1

NUMERICAL SIMULATION OF VISCOUS FLOW IN CHANNELS WITH MOVING WALLS

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Introduction

The subject of this paper is the numerical simulation of viscous incompressible flow in two-dimensional channels with moving walls. The Navier-Stokes equations and the continuity equation are written in the ALE (Arbitrary Lagrangian-Eulerian) form and discretized in space by conforming finite elements satisfying the Babuška-Brezzi condition and in time by the second order backward difference formula (BDF). The applicability of the developed method is proved by the solution of a test problem of flow in a channel with a wall moving in a prescribed way.

1. Formulation of the problem

We consider the flow in a bounded 2D domain Ω_t depending on time t with boundary $\partial\Omega_t = \Gamma_D \cup \Gamma_O \cup \Gamma_{W_t}$, where Γ_D represents inlet or parts of imepermeable fixed walls, Γ_O is outlet and Γ_{W_t} represents moving impermeable walls.

The dependence of the domain on time is taken into account with the aid of a regular ALE mapping $\mathcal{A}_t : \overline{\Omega}_0 \to \overline{\Omega}_t$, i.e. $\boldsymbol{X} \mapsto \boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$. Further, we define the ALE velocity: $\tilde{\boldsymbol{w}}(\boldsymbol{X}, t) = \frac{\partial}{\partial t}\boldsymbol{x}(\boldsymbol{X}, t) = \frac{\partial}{\partial t}\mathcal{A}_t(\boldsymbol{X}),\ \boldsymbol{w}(\boldsymbol{x}, t) = \tilde{\boldsymbol{w}}(\mathcal{A}_t^{-1}(\boldsymbol{x}), t),\ t \in [0, T],\ \boldsymbol{x} \in \overline{\Omega}_t$ and the ALE derivative of a function $f = f(\boldsymbol{x}, t)$: $\frac{D^A}{Dt} f(\boldsymbol{x}, t) = \frac{\partial \tilde{f}}{\partial t}(\boldsymbol{X}, t)|_{X=\mathcal{A}_t^{-1}(x)}$, where $\tilde{f}(\boldsymbol{X}, t) = f(\mathcal{A}_t(\boldsymbol{X}), t),\ \boldsymbol{X} \in \Omega_0$.

The Navier-Stokes system attains the following ALE form:

$$\mathrm{div}\boldsymbol{u} = 0, \quad \frac{D^A}{Dt}\boldsymbol{u} + ((\boldsymbol{u} - \boldsymbol{w}) \cdot \nabla)\,\boldsymbol{u} + \nabla p - \nu\Delta\boldsymbol{u} = 0. \tag{1}$$

Here $\boldsymbol{u}$ is the fluid velocity, p is the pressure and ν is the kinematic viscosity. System (1) is equipped with the following initial and boundary conditions:

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega_0, \tag{2}$$

$$\boldsymbol{u}|_{\Gamma_D \times (0,T)} = \boldsymbol{u}_D, \quad \boldsymbol{u}|_{\Gamma_{W_t} \times (0,T)} = \boldsymbol{w}|_{\Gamma_{W_t}, \times (0,T)}, \tag{3}$$

$$-p\boldsymbol{n} + \nu\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} = -p_{ref}\boldsymbol{n} \quad \text{on } \Gamma_O \times (0, T), \tag{4}$$

where p_{ref} is a given reference pressure.

Now we describe the construction of the ALE mapping. We assume that the inlet and outlet are straight segments given by the conditions $X_1 = a$ and $X_1 = b$, respectively, where $a,\ b \in \mathrm{R},\ a < b$ and the walls are represented by the graphs of smooth functions

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$$x_2 = \phi(X_1, t),\ X_1 \in [a, b], t \in [0, T]\ \text{(upper wall)},$$

$$x_2 = \varphi(X_1, t),\ X_1 \in [a, b], t \in [0, T]\ \text{(lower wall)},$$

where $\phi(X_1, t) > \varphi(X_1, t)$ for all $X_1 \in [a, b],\ t \in [0, T]$. The ALE mapping is given by the relations

$$x_1 = X_1,\ \ x_2(\boldsymbol{X}, t) = \varphi(X_1, t) + \frac{X_2 - \varphi(X_1, 0)}{\phi(X_1, 0) - \varphi(X_1, 0)}(\phi(X_1, t) - \varphi(X_1, t)). \tag{5}$$

2. Discretization

For the time discretization we introduce a time partition $t_k = k\tau$, $k = 0, 1, \ldots,\ \tau > 0$. We use the definition of the ALE derivative, put

$$\boldsymbol{x}^{n+1} = \mathcal{A}_{t_{n+1}}(\boldsymbol{X}),\ \boldsymbol{x}^n = \mathcal{A}_{t_n}(\boldsymbol{X}),\ \boldsymbol{x}^{n-1} = \mathcal{A}_{t_{n-1}}(\boldsymbol{X}), \quad \boldsymbol{X} \in \Omega_0$$

and use the approximations

$$\boldsymbol{w}^n \approx \boldsymbol{w}(t_n),\ p^n \approx p(t_n),\ \boldsymbol{u}^n \approx \boldsymbol{u}(t_n), \tag{6}$$

Then the second order backward difference formula leads to the approximation of the ALE derivative in the form

$$\frac{D^A}{Dt}\boldsymbol{u}(\boldsymbol{x}^{n+1}, t_{n+1}) \approx \frac{3\boldsymbol{u}^{n+1}(\boldsymbol{x}^{n+1}) - 4\boldsymbol{u}^n(\boldsymbol{x}^n) + \boldsymbol{u}^{n-1}(\boldsymbol{x}^{n-1})}{2\tau}. \tag{7}$$

Substituting in the system (1), we get the system for the unknown functions $\boldsymbol{u}^{n+1}$ a p^{n+1}:

$$\frac{3\boldsymbol{u}^{n+1}(\boldsymbol{x}^{n+1}) - 4\boldsymbol{u}^n(\boldsymbol{x}^n) + \boldsymbol{u}^{n-1}(\boldsymbol{x}^{n-1})}{2\tau} - \nu\Delta\boldsymbol{u}^{n+1}(\boldsymbol{x}^{n+1}) +$$

$$+ \left(\left(\boldsymbol{u}^{n+1}(\boldsymbol{x}^{n+1}) - \boldsymbol{w}^{n+1}(\boldsymbol{x}^{n+1})\right) \cdot \nabla\right) \boldsymbol{u}^{n+1}(\boldsymbol{x}^{n+1}) + \nabla p^{n+1}(\boldsymbol{x}^{n+1}) = 0, \tag{8}$$

If we define the function

$$\hat{\boldsymbol{u}}^i(\boldsymbol{x}^{n+1}) = \boldsymbol{u}^i\left(\mathcal{A}_{t_i}\left(\mathcal{A}^{-1}_{t_{n+1}}(\boldsymbol{x}^{n+1})\right)\right),\ \boldsymbol{x}^{n+1} \in \Omega_{t_{n+1}},$$

we can formulate the problem to find the functions $\boldsymbol{u}^{n+1} : \Omega_{t_{n+1}} \to \mathrm{R}^2$ a $p^{n+1} : \Omega_{t_{n+1}} \to \mathrm{R}$ satisfying in $\Omega_{t_{n+1}}$ the equations

$$\frac{3\boldsymbol{u}^{n+1} - 4\hat{\boldsymbol{u}}^n + \hat{\boldsymbol{u}}^{n-1}}{2\tau} + \left(\left(\boldsymbol{u}^{n+1} - \boldsymbol{w}^{n+1}\right) \cdot \nabla\right) \boldsymbol{u}^{n+1} + \nabla p^{n+1} - \nu\Delta\boldsymbol{u}^{n+1} = 0, \tag{9}$$

$$\mathrm{div}\boldsymbol{u}^{n+1} = 0 \tag{10}$$

and the Dirichlet boundary conditions (3) considered at time $t = t_{n+1}$.

The space discretization is carried out by the finite element method. It is based on a weak formulation. In what follows, we shall carry out the space discretization of the problem to find approximations of the functions $\boldsymbol{u} := \boldsymbol{u}^{n+1}$ and $p := p^{n+1}$ defined in the domain $\Omega_{t_{n+1}}$, satisfying system (9) and the boundary conditions (3) – (4). To this end, we reformulate this problem in a weak sense. Let us set $\Omega = \Omega_{t_{n+1}}$ and define the velocity spaces $W = (H^1(\Omega))^2, X =$

$\{\mathbf{v} \in W; \mathbf{v}|_{\Gamma_D \cap \Gamma_{Wt}} = 0\}$ and the pressure space $M = L_0^2(\Omega) = \{q \in L^2(\Omega); \int_\Omega q\,dx = 0\}$. Then it is easy to find that the solution $U = (\boldsymbol{u}, p)$ of our problem satisfies the identity

$$a(U, U, V) = f(V), \qquad \forall\, V = (\mathbf{v}, q) \in (X, M). \tag{11}$$

Here

$$\begin{aligned} a(U^*, U, V) &= \frac{3}{2\tau}(\boldsymbol{u}, \mathbf{v}) + \nu(\nabla \boldsymbol{u}, \nabla \mathbf{v}) + \Big(\Big((\boldsymbol{u}^* - \boldsymbol{w}^{n+1}) \cdot \nabla\Big)\, \boldsymbol{u}, \mathbf{v}\Big) \\ &\quad - (p, \nabla \cdot \mathbf{v}) + (\nabla \cdot \boldsymbol{u}, q)\,, \\ f(V) &= \frac{1}{2\tau}\Big(4\hat{\boldsymbol{u}}^n - \hat{\boldsymbol{u}}^{n-1}, \mathbf{v}\Big) - \int_{\Gamma_O} p_{ref}\mathbf{v} \cdot \mathbf{n}\, dS, \\ &\quad U = (\boldsymbol{u}, p),\ V = (\mathbf{v}, q),\ U^* = (\boldsymbol{u}^*, p), \end{aligned} \tag{12}$$

where by $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$. Moreover, we require that $\boldsymbol{u}$ satisfies the Dirichlet boundary conditions (3). The couple $(\boldsymbol{u}, p)$ represents the solution on the time level t_{n+1}, i.e. $\boldsymbol{u}^{n+1} := \boldsymbol{u}$ and $p^{n+1} := p$.

In order to apply the Galerkin FEM, we shall restrict the weak formulation from the spaces W, X, M to approximate spaces $W_h, X_h, M_h,\ h \in (0, h_0),\ h_0 > 0, X_h = \{\mathbf{v}_h \in W_h; \mathbf{v}_h|_{\Gamma_D \cap \Gamma_{Wt}} = 0\}$. Hence, we want to find $U_h = (\boldsymbol{u}_h, p_h) \in W_h \times M_h$ such that $\boldsymbol{u}_h$ satisfies approximately conditions (3) and

$$a(U_h, U_h, V_h) = f(V_h), \qquad \forall\, V_h = (\mathbf{v}_h, q_h) \in X_h \times M_h. \tag{13}$$

The couple (X_h, M_h) of the finite element spaces should satisfy the *Babuška–Brezzi (BB) condition*, which guarantees the stability of the scheme: there exists a constant $c > 0$ such that

$$\sup_{\boldsymbol{w} \in X_h} \frac{(p, \nabla \cdot \boldsymbol{w})}{|\boldsymbol{w}|_{H^1(\Omega)}} \geq c\|p\|_{L^2(\Omega)}, \qquad \forall p \in M_h, \quad h \in (0, h_0). \tag{14}$$

We proceed in the following way. Assuming that Ω is polygonal, by $\mathcal{T}_h$ we denote a triangulation of Ω with standard properties from the FEM. The pressure space M is then approximated by the space of piecewise polynomial functions of degree $\leq k$:

$$p \approx p_h \in M_h = \{q \in M \cap C(\overline{\Omega}); q|_K \in P^k(K), \forall K \in \mathcal{T}_h\} \tag{15}$$

and the velocity space W and X are approximated by the spaces of piecewise polynomial functions of degree $\leq k+1$:

$$\begin{aligned} \boldsymbol{u} \approx \boldsymbol{u}_h \in W_h &= \{\mathbf{v} \in W \cap \Big(C(\overline{\Omega})\Big)^2; \mathbf{v}|_K \in \Big(P^{k+1}(K)\Big)^2, \forall K \in \mathcal{T}_h\} \\ X_h &= W_h \cap W. \end{aligned} \tag{16}$$

This couple (X_h, M_h) satisfies the BB condition.

In practical computations we use the Taylor-Hood P_2/P_1 elements.

2. Stabilization of the FEM

The standard Galerkin discretization (13) may produce approximate solutions suffering from spurious oscillations for high Reynolds numbers. In order to avoid this drawback, we apply the stabilization via streamline-diffusion/Petrov-Galerkin technique (see, e.g., [2], [3]). We define the stabilization terms

$$\begin{aligned}
\mathcal{L}_h(U^*,U,V) &= \sum_{K\in\mathcal{T}_h} \delta_K\Big(\frac{3}{2\tau}\boldsymbol{u} - \nu\triangle\boldsymbol{u} + (\overline{\mathbf{w}}\cdot\nabla)\,\boldsymbol{u} + \nabla p, (\overline{\mathbf{w}}\cdot\nabla)\mathbf{v}\Big)_K, \\
\mathcal{F}_h(V) &= \sum_{K\in\mathcal{T}_h} \delta_K\Big(\frac{1}{2\tau}\left(4\hat{\boldsymbol{u}}^n - \hat{\boldsymbol{u}}^{n-1}\right), (\overline{\mathbf{w}}\cdot\nabla)\mathbf{v}\Big)_K, \qquad (17)\\
& U = (\boldsymbol{u},p),\;\; V = (\mathbf{v},q),\;\; U^* = (\boldsymbol{u}^*,p),
\end{aligned}$$

where the function $\overline{\mathbf{w}}$ stands for the transport velocity $\overline{\mathbf{w}} = \boldsymbol{u}^* - \boldsymbol{w}^{n+1}$, $(\cdot,\cdot)_K$ denotes the scalar product in $L^2(K)$ and $\delta_K \geq 0$ are suitable parameters. Moreover, we introduce the pressure stabilization terms

$$\mathcal{P}_h(U,V) = \sum_{K\in\mathcal{T}_h} \tau_K(\nabla\cdot\boldsymbol{u}, \nabla\cdot\mathbf{v})_K, \quad U = (\boldsymbol{u},p), \quad V = (\mathbf{v},q), \qquad (18)$$

with suitable parameters $\tau_K \geq 0$.

The *stabilized discrete problem* reads: Find $U_h = (\boldsymbol{u}_h, p_h) \in W_h \times M_h$ such that $\boldsymbol{u}_h$ satisfies approximately conditions (3) and

$$\begin{aligned}
& a(U_h,U_h,V_h) + \mathcal{L}_h(U_h,U_h,V_h) + \mathcal{P}_h(U_h,V_h) = f(V_h) + \mathcal{F}_h(V_h), \qquad (19)\\
& \quad \forall V_h \in X_h \times M_h.
\end{aligned}$$

The parameter δ_K is defined by

$$\delta_K = \delta^* h_K^2, \qquad (20)$$

where h_K is the size of the element K measured in the direction of $\overline{\mathbf{w}}$. The parameter $\delta^* \in (0,1]$ is an additional free parameter. Further, we put

$$\tau_K = \tau^* \in (0,1]. \qquad (21)$$

The nonlinear problem (19) is (on each time level) solved iteratively. Starting from an initial approximation $U_h^{(0)}$ and assuming that already iterate $U_h^{(k)}$ has been computed, we define $U_h^{(k+1)} \in W_h \times M_h$ by

$$\begin{aligned}
& a(U_h^{(k)}, U_h^{(k+1)}, V_h) + \mathcal{L}_h(U_h^{(k)}, U_h^{(k+1)}, V_h) \qquad (22)\\
& + \mathcal{P}_h(U_h^{(k+1)}, V_h) = f(V_h) + \mathcal{F}_h(V_h), \\
& \quad \forall V_h \in X_h \times M_h.
\end{aligned}$$

For each time level t_{n+1} we set

$$U_h^{(0)} := (2\hat{\boldsymbol{u}}^n - \hat{\boldsymbol{u}}^{n-1}, \hat{p}^n). \qquad (23)$$

As numerical experiments show, only a few iterations (22) have to be computed on each time level.

Obviously, problem (22) is linear. It is equivalent to the linear algebraic system

$$S\underline{u} + 2\tau(B+C)\underline{p} = f, \qquad B^T\underline{u} = 0, \qquad (24)$$

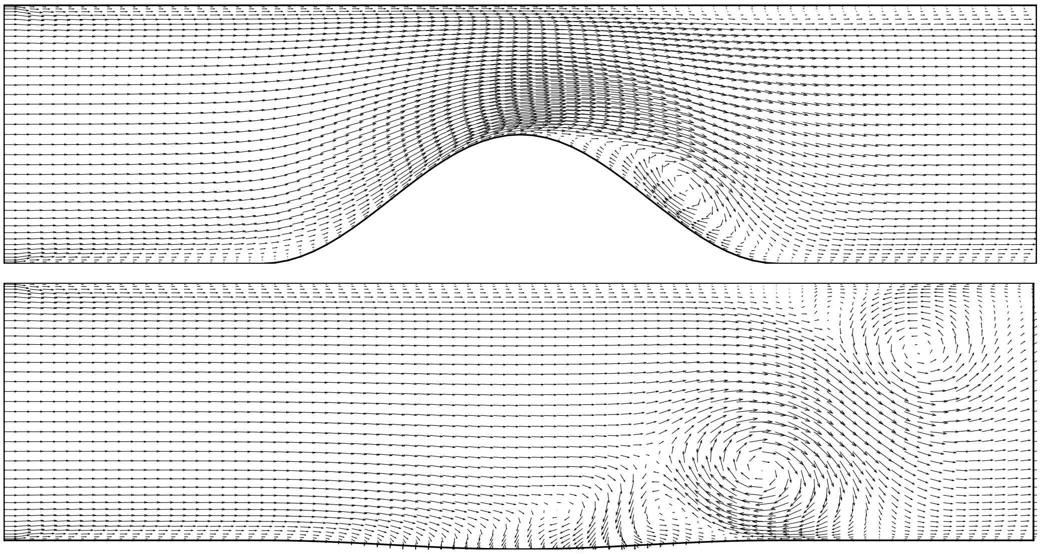

Figure 1: Velocity vectors at $t = 1.7$ and $t = 3.3$

where $\underline{u} \in R^{n_h}$ and $\underline{p} \in R^{m_h}$ are vectors whose components represent degrees of freedom defining the velocity $\boldsymbol{u}$ and the pressure p, respectively, S is a nonsingular $n_h \times n_h$ matrix and B and C are $n_h \times m_h$ matrices. The solution of this system was realized by the direct solvers UMFPACK ([1]), which works sufficiently fast for systems with up to 10^5 equations.

4. Test problem

In the test problem we considered the following data: $a = -2,\ b = 2,\ \varphi(X_1, t) = \sin t\ \left(\cos(\pi X_1) + 1\right)/4, \quad X_1 \in [-1, 1], \varphi(X_1, t) = 0, \quad X_1 \in [-2, 1) \cup (-1, 2], \phi(X_1, t) = 1, \quad X_1 \in [-2, 2],\ t \in [0, T]$. Further, we set $p_{ref} = 0$, $\boldsymbol{u}_0 = (1, 0), \boldsymbol{u}_D = (1, 0)$ at the inlet, otherwise $\boldsymbol{u}_D = (0, 0)$, $\tau = 0.01, \nu = 0.001$.

Figure 1 shows the velocity field at time $t = 1.7$ and $t = 3.3$.

References

[1] T.A. Davis: *UMFPACK V4.0,* http://www.cise.ufl.edu/research/sparse/unfpack, University of Florida.

[2] T. Gelhard, G. Lube, M.A. Olshanskii, J.-H. Starcke: Stabilized finite element schemes with LBB-stable elements for incompressible flows. J. Comput. Appl. Math. 177 (2005), 243-267.

[3] P. Sváček, M. Feistauer, J. Horáček: Numerical simulation of flow induced airfoil vibrations with large amplitudes. J. of Fluids and Structures, 23 (2007), 391-411.