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An L^q-approach with generalized anisotropic weights of the weak solution of the Oseen flow around a rotating body in the whole space

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Abstract

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1 Introduction

In the present paper we will study the stationary Oseen system in the whole three-dimensional space. We consider a three dimensional rigid body rotating with angular velocity $\omega = e_3 = (0, 0, 1)$ and that the complement of this

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body is filled with a viscous incompressible fluid modelled by the Navier-Stokes equations. Then, given the coefficient of viscosity $\nu > 0$ and an external force $\tilde{f} := \tilde{f}(y,t)$, we are looking for the velocity $v := v(y,t)$ and the pressure $q := q(y,t)$ solving the nonlinear system

$$\begin{aligned} v_t - \nu\Delta v + (v\cdot\nabla)v + \nabla q = \tilde{f} \quad & \text{in } D(t),\ t>0, \\ \operatorname{div} v = 0 \quad & \text{in } D(t),\ t>0, \\ v(y,t) = \omega\wedge y \quad & \text{on } \partial D(t),\ t>0, \\ v(y,t) \to v_\infty \quad & \text{as } |y|\to\infty, \end{aligned} \tag{1.1}$$

with $v_\infty \neq 0$. Here the time-dependent exterior domain $D(t)$ is given, due to the rotation with the angular velocity $\omega = e_3$, by

$$D(t) := O(t)D,$$

where $D \subset \mathbb{R}^3$ is a fixed exterior domain and $O(t)$ denotes the orthogonal matrix

$$O(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2}$$

After the change of variables $x := O(t)^T y$ and passing to the new functions $u(x,t) := O(t)^T(v(y,t) - v_\infty)$ and $p(x,t) := q(y,t)$, as well as to the force term $f(x,t) := O(t)^T\tilde{f}(y,t)$, we arrive at the modified Navier-Stokes system

$$\begin{aligned} u_t - \nu\Delta u + u\cdot\nabla u - (\omega\wedge x)\cdot\nabla u & \\ +(O(t)^T v_\infty)\cdot\nabla u + \omega\wedge u + \nabla p = f \quad & \text{in } D, \\ \operatorname{div} u = 0 \quad & \text{in } D, \\ u(x,t) + O(t)^T v_\infty = \omega\wedge x \quad & \text{on } \partial D, \\ u(x,t) \to 0 \quad & \text{at } \infty, \end{aligned} \tag{1.3}$$

for all $t > 0$ in the exterior time-independent domain D. Note that because of the new coordinate system attached to the rotating body, the equation $(1.3)_1$ contains three new terms, the classical Coriolis force term $\omega\wedge u$ (up to a multiplicative constant), and the terms $(\omega\wedge x)\cdot\nabla u$ and $(O(t)^T v_\infty)\cdot\nabla u$ which are not subordinate to the Laplacian in unbounded domains. As an important step in the study of problem (1.3) one considers its linearized and steady version, i.e. the following stationary Oseen like problem

$$
\begin{aligned}
-\nu\Delta u + k\partial_3 u - (\omega\wedge x)\cdot\nabla u & \\
+\omega\wedge u + \nabla p = f & \quad \text{in } \Omega \\
\operatorname{div} u = 0 & \quad \text{in } \Omega \\
u \to 0 & \quad \text{at } \infty
\end{aligned}
\tag{1.4}
$$

in the case $\Omega = \mathbb{R}^3$, but complemented in the case $\Omega = D$ by the boundary condition $u(x,t) = \omega\wedge x - u_\infty$ for $x \in \partial D$.

The strong solution of the Oseen like problem in the whole space case was shown by Farwig in [3], [4], proving estimates for $\|\nu\nabla^2 u\|_q$, $\|k\partial_3 u\|_q$, $\|\nabla p\|_q$ and $\|(\omega\wedge x)\cdot\nabla u - \omega\wedge u\|_q$. These estimates have been generalized by Farwig, Krbec and Nečasová [9] for L^q spaces with weights.

For related results on weak solutions in L^q and in Lorentz spaces we refer to Farwig and Hishida [5] and Hishida [18].

The weak solutions to problem (1.4) in L^q setting were analyzed in [23] and [24]. The a priori estimates

$$
\|\nabla u\|_q + \|p\|_q + \|(\omega\wedge x)\cdot\nabla u - \omega\wedge u\|_{-1,q} \;\le\; c\|f\|_{-1,q}, \tag{1.5}
$$

with some constant $c > 0$, which depends on q, have been proved there.

The aim of this paper is firstly to generalize the previous results (1.5) to weighted L^q-spaces for the whole space $\mathbb{R}^3$, and secondly to investigate the regularity of the solutions. It is easy to see that one can compute explicitly the Fourier transform of u solving the problem (1.4) after rewriting it in cylindrical coordinates $(r,\theta,x_3) \in [0,\infty)\times[0,2\pi)\times\mathbb{R}$, in a perfect agreement with the geometry of the problem. For $\hat{u} = \hat{u}(|\xi|,\xi_3,\varphi)$ in the Fourier space, we get the equations

$$
(\nu|\xi|^2 + ik\xi_3)\hat{u} - \partial_\varphi\hat{u} + e_3\wedge\hat{u} = \hat{f} - i\xi\hat{p} \text{ and } i\xi\cdot\hat{u} = 0.
$$

Then, from the unique 2π-periodic solution with respect to φ, we will take advantage of an integral representation of u : Then we follow the way used by Farwig, Hishida, Müller [6], also used by us in [23, 24], and we apply the Littlewood-Paley theory in weighted L^q setting. The theory requires general appropriate weight functions, we will use the one-sides Muckenhaupt weights. We reserve a section to recall all properties we need concerning these weights.

Applying the operator *curl* to the Oseen type equation in (1.4), we are always in the same situation, so we can iterate the application of the operator *curl* and take again advantage of successive integral representations to deduce boundness and regularity properties of the solutions.

To get same respective results for solutions to the stationary Oseen like problem in the case of exterior domains, one essentially need the results obtained in the whole space. The so-called localization procedure is then applied, this now is well known. For a complete study, see e.g.[20].

Therefore the main results of this paper are given in the case of the whole space.

2 Main results

In fact, the problem in exterior domains and the anisotropy of the wake region in the downstream direction impose the choice of appropriate weight functions : As an example of anisotropic weight functions to reflect the decay properties near the infinity, we consider

$$w(x) := \eta_\beta^\alpha(x) = (1+|x|)^\alpha (1+s(x))^\beta, \quad s(x) = |(x_1,x_2,x_3)| - x_3, \tag{2.1}$$

introduced in [2] to analyze the Oseen equations; see also [21]–[22]. If $|\beta| < 1$ and $|\alpha+\beta| < 3$, we know that such weights belongs to the Muckenhoupt class $A_2(\mathbb{R}^3)$, the corresponding weighted Lebesgue space being $L^2_w(\mathbb{R}^3)$.

For a L^q approach, let us denote by $A_q(\mathbb{R}^3)$, A_q^- and $\widetilde{A}_q^-$ the respective Muckenhoupt classes, the classical class in $\mathbb{R}^3$, the one-sided class in $\mathbb{R}$, and a convenient subclass of $A_q(\mathbb{R}^3)$ one-sided in x_3-direction (precise definitions are given in section 4).

We now formulate the main results of the paper.

Theorem 2.1. *Let $q \in (1,\infty)$. Let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ be a non-negative weight function independent of the angular variable θ and let $\widetilde{A}_q^-$ the convenient subclass of $A_q(\mathbb{R}^3)$ defined in section 4 . We assume that w satisfies the following condition depending on q:*

$$\begin{aligned} w^\tau \in \widetilde{A}_s^- \quad & \textit{with} \quad s = \tau q/2 \\ & \textit{for some} \quad \tau \in [1,\infty) \quad \textit{if } 2 \le q < \infty, \\ & \qquad \textit{or } \tau \in \left(\tfrac{2}{q}, \tfrac{2}{2-q}\right] \quad \textit{if } 1 < q < 2. \end{aligned} \tag{2.2}$$

Then, given $f \in \widehat{H}_w^{-1,q}(\mathbb{R}^3)^3$, *there exists a weak solution* $\{u,p\} \in \widehat{H}_w^{1,q}(\mathbb{R}^3)^3 \times L_w^q(\mathbb{R}^3) \subset L^1_{\mathrm{loc}}(\mathbb{R}^3)^3 \times L^1_{\mathrm{loc}}(\mathbb{R}^3)$ *of*

$$\begin{aligned} -\nu\Delta u + k\partial_3 u - (\omega \wedge x)\cdot\nabla u + \omega\wedge u + \nabla p = f \quad & \textit{in } \mathbb{R}^3 \\ \operatorname{div} u = 0 \quad & \textit{in } \mathbb{R}^3, \end{aligned} \tag{2.3}$$

satisfying the estimate

$$\|\nu\nabla u\|_{q,w} + \|p\|_{q,w} + \|(\omega\wedge x)\cdot\nabla u - \omega\wedge u\|_{-1,q,w} \le c\|f\|_{-1,q,w}, \tag{2.4}$$

with a constant $c = c(q,w) > 0$ *independent of* ν, k *and* ω.

Moreover, if $f \in L_w^q(\mathbb{R}^3)^3$, *we also have the estimates*

$$\|\nu\nabla^2 u\|_{q,w} + \|\nabla p\|_{q,w} \le c\|f\|_{q,w}. \tag{2.5}$$

Remark 2.2. Theorem 2.1 involves the notion of weak solution : We call $\{u,p\} \in \widehat{H}_w^{1,q}(\mathbb{R}^3)^3 \times L_w^q(\mathbb{R}^3)$ a weak solution to (1.4) if

(1) $\operatorname{div} u = 0 \quad \text{in} \quad L_w^q(\mathbb{R}^3),$ (2.6)

(2) $(\omega\wedge x)\cdot\nabla u - \omega\wedge u \in \widehat{H}_w^{-1,q}(\mathbb{R}^3)^3,$

(3) $\{u,p\}$ satisfies the following equation in the sense of distributions,

$$\begin{aligned} \nu < \nabla u, \nabla\varphi > +k < \partial_3 u, \varphi > - < (\omega\wedge x)\cdot\nabla u - \omega\wedge u, \varphi > & \\ =< f,\varphi >, \qquad \varphi \in C^\infty_{0,\sigma}(\mathbb{R}^3)^3. & \end{aligned} \tag{2.7}$$

In fact, by density, equation (2.7) holds for all $\varphi \in \widehat{H}_w^{1,q/(q-1)}(\mathbb{R}^3)^3$, $\operatorname{div}\varphi = 0$.

Remark 2.3. For $f \in L_w^q(\mathbb{R}^3)$, the obtained more regular solution $\{u,p\}$ belongs to $\widehat{H}_w^{2,q}(\mathbb{R}^3)^3 \times H_w^{1,q}(\mathbb{R}^3)$, then it is a strong solution. The strong solvability in corresponding weighted spaces was already studied in [9].

Corollary 2.4. *The a priori estimate* (2.4) *holds for the anisotropic weights* $w = \eta_\beta^\alpha$, *see* (2.1), *provided that the parameters* α *and* β *satisfy*

$$\begin{array}{llll} \textit{if } 2 \le q < \infty \ , & -\frac{q}{2} < \alpha < \frac{q}{2}, & 0 \le \beta < \frac{q}{2} & \textit{and} \quad \alpha+\beta > -1 \\ \textit{if } 1 < q < 2 \ , & -\frac{q}{2} < \alpha < q-1, & 0 \le \beta < q-1 & \textit{and} \quad \alpha+\beta > -\frac{q}{2}. \end{array}$$

Remark 2.5. Note that the condition $\beta \geq 0$ will reflect the existence of a wake region in the downstream direction $x_3 > 0$ where the solution of the original nonlinear problem (1.1) will decay slower than in the upstream direction $x_3 < 0$.

Theorem 2.6. *Let $f \in \widehat{H}_w^{-1,q}(\mathbb{R}^3)^3$, the solution $\{u, p\} \in \widehat{H}_w^{1,q}(\mathbb{R}^3)^3 \times L_w^q(\mathbb{R}^3)$ given by Theorem 2.1 is unique up to a constant for u.*

Remark 2.7. Because the gradient of the velocity should be bounded in L^q, it is easy to see that the velocity which differs in the terms $Ce_3 + \omega \wedge x$ can be only corrected by the part Ce_3.

Theorem 2.8. *Let $n \geq 1$ and let the assumptions of Theorem 2.1 hold. If f satisfies $\operatorname{curl}^n f \in L_w^q(\mathbb{R}^3)^3$, then*

$$\|\nu \nabla^{n+2} u\|_{q,w} + \|\nabla^n p\|_{q,w} \leq c \|\operatorname{curl}^n f\|_{q,w}, \tag{2.8}$$

with a constant $c = c(q, w) > 0$ independent of ν, k and ω.

Remark 2.9. The remarkable fact that the operator *curl* commute with all operators in the left hand side of the equation $(2.3)_1$ allows to extend and to iterate the regularity properties of the obtained solution $\{u, p\}$.

Theorem 2.10. *Let the assumptions of Theorem 2.1 hold, and $f \in L_w^q(\mathbb{R}^3)^3$. Given $g \in W_{\text{loc}}^{1,1}(R^3)$ with $\nu \nabla g + (\omega \wedge x) g \in L_w^q(R^3)$, there exists a solution $(u, p) \in \widehat{H}_w^{1,q}(\mathbb{R}^3)^3 \times L_w^q(\mathbb{R}^3) \subset L_{\text{loc}}^1(\mathbb{R}^3)^3 \times L_{\text{loc}}^1(\mathbb{R}^3)$ of*

$$\begin{aligned} -\nu \Delta u + k\partial_3 u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p &= f \quad \text{in } \mathbb{R}^3 \\ \operatorname{div} u &= g \quad \text{in } \mathbb{R}^3, \end{aligned} \tag{2.9}$$

satisfying the estimate

$$\begin{aligned} \|\nu \nabla u\|_{q,w} + \|p\|_{q,w} + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_{-1,q,w} \\ \leq c(\|f\|_{-1,q,w} + \|\nu \nabla g + (\omega \wedge x) g\|_{-1,q,w}), \end{aligned} \tag{2.10}$$

with a constant $c = c(q, w) > 0$ independent of ν, k and ω.

Remark 2.11. The incorporated problem $\operatorname{div} u = g$ must be solved in weighted Sobolev spaces of negative order : So we will investigate the estimate of the Bogowskii's operator, which was already studied by Hieber and his collaborators [17], and we will extend their results to the negative spaces with weights (see section 4.3).

3 Integral representations of the velocity

We work in the space of tempered distributions because we have in mind applications of the Fourier transform.

Using the fact that the space $\{\, g \mid\ g = \nabla \cdot G,\quad G \in C_0^\infty\left(\mathbb{R}^3\right)^{3\times 3}\}$ is dense in $\widehat{H}_w^{-1,q}(\mathbb{R}^3)^3$, we can write either f in the divergence form in the Oseen system (1.4) assuming first $G \in C_0^\infty\left(\mathbb{R}^3\right)^{3\times 3}$.

3.1 Formal computations

We just recall some main line which was already proved in L^q setting by Farwig in the case of strong solution and extend to weighted case by Farwig, Nečasová. Krbec [9]. The weak L^q setting was investigated by Kračmar, Nečasová, Penel [22].

One derives the following formal expressions of $\widehat{u}$, u, and ∇u :

$$\widehat{u}(\xi) = \int_0^\infty e^{-(\nu|\xi|^2+ik\xi_3)t} O_\omega^T(t)\, \widehat{g}(O_\omega(t)\xi)\, dt,$$

yielding $u(\cdot)$ in the form

$$u(x) = \int_0^\infty E_t * O_\omega^T(t)\, g(O_\omega(t). - kte_3)(x)\, dt,$$

where

$$E_t(x) = \frac{1}{(4\pi\nu t)^{3/2}} e^{-\frac{|x|^2}{4\nu t}}.$$

Observing the previous integral the solution can be rewritten as

$$u(x) = \int_{\mathbb{R}^3} \Gamma\,(x,y)\ \nabla \cdot G\,(y)\ dy,$$

where

$$\Gamma(x,y) = \int_0^\infty E_t\,(O_\omega(t)x - y - kte_3)\ O_\omega^T(t)\, dt.$$

Therefore we can compute the gradient of u,

$$\nabla u(x) = -\int_{\mathbb{R}^3} \nabla_x \nabla_y \Gamma\,(x,y) : G\,(y)\ dy,$$

For more details we refer to [23].

In the cylindrical coordinates $(r, \theta, x_3) \in (0, \infty) \times [0, 2\pi) \times \mathbb{R}$, the term $(e_3 \wedge x) \cdot \nabla u = -x_2 \partial_1 u + x_1 \partial_2 u$ may be rewritten in the form $(e_3 \wedge x) \cdot \nabla u = \partial_\theta u$ using the angular derivative ∂_θ applied to $u(r, \theta, x_3)$. So

$$-\nu \Delta u + k \partial_3 u - \widetilde{\omega}(\partial_\theta u - e_3 \wedge u) = F := f - \nabla p \quad \text{in} \quad \mathbb{R}^3, \tag{3.1}$$

Now applying the operator curl^k, we get in $\mathbb{R}^3$

$$-\nu \Delta \mathrm{curl}^k u + k \partial_3 \mathrm{curl}^k u - \widetilde{\omega}(\partial_\theta \mathrm{curl}^k u - e_3 \wedge \mathrm{curl}^k u) = \mathrm{curl}^k F := \mathrm{curl}^k (f - \nabla p). \tag{3.2}$$

Setting $v = \mathrm{curl}^k u$, $H = \mathrm{curl}^k F$ and applying the same method as before we obtain

$$\widehat{v}(\xi) = \int_0^\infty e^{-(\nu |\xi|^2 + ik\xi_3)t} O_\omega^T(t)\, \widehat{H}(O_\omega(t)\xi)\, dt. \tag{3.3}$$

3.2 Littlewood-Paley decomposition

In this section we are interested in each component of $TG\,(x) := \triangle \int_{\mathbb{R}^3} \Gamma(x, y) : G(y)\, dy$, say $TG_{i,k}(\cdot)$. Then we will estimate its L^q-norm by the L^q-norm of $G_{i,k}(\cdot)$, in order to apply the results for L^q-estimate of ∇u. To this end, we follow the way used by Farwig, Hishida, Müller [6] and by Hishida [18] because till now we do not have a more direct analysis. The basic tool is the equivalence of L^q-norms given by the theorem of E. M. Stein Chapter I, Section 8.23 [34],

$$c_1 \|F\|_q \le \left\|S(F)^{1/2}\right\|_q \le c_2 \|F\|_q\,,$$

where

$$S(F)(x) = \int_0^\infty |\varphi(t, \cdot) * F(x)|^2 \frac{dt}{t}$$

is the so called square operator, the function $\varphi(\cdot)$ being appropriate in $C_0^\infty((0, \infty); \mathcal{S}\,(\mathbb{R}^3))$. The function $F(\cdot)$ will play the role of $TG_{i,k}(\cdot)$.
The necessary properties of $\varphi(t, \cdot)$ for $t > 0$ are

$$\begin{aligned}
&\operatorname{supp} \widehat{\varphi}(t, \cdot) \subset \{\xi \in \mathbb{R}^n : \ \frac{1}{2\sqrt{t}} < |\xi| < \frac{2}{2\sqrt{t}}\} \\
&\int_0^\infty \widehat{\varphi}(t, \xi)^2 \frac{dt}{t} = 1 \\
&\int_{\mathbb{R}^3} \varphi(t, x) dx = 0
\end{aligned}$$

By means of the Fourier transform we have

$$\widehat{TG}\,(\xi) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty |\xi|^2 \exp\left(-\nu\,|\xi|^2\,t\right) O^T_\omega(t)\,\widehat{G}\left((O_\omega(t)\cdot -k\,t\,e_3)\,\xi\right)\,dt,$$

which we can rewrite as

$$\widehat{TG}\,(\xi)$$
$$= \frac{1}{\nu(2\pi)^{3/2}} \int_0^\infty |\xi|^2 \exp\left(-\,|\xi|^2\,t\right) O^T_{\omega/\nu}(t)\,\widehat{G}\left(\left(O_{\omega/\nu}(t)\cdot -k\,\frac{t}{\nu}\,e_3\right)\xi\right)\,dt.$$

We define $\psi \in \mathcal{S}(\mathbb{R}^3)$ by its Fourier transform

$$\widehat{\psi}(\xi) = \frac{1}{(2\pi)^{3/2}}|\xi|^2 e^{-|\xi|^2} \quad \text{and} \quad \widehat{\psi}_t(\xi) = \widehat{\psi}(\sqrt{t}\xi) \text{ for } t>0, \tag{3.4}$$

and so for all $t>0$

$$\psi_t(x) = t^{-3/2}\psi\left(\frac{x}{\sqrt{t}}\right), \quad \widehat{\psi}_t(\xi) = \frac{1}{(2\pi)^{3/2}} t|\xi|^2 e^{-\nu\,t\,|\xi|^2}. \tag{3.5}$$

So we get

$$\widehat{TG}(\xi) = \frac{1}{\nu}\int_0^\infty \widehat{\psi}_t(\xi)\,O^T_{\omega/\nu}(t)\,\widehat{G}\left(\left(O_{\omega/\nu}(t). - k\frac{t}{\nu}e_3\right)\xi\right)\frac{dt}{t}. \tag{3.6}$$

We introduce all the ingredients we need to use a Littlewood-Paley decomposition. We define the multiplier operator Δ_j such that

$$\widehat{\Delta_j f(\xi)} := \widehat{\chi}^j(\xi)\widehat{f}(\xi) \tag{3.7}$$

where

$$\widehat{\chi}^j(\xi) = \widehat{\chi}_0\Big(\frac{\xi}{2^{j+1}}\Big) - \widehat{\chi}_0\Big(\frac{|\xi|}{2^j}\Big) \tag{3.8}$$

with

$$\widehat{\chi}_0(.) : |\xi| \to R, \chi \in C^\infty, \widehat{\chi}_0\Big|_{\{|\xi|\le\frac{1}{2}\}} = 1, \widehat{\chi}_0\Big|_{\{|\xi|\ge 1\}} = 0. \tag{3.9}$$

Note that

$$\sum_{j\in Z} \widehat{\chi}^j(\xi) = 1. \tag{3.10}$$

and

$$f(.) = \sum_{j\in Z} \Delta_j f(.). \tag{3.11}$$

We introduce χ^j for $\xi \in \mathbb{R}^3$ and $j \in \mathbb{Z}$ by its Fourier transform

$$\widehat{\chi}^j(\xi) = \widetilde{\chi}(2^{-j}|\xi|), \quad \xi \in \mathbb{R}^3,$$

yielding $\sum_{j=-\infty}^{\infty} \widehat{\chi}_j = 1$ on $\mathbb{R}^3 \setminus \{0\}$ and

$$\operatorname{supp} \widehat{\chi}^j \subset A(2^{j-1}, 2^{j+1}) := \{\xi \in \mathbb{R}^3 : 2^{j-1} < |\xi| < 2^{j+1}\}. \tag{3.12}$$

Using χ^j we define for $j \in \mathbb{Z}$

$$\psi^j = \frac{1}{(2\pi)^{n/2}} \chi_j * \psi_t, \qquad \widehat{\psi^j} = \widehat{\Delta_j \psi(.)} = \widehat{\chi}^j \cdot \widehat{\psi}_t. \tag{3.13}$$

Obviously, $\sum_{j=-\infty}^{\infty} \psi^j = \psi$ on $\mathbb{R}^3$. We start with the procedure of the Littlewood-Paley decomposition of $F = TG_{i,k}$. For $G_{i,k} \in \mathcal{S}'(\mathbb{R}^3)$ the property $TG_{i,k} \in L^p(\mathbb{R}^3)$ is equivalent to

$$TG_{i,k} = \sum_{j=-\infty}^{+\infty} \Delta_j TG_{i,k} \quad \text{and} \quad \left(\sum_{j=-\infty}^{+\infty} |\Delta_j TG_{i,k}|^2 \right)^{1/2} \in L^p\left(\mathbb{R}^3\right),$$

where

$$\Delta^j = \mathcal{F}^{-1} \widehat{\psi}^j \left(\frac{\xi}{2^j} \right) \mathcal{F},$$

$$\Delta_t^j = \mathcal{F}^{-1} \widehat{\psi}_t^j \left(\frac{\xi}{2^j} \right) \mathcal{F}.$$

We denote by $\Delta\Gamma$ the operator

$$\Delta\Gamma = \sum_{j\in Z} \Delta_j \Delta\Gamma \tag{3.14}$$

yielding

$$\Delta \int_{\mathbb{R}^3} \Gamma(x,y) : G(y)dy, \quad G \in C_0^\infty(\mathbb{R}^3)^9, (G_{ik})_{1\le i\le 3, 1\le k\le 3}. \tag{3.15}$$

We set the linear operator

$$TG_{ik}(x) = \int_{\mathbb{R}^3} \Delta\Gamma(x,y)_{ki} G_{ik}(y) dy. \tag{3.16}$$

Since formally $T = \sum_{j=-\infty}^{\infty} T_j$, we have to prove that this infinite series converges even in the operator norm on L^q see proof in section 5.

4 Muckenhoupt weights

In this section, we fix the notations, we recall the definitions and properties of Muckenhoupt classes.

Let $q \in (1, \infty)$. w being a non negative weight function from $L^1_{loc}(\mathbb{R}^3)$, the weighted Lebesgue space is

$$L^q_w(\mathbb{R}^3) = L^q_w := \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^3) : \ \|f\|_{q,w} = \Big(\int_{\mathbb{R}^3} |f(x)|^q w(x)\, dx \Big)^{1/q} < \infty \Big\},$$

In the case of a bounded domain Ω, $L^q_w(\Omega)$ is similarly defined but as a subset of $L^1_{\mathrm{loc}}(\overline{\Omega})$.

4.1 Function Spaces and weight functions

We now introduce different Sobolev spaces, all subspaces of $L^1_{\mathrm{loc}}(\overline{\Omega})$,

- the Sobolev space $H^{n,q}_w(\Omega) := \{ f \in L^1_{\mathrm{loc}}(\overline{\Omega}) \mid \nabla^j f \in L^q_w(\Omega),\ j \leq n \}$, equipped with the norm $\|f\|_{n,q,w} := \sum_{j=0}^n \|\nabla^j f\|_{q,w}$.

- the homogeneous Sobolev space $\widehat{H}^{n,q}_w(\Omega) := \{ f \in L^1_{\mathrm{loc}}(\overline{\Omega}) \mid \nabla^n f \in L^q_w(\Omega) \}$, equipped with the norm $\|\nabla^n f\|_{q,w}$.

- the spaces $H^{n,q}_{w,0}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^{n,q}_w}}$ and $\widehat{H}^{n,q}_{w,0}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{\widehat{H}^{n,q}_w}}$, where $C_0^\infty(\Omega)$ denote the space of smooth and compactly supported functions.

The divergence free counterpart is usually denoted by $C^\infty_{0,\sigma}(\Omega)$, so we always can add an index σ to the associated spaces like $H^{n,q}_{w,0,\sigma}(\Omega)$ or $\widehat{H}^{n,q}_{w,0,\sigma}(\Omega)$.

It is easy to see that

$$(L^q_w(\Omega))' = L^{q'}_{w'}(\Omega) \quad \text{with} \quad \frac{1}{q} + \frac{1}{q'} = 1 \quad \text{and} \quad w' = w^{-\frac{1}{q-1}}. \tag{4.1}$$

The choice of weight functions w should reflect the anisotropy of the flow and the existence of a wake region. But to study the Oseen rotating problem the weight functions of Muckenhoupt class in the sense of Kurtz see [26] is not enough in the comparison of Stokes case. Therefore we have to introduce the more general definition of Muckenhoupt class in the sense of Sawyer see [32].

Definition 4.1. Let $\mathcal{R}$ be a collection of bounded sets R in $\mathbb{R}^3$, each of positive Lebesgue measure $|R|$. A non negative weight function $w \in L^1_{\mathrm{loc}}$ belongs to the *Muckenhoupt class* $A_q(\mathbb{R}^3, \mathcal{R})$, $1 \le q < \infty$, if there exists a constant $C > 0$ such that

$$\sup_R \left(\frac{1}{|R|} \int_R w(x)\, dx \right) \left(\frac{1}{|R|} \int_R w^{-1/(q-1)}\, dx \right)^{q-1} \le C \quad \text{for any } R \in \mathcal{R}$$

if $1 < q < \infty$, and

$$\sup_{R \in \mathcal{R},\, R \ni x_0} \frac{1}{|R|} \int_R w(x)\, dx \le C w(x_0) \quad \text{for a.a. } x_0 \in \mathbb{R}^n$$

if $q = 1$, respectively.

Two variants of this classical Muckenhoupt class are $A_q(\mathbb{R}^3, \mathcal{C})$, where $\mathcal{C}$ is the set of all cubes $Q \subset \mathbb{R}^3$ with positive Lebesgue measure $|Q|$, and $A_q(\mathbb{R}^3, \mathcal{J})$, where $\mathcal{J}$ is the set of all bounded nondegenerate rectangles in $\mathbb{R}^3$ with edges parallel to the coordinate axes. For these subclasses, we obviously have $A_q(\mathbb{R}^3, \mathcal{J}) \subsetneq A_q(\mathbb{R}^3, \mathcal{C})$.

By [34] for $1 < q < \infty$ and $w \in A_q(\mathbb{R}^3, \mathcal{J})$, there exists $1 \le s < q$ such that $w \in A_s(\mathbb{R}^3, \mathcal{J})$.

Let $A_q(\mathbb{R}^3) = A_q(\mathbb{R}^3, \mathcal{J})$.

Proposition 4.2. *(1) Let μ be a nonnegative regular Borel measure such that the strong centered Hardy-Littlewood maximal operator*

$$\mathcal{M}_{\mathcal{J}} \mu(x) = \sup_{R \ni x} \frac{1}{|R|} \int_R d\mu$$

is finite for almost all $x \in \mathbb{R}^3$. Then $(\mathcal{M}_{\mathcal{J}}\mu)^\gamma \in A_1(\mathbb{R}^3)$ for all $\gamma \in [0,1)$.

(2) For all $1 < q < \tau$ we have $A_1(\mathbb{R}^3) \subset A_q(\mathbb{R}^3) \subset A_\tau(\mathbb{R}^3)$.

(3) Let $1 < q < \infty$ and $w \in A_q(\mathbb{R}^3)$. Then there are $w_1, w_2 \in A_1(\mathbb{R}^3)$ such that

$$w = w_1 \cdot w_2^{1-q}.$$

Conversely, given $w_1, w_2 \in A_1$, the weight $w = w_1 w_2^{1-q}$ belongs to A_q.

Proof. See [16, Chapter IV, §6]. Note that the claim (3) is a variant of Jones' factorization theorem, see [16, Chapter IV, Theorem 6.8]. ∎

For a rapidly decreasing function $u \in \mathcal{S}(\mathbb{R}^3)$ or simply for $u \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, we also denote by the operator $\mathcal{M}$ the centered Hardy-Littlewood maximal operator

$$\mathcal{M}u(x) = \sup_{R \ni x} \frac{1}{|R|} \int_R |u(y)|\, dy, \quad x \in \mathbb{R}^3.$$

Theorem 4.3. *Let $1 < q < \infty$ and $w \in A_q(\mathbb{R}^3, \mathcal{J})$.*

(i) The operator $\mathcal{M}$ is a bounded operator from L^q_w to L^q_w.

(ii) Let m be any function of class C^n in each "quadrant" of $\mathbb{R}^3$ and let $B \geq 0$ be a constant such that $\|m\|_\infty \leq B$ and

$$\sup_{x_{k+1},\dots,x_3} \int_{\mathcal{I}} \left| \frac{\partial^k m(x)}{\partial x_1 \cdots \partial x_k} \right| dx_1 \cdots dx_k \leq B$$

for any dyadic interval $\mathcal{I}$ in $\mathbb{R}^k$, $1 \leq k \leq 3$, and also for any permutation of the variables $x_1, \dots, x_k$ within $x_1, \dots, x_3$.

Then, if $1 < p < \infty$ and $w \in A_p(\mathbb{R}^3, \mathcal{J})$, m defines a bounded multiplier operator from $L^p_w(\mathbb{R}^3)$ to $L^p_w(\mathbb{R}^3)$.

Proof. (i) See [16, Theorem IV 2.8], [27, Theorem 9]. For the proof of (ii) see [26]. ∎

4.2 One-sided Muckenhoupt weights and maximal operators

We now have to introduce *one-sided Muckenhoupt weights* and *one-sided maximal operators* on the real line, because we are interested in the wake region precisely in the downstream direction $x_3 > 0$ resp. in the upstream (opposite) one; see Theorem 4.5 and Lemma 4.6 below.

Definition 4.4. (i) For every locally integrable function u on the real line let M^+u be defined by

$$M^+u(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |u(t)|\, dt.$$

Analogously,

$$M^-u(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |u(t)|\, dt.$$

(ii) A non negative weight function $w \in L^1_{\rm loc}$ belongs to the *class* A_1^- if there exists a constant $c > 0$ such that

$$M^+w(x) \le cw(x) \text{ for almost all} x \in \mathbb{R}.$$

Analogously, $w \in A_1^+$ if and only if $M^-w(x) \le cw(x)$ for almost all $x \in \mathbb{R}$. The smallest constant $c \ge 0$ satisfying $M^\pm w(x) \le cw(x)$ for almost all $x \in \mathbb{R}$ is called the $A_1^\mp$-*constant of* the weight w.

(iii) A non negative weight function $w \in L^1_{\rm loc}$ belongs to the *one-sided Muckenhoupt class* A_q^+, $1 < q < \infty$, if there exists a constant $C > 0$ such that for all $x \in \mathbb{R}$

$$\sup_{h>0} \left(\frac{1}{h}\int_{x-h}^{x} w(t)\, dt\right)\left(\frac{1}{h}\int_x^{x+h} w(t)^{-1/(q-1)}\, dt\right)^{q-1} \le C.$$

The smallest constant $C \ge 0$ satisfying this estimate is called the A_q^+-*constant* of the weight w. By analogy, one define the set of weights A_q^- and the A_q^--constant of a weight in A_q^-.

We recall the following duality property: $w \in A_q^+$ if and only if $w^{-q'/q} = w^{-1/(q-1)} \in A_{q'}^-$. Moreover we will need the following results:

Theorem 4.5 (Theorem 1 of [32]). *Let* $1 < p < \infty$ *and* $p' = \frac{p}{p-1}$.

(i) Let $w_1 \in A_1^+$, $w_2 \in A_1^-$. *Then* $w_1 \cdot w_2^{1-p} \in A_p^+$. *Conversely, given* $w \in A_p^+$ *there exist* $w_1 \in A_1^+$, $w_2 \in A_1^-$ *such that* $w = w_1 \cdot w_2^{1-p}$.

(ii) The operator M^+ *is continuous from* $L^p_w(\mathbb{R})$ *to itself if and only if* $w \in A_p^+$. *Analogously,* $M^- : L^p_w(\mathbb{R}) \to L^p_w(\mathbb{R})$ *if and only if* $w \in A_p^-$.

Obviously, $A_p(\mathbb{R}) \subset A_p^\pm$ where $A_p(\mathbb{R})$ denotes the usual Muckenhoupt class on the real line. Hence $|x|^\alpha, (1+|x|)^\alpha \in A_p^\pm$ if $-1 < \alpha < p-1$, $1 < p <$

∞. However, in view of the anisotropic weight $w = \eta_\beta^\alpha$ on $\mathbb{R}^3$, see (2.1), we have to consider also one-dimensional anisotropic weight functions such as

$$\widetilde{w}_{\alpha,\beta}(x) := \widetilde{w}_{\alpha,\beta}(x;r) = (r^2+x^2)^{\alpha/2}(\sqrt{r^2+x^2}-x)^\beta, \quad x \in \mathbb{R},\ r>0. \quad (4.2)$$

Lemma 4.6. (i) *For every $r>0$ the univariate weight $\widetilde{w}_{\alpha,\beta}(x;r)$ lies in A_1^- if and only if $\beta \geq 0$, $\alpha \leq \beta$ and $\alpha+\beta > -1$. Moreover, the A_1^--constant of $\widetilde{w}_{\alpha,\beta}$ is uniformly bounded in r.*

(ii) For every $r>0$ the univariate weight

$$w_{\alpha,\beta}(x) := w_{\alpha,\beta}(x;r) = (1+r^2+x^2)^{\alpha/2}(1+\sqrt{r^2+x^2}-x)^\beta$$

lies in A_1^- with an A_1^--constant independent of $r>0$ if and only if

$$\alpha \leq 0 \leq \beta \text{ and } \alpha+\beta > -1. \quad (4.3)$$

(iii) Let $1<p<\infty$. Then for every $r>0$

$$\begin{aligned} &w_{\alpha,\beta}(\cdot\,;r) \in A_p^+ \quad \text{for} \quad \alpha > -1, \quad \beta \leq 0, \quad \alpha+\beta < p-1 \\ &w_{\alpha,\beta}(\cdot\,;r) \in A_p^- \quad \text{for} \quad \alpha < p-1, \quad \beta \geq 0, \quad \alpha+\beta > -1. \end{aligned} \quad (4.4)$$

Moreover, the $A_p^\pm$-constant is uniformly bounded in $r>0$.

Proof. See [9]. ■

4.3 Equation $\operatorname{div} u = g$ in negative Sobolev spaces

We investigate the estimate of the Bogovskii's operator in negative spaces with weights, which was already studied by Hieber and his collaborators [17] and we extend their results to the negative spaces with weights.

Theorem 4.7. *For every $f \in L^q_w(\mathbb{R}^n)$ there exists*

$$(v,p) \in \widehat{H}^{2,q}_w(\mathbb{R}^n) \times \widehat{H}^{1,q}_w(\mathbb{R}^n)$$

that fulfills the equation

$$-\Delta v + \nabla p = f, \quad \operatorname{div} v = 0.$$

Moreover, v is unique modulo polynomials of degree 1 and p is unique modulo constants.

This solution fulfills the a priori estimate

$$\|v\|_{\widehat{H}^{2,q}_w} + \|p\|_{\widehat{H}^{1,q}_w} \leq c\|f\|_{L^q_w}.$$

Proof. First we assume that $f \in L^q(\mathbb{R}^n) \cap L^q_w(\mathbb{R}^n)$. Then by Galdi [15, Theorem IV.2.1] there exists a strong solution $(v,p) \in \widehat{H}^{2,q}(\mathbb{R}^n) \times \widehat{H}^{1,q}(\mathbb{R}^n)$ to the Stokes problem. Obviously for every $i \in \{1, ..., n\}$ the functions $(\partial_i v, \partial_i p)$ are weak solutions to

$$-\Delta \partial_i v + \nabla \partial_i p = \partial_i f, \quad \operatorname{div} \partial_i v = 0$$

with $\partial_i f \in \widehat{H}^{-1,q}_w(\mathbb{R}^n)$. Then by Fröhlich [12, Theorem 5.1 and Corollary 5.1] one obtains

$$(\partial_i v, \partial_i p) \in \widehat{H}^{1,q}_w(\mathbb{R}^n) \times L^q_w(\mathbb{R}^n)$$

and the estimate

$$\|\partial_i v\|_{\widehat{H}^{1,q}_w} + \|\partial_i p\|_{L^q_w} \leq c\|\partial_i f\|_{\widehat{H}^{-1,q}_w}.$$

Since this is true for every $i \in \{1, ..., n\}$, we have shown that

$$(v,p) \in \widehat{H}^{2,q}_w(\mathbb{R}^n) \times \widehat{H}^{1,q}_w(\mathbb{R}^n)$$

and that it fulfills the a priori estimate.

The result for a general $f \in L^q_w(\mathbb{R}^n)$ follows by a classical density argument.

To show the uniqueness let $(u,p) \in \widehat{H}^{2,q}_w(\mathbb{R}^n) \times \widehat{H}^{1,q}_w(\mathbb{R}^n)$ with $-\Delta u + \nabla p = 0$ and $\operatorname{div} u = 0$. Then $\nabla u \in \widehat{H}^{1,q}_w(\mathbb{R}^n)$ is a weak solution to the Stokes problem with vanishing data. Thus we obtain from the uniqueness of weak solutions that $\nabla u = \text{const}$. Consequently u is a linear polynomial. Analogously for the pressure. ∎

Theorem 4.8. *For every $g \in \widehat{H}^{-1,q}_w(\mathbb{R}^3)$ there exists a function $u \in L^q_w(\mathbb{R}^3)$ that solves*

$$\operatorname{div} u = g \quad \textit{and} \quad \|u\|_{q,w} \leq c\|g\|_{H^{-1,q}_w(\mathbb{R}^3)}$$

with c independent of g.

Proof. On $L^{q'}_{w'}(\mathbb{R}^3)$ we define the functional

$$U := [v \mapsto < g, \psi >],$$

where for $v \in L^{q'}_{w'}(\mathbb{R}^3)$ the function $\psi \in \widehat{H}^{1,q'}_{w'}(\mathbb{R}^3)$ is given by the solution to the problem

$$-\Delta \phi + \nabla \psi = v \quad \text{and} \quad \operatorname{div} \phi = 0, \tag{4.5}$$

which exists by Theorem 4.7. The function ψ is unique only modulo constants, however, in $\widehat{H}^{1,q'}_{w'}(\mathbb{R}^3)$, where g acts, we consider cosets with respect to constants. Thus $< g, \psi >$ is well-defined.

Then we obtain

$$| < U, v > | \leq \|g\|_{(H^{1,q'}_{w'})'} \|\psi\|_{H^{1,q'}_{w'}} \leq c\|g\|_{(H^{1,q'}_{w'})'} \|v\|_{L^{q'}_{w'}} \tag{4.6}$$

and this shows that $U \in (L^{q'}_{w'}(\mathbb{R}^3))' = L^q_w(\mathbb{R}^3)$.

Setting $v = \nabla\psi$, the solution to (4.5) that corresponds to v is $(0, \psi)$ by the uniqueness result in Theorem 4.7. Thus we obtain

$$< U, \nabla\psi >=< U, v >=< g, \psi >$$

and this shows that U solves the problem. Using (4.6) one obtains the announced estimate. ∎

Remark 4.9. As a corollary, we easily obtain the solution to the equation $\operatorname{div} u = g$ relative to exterior domains, also in negative spaces with weights.

Corollary 4.10. *Let D be an exterior $C^{1,1}$-domain in $\mathbb{R}^3$. For every $g \in \widehat{H}^{-1,q}_w(D)$ there exists a function $u \in L^q_w(D)$ that solves*

$$\operatorname{div} u = g \quad \textit{and} \quad \|u\|_{q,w} \leq c\|g\|_{H^{-1,q}_w(D)}$$

with c independent of g.

Proof. Extending every $\psi_0 \in \widehat{H}^{1,q'}_{w',0}(D)$ by 0 on $\mathbb{R}^3 \setminus D$ the functional g can be considered to act on a closed subspace of $\widehat{H}^{1,q'}_{w'}(\mathbb{R}^3)$. Thus, by the Hahn-Banach theorem g can be extended to a functional, denoted by G, in $\widehat{H}^{-1,q}_w(\mathbb{R}^3)$ with the same norm. We now may set $v = \nabla\psi_0$ and restrict to D the functional $U := [v \in L^{q'}_{w'}(\mathbb{R}^3) \mapsto < G, \psi_0 >]$; let $u := U|_D$. By the uniqueness result in Theorem 4.7, the solution to (4.5) that corresponds to v is $(0, \psi_0)$. Thus we obtain

$$< u, \nabla\psi_0 >_D =< U, \nabla\psi_0 >_{\mathbb{R}^3} =< u, v >_{\mathbb{R}^3} =< G, \psi_0 >_{\mathbb{R}^3} =< g, \psi_0 >_D$$

and this shows that u solves the problem. Using (4.6) one obtains the estimate

$$\|u\|_{L^q_w(D)} \leq \|U\|_{L^q_w(\mathbb{R}^n)} \leq c\|G\|_{H^{-1,q}_w(\mathbb{R}^n)} = c\|g\|_{H^{-1,q}_w(D)}.$$

∎

5 Proofs of the main theorems

To introduce a weighted Littlewood-Paley decomposition of L^q_w choose $\widetilde{\varphi} \in C_0^\infty(\frac{1}{2}, 2)$ such that $0 \le \widetilde{\varphi} \le 1$ and $\int_0^\infty \widetilde{\varphi}(s)^2 \frac{ds}{s} = \frac{1}{2}$. Then define $\varphi \in \mathcal{S}(\mathbb{R}^3)$ by its Fourier transform $\widehat{\varphi}(\xi) = \widetilde{\varphi}(|\xi|)$ yielding for every $s > 0$

$$\widehat{\varphi}_s(\xi) = \widetilde{\varphi}(\sqrt{s}|\xi|), \quad \operatorname{supp} \widehat{\varphi}_s \subset A\Big(\frac{1}{2\sqrt{s}}, \frac{2}{\sqrt{s}}\Big) \tag{5.1}$$

and the normalization $\int_0^\infty \widehat{\varphi}_s(\xi)^2 \frac{ds}{s} = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

Theorem 5.1. *Let $1 < q < \infty$ and $w \in A_q(\mathbb{R}^3)$. Then there are constants $c_1, c_2 > 0$ depending on q, w and φ such that for all $f \in L^q_w$*

$$c_1 \|f\|_{q,w} \le \left\| \Big(\int_0^\infty |\varphi_s * f(\cdot)|^2 \frac{ds}{s} \Big)^{1/2} \right\|_{q,w} \le c_2 \|f\|_{q,w} \tag{5.2}$$

where $\varphi_s \in \mathcal{S}(\mathbb{R}^n)$ is defined by (5.1).

Proof. See [31, Proposition 1.9, Theorem 1.10], and also [26], [33]. ∎

Applying the previous theorem to the operator $T_j G_{ik}$, we get the following estimate

$$c_1 \|T_j G_{ik}\|_{q,w} \le \|(\int_0^\infty |(\varphi(t,.) * T_j G_{ik}(x)|^2 \frac{dt}{t})^{1/2}\|_{q,w} \le c_2 \|T_j G_{ik}\|_{q,w}.$$

5.1 An intermediate result

As a preliminary version of Theorem 2.1 we prove the following proposition. The extension to more general weights based on complex interpolation of L^q_w-spaces will be postponed to 5.2

Proposition 5.2. *Let the weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ be independent of the angle θ and define $w_r(x_3) := w(x_1, x_2, x_3)$ for fixed $r = |(x_1, x_2)| > 0$. Assume that*

$$\begin{array}{lll} w \in \widetilde{A}^-_{q/2} & \text{if} & q > 2, \\ w \in \widetilde{A}^-_1 \ \ \text{or} \ \ \frac{1}{w} \in \widetilde{A}^+_1 & \text{if} & q = 2, \\ w^{2/(2-q)} \in \widetilde{A}^-_{q/(2-q)} & \text{if} & 1 < q < 2. \end{array} \tag{5.3}$$

Then the linear operator T defined by (3.16) *satisfies the estimate*

$$\|TG\|_{q,w} \le c\|G\|_{q,w} \quad \text{for all} \quad f \in L^q_w \tag{5.4}$$

with a constant $c = c(q,w) > 0$ independent of f.

For later use we recall the following lemma, see [6].

Lemma 5.3. *The functions Δ^j, Δ^j_t, $j \in \mathbb{Z}$, $t > 0$, have the following properties:*

(i) $\operatorname{supp} \widehat{\Delta}^j_t \subset A\left(\frac{2^{j-1}}{\sqrt{t}}, \frac{2^{j+1}}{\sqrt{t}}\right)$.

(ii) For $m > \frac{n}{2}$ let $h(x) = (1+|x|^2)^{-m}$ and $h_t(x) = t^{-n/2}h(\frac{x}{\sqrt{t}})$, $t > 0$. Then there exists a constant $c > 0$ independent of $j \in \mathbb{Z}$ such that

$$\begin{array}{rcl} |\Delta^j(x)| & \le & c2^{-2|j|}\, h_{2^{-2j}}(x), \quad x \in \mathbb{R}^n, \\ \|\Delta^j\|_1 & \le & c2^{-2|j|}\,. \end{array}$$

Proof. See [6]. ∎

Proof. Step 1. First we consider the case $q > 2$, $w \in \widetilde{A}^-_{q/2} \subset A_q$, and define the sublinear operator $\mathcal{M}^j$, a modified maximal operator, by

$$\mathcal{M}^j g(x) = \sup_{s>0} \int_{A_s} \left(|\Delta^j_t| * |\varphi|\right)\left(O^T_{\omega/\nu}(t)x + \frac{k}{\nu}te_3\right)\frac{dt}{t}\,, \tag{5.5}$$

where $A_s = [\frac{s}{16}, 16s]$.

First step. We will prove the preliminary estimate

$$\|T_j G_{ik}\|_{q,w} \le c\|\Delta^j\|_1^{1/2}\|\mathcal{M}^j\|^{1/2}_{L_v^{(q/2)'}}\|G_{ik}\|_{q,w}\,, \quad j \in \mathbb{Z}, \tag{5.6}$$

where v denotes the θ-independent weight

$$v = w^{-\left(\frac{q}{2}\right)'/\left(\frac{q}{2}\right)} = w^{-\frac{2}{q-2}} \in \widetilde{A}^+_{(q/2)'} = \widetilde{A}^+_{q/(q-2)}. \tag{5.7}$$

To prove (5.6) we use the Littlewood-Paley decomposition of L^q_w, see (5.2), applied to T_jG_{ik}. By a duality argument we find some function $0 \le g \in L_v^{(q/2)'} = \left(L_w^{(q/2)}\right)^*$ with $\|g\|_{(q/2)',v} = 1$ such that

$$\left\|\int_0^\infty |\varphi_s * T_jG_{ik}(\cdot)|^2 \frac{ds}{s}\right\|_{q/2,w} = \int_0^\infty \int_{\mathbb{R}^3} |\varphi_s * T_jG_{ik}(x)|^2 g(x)\,dx\,\frac{ds}{s}\,. \tag{5.8}$$

To estimate the right-hand side of (5.8) note that

$$\varphi_s * T_j G_{ik}(x) = \int_0^\infty O^T_{\omega/\nu}(t)(\varphi_s * \Delta^j_t * G_{ik})\Big(O_{\omega/\nu}(t)x - \frac{k}{\nu} t e_3\Big)\,\frac{dt}{t}\,,$$

where $\varphi_s * \Delta^j_t = 0$ unless $t \in A(s,j) := [2^{2j-4}s, 2^{2j+4}s]$. Since $\int_{t\in A(s,j)} \frac{dt}{t} = \log 2^8$ for every $j \in \mathbb{Z}$, $s > 0$, we get by the inequality of Cauchy-Schwarz and the associativity of convolutions that

$$\begin{aligned}
|\varphi_s * T_j G_{ik}(x)|^2 &\le c \int_{A(s,j)} \left| \big(\Delta^j_t * (\varphi_s * G_{ik})\big) \left(O_{\omega/\nu}(t)x - \frac{k}{\nu} t e_3\right)\right|^2 \frac{dt}{t} \\
&\le c\|\Delta^j\|_1 \int_{A(s,j)} \big(|\Delta^j_t| * |\varphi_s * G_{ik}|^2\big)\Big(O_{\omega/\nu}(t)x - \frac{k}{\nu} t e_3\Big)\,\frac{dt}{t}\,;
\end{aligned}$$

here we used the estimate $|(\Delta^j_t * (\varphi_s * G_{ik}))(y)|^2 \le \|\Delta^j_t\|_1 (|\Delta^j_t| * |\varphi_s * G_{ik}|^2\big)(y)$ and the identity $\|\Delta^j_t\|_1 = \|\Delta^j\|_1$, see Theorem 5.2. Thus

$$\begin{aligned}
&|T_j G_{ik}\|^2_{q,w} \\
&\le c \int_0^\infty \int_{A(s,j)} \int_{R^n} (|\Delta^j_t| * |\varphi_s * G_{ik}|^2(x) \\
&\qquad \int_{A(s,j)} \big(|\psi^j_t| * G_{ik}\big)\Big(O^T_{\omega/\nu}(t)x - \frac{k}{\nu} t e_3\Big)\,\frac{dt}{t}\frac{ds}{s} dx,
\end{aligned} \tag{5.9}$$

since Δ^j_t is radially symmetric. By definition of $\mathcal{M}^j$ the innermost integral is bounded by $\mathcal{M}^j g(x)$ uniformly in $s > 0$. Hence we may proceed in (5.9) using Hölder's inequality as follows:

$$\begin{aligned}
\|T_j G_{ik}\|^2_{q,w} &\le c\|\Delta^j\|_1 \int_{\mathbb{R}^3} \left(\int_0^\infty |\varphi_s * G_{ik}|^2(x)\,\frac{ds}{s}\right) \mathcal{M}^j g(x)\,dx \\
&\le c\|\Delta^j\|_1 \left\| \int_0^\infty |\varphi_s * G_{ik}|^2(x)\,\frac{ds}{s}\right\|_{q/2,w} \|\mathcal{M}^j g\|_{(q/2)',v}.
\end{aligned} \tag{5.10}$$

Now (5.2) and the normalization $\|g\|_{(q/2)',v} = 1$ complete the proof of (5.6).

Step 2. We estimate $\|\mathcal{M}^j g\|_{(q/2)',v}$. For functions γ depending on θ, x_3 only let $\mathcal{M}_{\text{hel}}$ denote the "helical" maximal operator

$$\mathcal{M}_{\text{hel}}\gamma(\theta, x_3) = \sup_{s>0} \frac{1}{s} \int_{A_s} |\gamma|\Big(\theta - \frac{\omega}{\nu} t, x_3 + \frac{k}{\nu} t\Big)\,dt,$$

where $A_s = \left[\frac{s}{16}, 16s\right]$. Then, writing $p := \left(\frac{q}{2}\right)'$, we claim that

$$\mathcal{M}^j g(x) \le c2^{-2|j|}\mathcal{M}(\mathcal{M}_{\mathrm{hel}}g)(x) \quad \text{ for a.a. } x \in \mathbb{R}^n, \tag{5.11}$$

$$\|\mathcal{M}^j g\|_{p,v} \le c2^{-2|j|}\|g\|_{p,v}, \tag{5.12}$$

where in (5.11) $\mathcal{M}_{\mathrm{hel}}g$ is considered as $\mathcal{M}_{\mathrm{hel}}g(r,\cdot,\cdot)$ for almost all $r > 0$.

To prove (5.11) we use the pointwise estimate $|\psi_t^j(x)| \le c2^{-2|j|}h_{t2^{-2j}}(x)$, see Lemma 5.1(ii). Hence

$$\mathcal{M}^j g(x) \le c2^{-2|j|}\sup_{s>0}\int_{A_s}(h_{t2^{-2j}} * |g|)\Big(O^T_{\omega/\nu}(t)x + \frac{k}{\nu}te_3\Big)\,\frac{dt}{t}.$$

Moreover, there exists a constant $c > 0$ independent of $s > 0$, $j \in \mathbb{Z}$, such that $h_{t2^{-2j}} \le ch_{s2^{-2j}}$ for all $t \in A_s$. Consequently,

$$\begin{aligned}\mathcal{M}^j g(x) &\le c2^{-2|j|}\,\sup_{s>0} h_{s2^{-2j}} * \int_{A_s}|g|\Big(O^T_{\omega/\nu}(t)x + \frac{k}{\nu}te_3\Big)\,\frac{dt}{t}\\ &\le c2^{-2|j|}\sup_{t>0} h_t * \mathcal{M}_{\mathrm{hel}}g(x).\end{aligned}$$

Since h is nonnegative, radially decreasing, and $\|h_t\|_1 = \|h\|_1 = c_0 > 0$ for all $t > 0$, a well-known convolution estimate, see [34], II §2.1, yields the pointwise estimate (5.11).

Step 3. Note that up to now we have not yet used any specific properties of the weight $v \in A_p$. To estimate $\mathcal{M}_{\mathrm{hel}}g$ we shall work with a suitable one-sided maximal operator since our weight belongs to a Muckenhoupt class in $\mathbb{R}^3$ but a problem occurs when the weight is considered with respect to x_3 only. This naturally corresponds to the physical circumstances of the problem, where in the Oseen case the wake should appear. To estimate $\mathcal{M}_{\mathrm{hel}}g$ we write $g_r(\theta, x_3) = g(r,\theta,x_3) = g(x)$ and $v_r(x_3) = v(x)$, $r = |(x_1,x_2)| > 0$, for the θ-independent weight v. Then by the 2π-periodicity of g_r and v_r with respect to θ we get for almost all $r > 0$

$$\begin{aligned}&\int_{\mathbb{R}}\int_0^{2\pi}\mathcal{M}_{\mathrm{hel}}g_r(\theta,x_3)^p v_r(x_3)\,d\theta\,dx_3\\ &\le \int_{\mathbb{R}}\int_0^{2\pi}\Big|\sup_{s>0}\frac{1}{s}\int_0^{16s}|g_r|\Big(\theta - \frac{\omega}{k}\Big(x_3 + \frac{k}{\nu}t\Big), x_3 + \frac{k}{\nu}t\Big)\,dt\Big|^p v_r(x_3)\,d\theta\,dx_3\\ &= \int_{\mathbb{R}}\int_0^{2\pi}\Big|\sup_{s>0}\frac{1}{s}\int_0^{16s}\gamma_{r,\theta}\Big(x_3 + \frac{k}{\nu}t\Big)\,dt\Big|^p d\theta\,v_r(x_3)\,dx_3\\ &= 16\int_0^{2\pi}\int_{\mathbb{R}}\big|M^+\gamma_{r,\theta}(x_3)\big|^p v_r(x_3)\,dx_3\,d\theta\end{aligned}$$

where $\gamma_{r,\theta}(y_3) = |g_r|(\theta - \frac{\omega}{k}y_3, y_3)$ and M^+ denotes the one-sided maximal operator, see Definition 4.4. Since $w_r \in A^-_{q/2}$, by (5.7) and Theorem 4.5 (i) $v_r = w_r^{-(q/2)'/(q/2)} \in A^+_{(q/2)'} = A^+_p$ with an A^+_p-constant uniformly bounded in $r > 0$. Then Theorem 2.3 (ii) yields the estimate

$$\begin{aligned}
&\int_{\mathbb{R}} \int_0^{2\pi} \mathcal{M}_{\mathrm{hel}} g_r(\theta, x_3)^p v_r(x_3)\, d\theta\, dx_3 \\
&\qquad \le c \int_0^{2\pi} \int_{\mathbb{R}} |\gamma_{r,\theta}(x_3)|^p v_r(x_3)\, dx_3\, d\theta = c\|g_r\|^p_{L^p(\mathbb{R}\times(0,2\pi),\, v_r(x_3))},
\end{aligned}$$

where $c > 0$ is independent of k, ν. Integrating with respect to $r\,dr$, $r \in (0, \infty)$, Fubini's theorem allows to consider an extension of $\mathcal{M}_{\mathrm{hel}}$ to a bounded operator from $L^p_v(\mathbb{R}^3)$ to itself with an operator norm bounded uniformly in k, ν. Moreover, $\mathcal{M} : L^p_v(\mathbb{R}^3) \to L^p_v(\mathbb{R}^3)$ is bounded by Theorem 4.5 (ii). Hence, (5.11) implies (5.12), and by (5.6) as well as Lemma 2.1 (ii) we get the estimate

$$\|T_j G_{ik}\|_{q,w} \le c 2^{-2|j|} \|G_{ik}\|_{q,w}$$

for all $G_{ik} \in L^q_w(\mathbb{R}^3)$ with a constant $c > 0$ independent of $j \in \mathbb{Z}$. Summarizing the previous inequalities we proved (5.4) for $q > 2$.

Step 4. Now let $q = 2$, $w \in \widetilde{A}^-_1$. In this case the Littlewood-Paley decomposition of $T_j f$ in L^2_w implies that

$$\|T_j G_{ik}\|^2_{2,w} \le c \int_0^\infty \int_{\mathbb{R}^n} |\varphi_s * T_j G_{ik}|^2(x)\, g(x)\, dx\, \frac{ds}{s}\,,$$

where

$$g \in L^\infty_v,\ v = \frac{1}{w} \quad \text{and} \quad \|g\|_{\infty,v} = \operatorname{ess\,sup}_{\mathbb{R}^3} |g\, v| = 1.$$

By the same reasoning as before we arrive at (5.6), i.e.,

$$\|T_j G_{ik}\|_{2,w} \le c 2^{-|j|} \|\mathcal{M}^j g\|^{1/2}_{\infty,v} \|G_{ik}\|_{2,w}, \tag{5.13}$$

and at (5.11). Concerning $\mathcal{M}_{\mathrm{hel}}$ we use the pointwise estimate $g_r(\theta, x_3) \le w_r(x_3)$ for a.a. $\theta \in (0, 2\pi)$, $x_3 \in \mathbb{R}$, and get that

$$\mathcal{M}_{\mathrm{hel}} g_r(\theta, x_3) \le \sup_{s>0} \frac{1}{s} \int_0^{16s} w_r\big(x_3 + \frac{k}{\nu} t\big)\, dt \le 16\, M^+ w_r(x_3) \le c w_r(x_3)$$

with a constant $c > 0$ independent of $r > 0$. Since w is an $A_1(\mathbb{R}^3)$-weight, (5.11) implies that

$$\mathcal{M}^j g(x) \le c 2^{-2|j|} \mathcal{M} w(x) \le c 2^{-2|j|} w(x)$$

and consequently that $\|\mathcal{M}^j g\|_{\infty,v} \le c 2^{-2|j|}$ with a constant $c > 0$ independent of $j \in \mathbb{Z}$. Hence $\|T_j G_{ik}\|_{2,w} \le c 2^{-2|j|}$ proving (5.4) when $q = 2$.

Step 5. The remaining estimates are proved by duality arguments. Obviously the dual operator to T is defined by

$$T^* G(x) = \int_0^\infty (\Delta_t * O_\omega(t) G)(O_\omega^T(t) x + kte_3) \frac{dt}{t},$$

which has the same structure as K, but with an "opposite orientation". Hence T^* is bounded on L^q_w for $q \ge 2$ and all weights $w \in \widetilde{A}^+_{q/2}$. Now let $1 < q < 2$ and $w^{2/(2-q)} \in \widetilde{A}^-_{q/(2-q)} = \widetilde{A}^-_{(q'/2)'}$. Then by simple duality arguments $w' = w^{-q'/q} \in \widetilde{A}^+_{(q'/2)}$ and

$$|\langle TG, g\rangle| = |\langle G, T^* g\rangle| \le \|G\|_{q,w} \|T^* g\|_{q',w'} \le c \|G\|_{q,w} \|g\|_{q',w'}.$$

Finally let $q = 2$ and $\frac{1}{w} \in \widetilde{A}^+_1$. As before,

$$|\langle TG, g\rangle| \le \|G\|_{2,w} \|T^* g\|_{2,1/w} \le c \|G\|_{2,w} \|g\|_{2,1/w}.$$

Now Proposition 3.1 is completely proved. ■

5.2 Proof of the existence Theorem

Lemma 5.4 **([1]).** *Let $1 \le p_1, p_2 < \infty$, let $0 < w_1, w_2$ be weight functions, $\delta \in (0,1)$, and*

$$\frac{1}{p} = \frac{1-\delta}{p_1} + \frac{\delta}{p_2}, \quad w^{\frac{1}{p}} = w_1^{\frac{1-\delta}{p_1}} \cdot w_2^{\frac{\delta}{p_2}}.$$

Then

$$\left[L^{p_1}_{w_1}, L^{p_2}_{w_2}\right]_\delta = L^p_w$$

in the sense of complex interpolation.

In the following we shall derive an anisotropic variant of Jones's factorization theorem tailored to our situation, when we need to work with one-sided Muckenhoupt weights with respect to x_3, satisfying the usual Muckenhoupt condition in three dimensions.

Lemma 5.5 (Anisotropic Version of Jones' Factorization Theorem)**.** *Suppose that $w \in \widetilde{A}_q^-$. Then there exist weights $w_1 \in \widetilde{A}_1^-$ and $w_2 \in \widetilde{A}_1^+$ such that*

$$w = w_1 \cdot w_2^{1-q}.$$

Here $\widetilde{A}_1^+$ is defined by analogy with $\widetilde{A}_1^-$, cf. Definition 1.2, by assuming for $w_2 \in \widetilde{A}_1^+$ that $(w_2)_r \in A_1^+$ with A_1^+-constant uniformly bounded in $r > 0$. An analogous result holds for $w \in \widetilde{A}_q^+$.

Now, using our previous results on the Bogovskii's operator we know that there is $G \in L^q_w(R^3)^9$ such that

$$\nabla \cdot G = f, \|G\|_{q,w,R^3} \le c\|f\|_{-1,q,R^3}.$$

Applying the density arguments the Theorem 2.1 is completely proved.

5.3 Proofs of the other results

Proof of Theorem 2.2

ATTENTION $\omega = e_3$

Introducing the adjoint problem

$$L^* v = -\Delta v + (\omega \wedge x) \cdot \nabla v - \omega \wedge v + \partial_3 v = \nabla \cdot G$$

with $G \in C_0^\infty(R^3)^9$. It has the solution

$$\widehat{v}(\xi) = \int_0^\infty e^{-\nu|x|^2 t} O\omega(t)(G(O_\omega^T(t). - kte_3))(\xi)dt.$$

Applying the duality method we get the uniqueness up to ω.

Proof of Theorem 2.4

Following the previous results now the pressure we can get as a solution of the following problem

$$\Delta p = \mathrm{div}\, f + \nu\Delta g + \partial_\theta g = \mathrm{div}\, G \text{ in } R^3.$$

Applying the Hörmander-Mikhlin multiplier (Theorem 3.2) and the result on the boundedness of the Bogovskii's operator from section 3.2 we get the estimate

$$
\begin{aligned}
&\|\nu\nabla u\|_{q,w} + \|p\|_{q,w} + \|(\omega \wedge x)\cdot\nabla u - \omega\wedge u\|_{-1,q,w} \\
&\le c(\|f\|_{-1,q,w} + \|\nu\nabla g + (\omega\wedge x)g\|_{-1,q,w}).
\end{aligned}
$$

Proof of Theorem 2.3

Introducing the integral operator

$$
\widehat{TH}(\xi) = \frac{1}{\nu(2\pi)^{3/2}}\int_0^\infty |\xi|^2 \exp\left(-|\xi|^2 t\right) O^T_{\omega/\nu}(t)\,\widehat{H}\left(\left(O_{\omega/\nu}(t)\cdot -k\,\frac{t}{\nu}\,e_3\right)\xi\right)\,dt
$$

we apply the Proposition 5.2 with for v and the right-hand side H we get

$$
\|TH\|_{q,w} \le c\|H\|_{q,w}.
$$

Substituting $v = curl^k u$ and $H = curl^k F$ it implies

$$
\|\nu\nabla^{k+2}u\|_{q,w} + \|\nabla^k p\|_{q,w} \le c\|\mathrm{curl}^k f\|_{q,w}. \tag{5.14}
$$

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