to thank Prof. van der Maas for supplying a copy of Hartelman's dissertation and for publishing invariant stochastic catastrophe theory tools on his web site. A support from the Czech Science Foundation under grants 402/06/0990 and Grant Agency of Charles University (GAUK) under grant 46108 and Centre of research MSMT LC06075 is gratefully acknowledged.

References

[1] COBB, L. *Parameter estimation for the cusp catastrophe model*. Behavioral Science 26, 1981,75--78.

[2] COBB, L., WATSON,B. *Statistical catastrophe theory: an overview*. Math. Model 1, 1980, 311--317.

[3] COBB, L., ZACKS, S. *Applications of Catastrophe Theory for Statistical Modeling in the Biosciences*. Journal of the American Statistical Association 392,1985, 793-802.

[4] HARTELMAN, P. A. *Stochastic Catastrophe Theory.* Dissertatie reeks 1997-2, Faculteit Psychologie, Universiteit van Amsterdam, 1997 .

[5] THOM, R. *Structural Stability and Morpohogenesis*, New York: Benjamin,1975.

[6] WAGENMARKERS, E.J., MOLENAAR, P.C.M., HARTELMAN, P.A.I., van der MAAS, H.L.J. *Transformation invariant stochastic catastrophe theory*. Physica D, 211, 263, 2005

[7] ZEEMAN, E. C., *On the unstable behaviour of stock exchanges*. Journal of Mathematical Economics 1(1), 1974, 39-49.