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THREE-FORMS AND ALMOST COMPLEX STRUCTURES ON SIX-DIMENSIONAL MANIFOLDS

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Abstract. This article deals with three-forms on six-dimensional manifolds, the first dimension where the classification of 3-forms is not trivial. It includes three classes of multisymplectic three-forms. We study the class which is closely related to almost complex structures.

Introduction

There is a growing interest in the study of three-forms among geometers and physisits in the recent years. There are various geometrical structures connected with different types of three-forms on manifolds.

The connections with totally skew-symmetric torsion, which is a three-form, play an important role in the research of Thomas Friedrich (a series of articles, see for example [F]).

Nigel Hitchin and his school also shows an interest in three-forms ([H], [W]). There are three orbits of the action of the group $GL(6,\mathbb{R})$ on the multisymplectic (full-rank) three-forms on a six-dimensional vector space. There is either a tangent, complex, or product structure connected with a three-form on a six-dimensional vector space The kind of structure depends on which of the three orbits the form belongs to. We speak about the forms of product type, of complex type or of tangent type acordingly). The notion of a three-form of the given type on the manifold can be defined in the obvious way. We study closely the three-forms of the complex type and we construct the associated complex structure in a different (and we think simpler) way than N. Hitchin in [H]. Further we investigate the interplay between the integrability of the complex structure associated with a given three-form of complex type and the existence of the linear symmetric connection, which preserves the form. The result is stated in the Theorem 13, which can be regarded as 'The Darboux theorem for the three-forms of complex type'.

Theorem. *Let ω be a real three-form of complex type on a six-dimensional differentiable manifold M. Let J be the almost complex structure on M such that for any vector fields $X_1, X_2, X_3 \in \mathfrak{X}(M)$*

$$\omega(JX_1, X_2, X_3) = \omega(X_1, JX_2, X_3) = \omega(X_1, X_2, JX_3).$$

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Then there exists a symmetric connection $\tilde{\nabla}$ *on* M *such that* $\tilde{\nabla}\omega = 0$ *if and only if the following conditions are satisfied*

(i) $d\omega = 0$,
(ii) *the almost complex structure* J *is integrable.*

The orbits of the three-forms on six dimensional spaces

Let V be a real vector space. Recall that a k-form ω ($k \geq 2$) are said to be multisymplectic if the homomorphism

$$\iota : V \to \Lambda^{k-1} V^*, \quad v \mapsto \iota_v \omega = \omega(v, \dots)$$

is injective. There is a natural action of the general linear group $G(V)$ on $\Lambda^k V^*$, and also on $\Lambda^k_{ms} V^*$, the subset of the multisymplectic forms. Two multisymplectic forms are called equivalent if they belong to the same orbit of the action. For any form $\omega \in \Lambda^k V^*$ we define a subset

$$\Delta(\omega) = \{v \in V; (\iota_v \omega) \wedge (\iota_v \omega) = 0\}.$$

If $\dim V = 6$ and $k = 3$ the subset $\Lambda^3_{ms} V^*$ consists of three orbits. Let $e_1, \dots, e_6$ be a basis of V and $\alpha_1, \dots, \alpha_6$ the corresponding dual basis. Representatives of the three orbits can be expressed in the form

(1) $\omega_1 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5 \wedge \alpha_6$,
(2) $\omega_2 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6$,
(3) $\omega_3 = \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6$.

We speak speak about multisymplectic forms of product type (first form), or of complex type (the second one), or of tangent type (the third one) accordning to which orbit they belong to. There is the following characterisation of the orbits:

(1) ω is of product type if and only if $\Delta(\omega) = V^a \cup V^b$, where V^a and V^b are three-dimensional subspaces satisfying $V^a \cap V^b = \{0\}$.
(2) ω is of complex type if and only if $\Delta(\omega) = \{0\}$.
(3) ω is of tangent type if and only if $\Delta(\omega)$ is a three-dimensional subspace.

The forms ω_1 and ω_2 have equivalent complexifications. From this point of view the forms of tangent type are exceptional. See [V]. for further details

A multisymplectic k-form on a manifold M is a section of $\Lambda^k T^* M$ such that its restriction to the tangent space $T_x M$ is multisymplectic for any $x \in M$, and is of type i in $x \in M$, $i = 1, 2, 3$, if the restriction to $T_x M$ is of type i. A multisymplectic form on M can change its type as can be seen from the example:

$$\sigma = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge dx_4 \wedge dx_5 + dx_2 \wedge dx_4 \wedge dx_6 +$$
$$\sin(x_3 + x_4) dx_3 \wedge dx_5 \wedge dx_6 + \sin(x_3 + x_4) dx_4 \wedge dx_5 \wedge dx_6,$$

a three-form on $\mathbb{R}^6$. Then σ is of type 3 on the submanifold given by the equation $x_3 + x_4 = k\pi$, $k \in \mathbb{N}$. If $x_3 + x_4 \in (k\pi, (k+1)\pi)$, k even, then σ is of type 1 and if $x_3 + x_4 \in (k\pi, (k+1)\pi)$, k odd, then σ is of type 2. We point out that σ is closed and invariant under the action of the group $(2\pi\mathbb{Z})^6$ and we can factor σ to get a form changing the type on $\mathbb{R}^6/(2\pi\mathbb{Z})^6$, which is the six-dimensional torus, that is σ is closed on a compact manifold. The goal of this paper is to study the forms of complex type. We denote $\omega = \omega_2$.

Three-forms of complex type on vector spaces

In this chapter , in Proposition 7, we associate a three-form of complex type on a six-dimensional vector space V with the complex structure on the vector space (thereby justifying the name). In Proposition 8 we associate the couple (a three-form and the corresponding complex structure) with the unique complex three-form on the complexification $V^{\mathbb{C}}$.

We need some considerations about a decomposion of the three-forms on the complex vector spaces with aditional complex structure first.

Let J be an automorphism of a six-dimensional real vector space V satisfying $J^2 = -I$. Further let $V^{\mathbb{C}} = V \oplus iV$ be the complexification of V. There is the standard decomposition $V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. Consider a non-zero form γ of type $(3,0)$ on $V^{\mathbb{C}}$ and set

$$\gamma_0 = \operatorname{Re}\gamma, \quad \gamma_1 = \operatorname{Im}\gamma.$$

For any $v_1 \in V$, $v_1 + iJv_1 \in V^{0,1}$, and consequently $\gamma(i(v_1+iJv_1), v_2, v_3) = 0$ for any $v_2, v_3 \in V$. This implies $\gamma_0(i(v_1 + iJv_1), v_2, v_3) = 0$ and $\gamma_1(i(v_1 + iJv_1), v_2, v_3) = 0$. Thus

$$0 = \gamma_0(i(v_1 + iJv_1), v_2, v_3) = \gamma_0(iv_1, v_2, v_3) - \gamma_0(Jv_1, v_2, v_3).$$

Similarly we can proceed with γ_1 and we get

$$\gamma_0(iv_1, v_2, v_3) = \gamma_0(Jv_1, v_2, v_3), \quad \gamma_1(iv_1, v_2, v_3) = \gamma_1(Jv_1, v_2, v_3),$$

for any $v_1, v_2, v_3 \in V$. Moreover

$$\begin{aligned}\gamma_0(w_1, w_2, w_3) &= \operatorname{Re}(-\gamma(i^2 w_1, w_2, w_3)) = \operatorname{Re}(-i\gamma(iw_1, w_2, w_3)) \\ &= \operatorname{Im}(\gamma(iw_1, w_2, w_3)) = \gamma_1(iw_1, w_2, w_3),\end{aligned}$$

for any w_1, w_2, $w_3 \in V^{\mathbb{C}}$ and that is $\gamma_1(w_1, w_2, w_3) = -\gamma_0(iw_1, w_2, w_3)$. Finally,

$$\begin{aligned}\gamma_0(Jv_1, v_2, v_3) &= \gamma_0(iv_1, v_2, v_3) = \operatorname{Re}(\gamma(iv_1, v_2, v_3)) = \operatorname{Re}(i\gamma(v_1, v_2, v_3)) \\ &= \operatorname{Re}(\gamma(v_1, iv_2, v_3)) = \operatorname{Re}(\gamma(v_1, Jv_2, v_3)) = \gamma_0(v_1, Jv_2, v_3).\end{aligned}$$

In a similar manner

$$\begin{aligned}\gamma_0(Jv_1, v_2, v_3) &= \gamma_0(v_1, Jv_2, v_3) = \gamma_0(v_1, v_2, Jv_3), \\ \gamma_1(Jv_1, v_2, v_3) &= \gamma_1(v_1, Jv_2, v_3) = \gamma_1(v_1, v_2, Jv_3),\end{aligned}$$

that is both forms γ_0 and γ_1 are pure with respect to the complex structure J.

We recall that a three-form ω on a vector space V is called pure with respect to an automorphism A of V, iff

$$\omega(AX_1, X_2, X_3) = \omega(X_1, AX_2, X_3) = \omega(X_1, X_2, AX_3), \quad \forall X_1, X_2, X_3 \in V.$$

1. Lemma. *The real three-forms $\gamma_0|V$ and $\gamma_1|V$ (on V) are multisymplectic.*

Proof. Let us assume that $v_1 \in V$ is a vector such that for any vectors $v_2, v_3 \in V$ $(\gamma_0|V)(v_1, v_2, v_3) = 0$ or equivalently $\gamma_0(v_1, v_2, v_3) = 0$. There are uniquely determined vectors $w_1, w_2, w_3 \in V^{1,0}$ such that

$$v_1 = w_1 + \bar{w}_1, \quad v_2 = w_2 + \bar{w}_2, \quad v_3 = w_3 + \bar{w}_3.$$

Then

$$\begin{aligned} 0 = \gamma_0(v_1, v_2, v_3) &= \mathrm{Re}(\gamma(w_1 + \bar{w}_1, w_2 + \bar{w}_2, w_3 + \bar{w}_3)) \\ &= \mathrm{Re}(\gamma(w_1, w_2, w_3)) = \gamma_0(w_1, w_2, w_3), \end{aligned}$$

for a fixed w_1, and arbitrary $w_2, w_3 \in V^{1,0}$. Because $iw_2 \in V^{1,0}$,

$$\gamma_0(iw_1, w_2, w_3) = \gamma_0(w_1, iw_2, w_3) = 0.$$

Moreover $\gamma_1(w, w', w'') = -\gamma_0(iw, w', w'')$ for any $w, w', w'' \in V^{\mathbb{C}}$, and we get

$$\gamma_1(w_1, w_2, w_3) = -\gamma_0(iw_1, w_2, w_3) = 0,$$

for arbitrary $w_2, w_3 \in V^{1,0}$. Thus

$$\gamma(w_1, w_2, w_3) = \gamma_0(w_1, w_2, w_3) + i\gamma_1(w_1, w_2, w_3) = 0,$$

for arbitrary $w_2, w_3 \in V^{1,0}$.

Because γ is a non-zero complex three-form on the complex three-dimensional vector space $V^{1,0}$, we find that $w_1 = 0$, and consequently $v_1 = 0$. This proves that the real three-form $\gamma_0|V$ is multisymplectic. Similarly the real three-form $\gamma_1|V$ is also multisymplectic.

2. Lemma. *The forms $\gamma_0|V$ and $\gamma_1|V$ satisfy $\Delta(\gamma_0|V) = \{0\}$ and $\Delta(\gamma_1|V) = \{0\}$.*

Proof. The complex three-form γ is decomposable, and therefore $\gamma \wedge \gamma = 0$. This implies that for any $w \in V^{\mathbb{C}}$ $(\iota_w\gamma) \wedge (\iota_w\gamma) = 0$. Similarly for any $w \in V^{\mathbb{C}}$ $(\iota_w\bar{\gamma}) \wedge (\iota_w\bar{\gamma}) = 0$. Obviously $\gamma_0 = (1/2)(\gamma + \bar{\gamma})$. Let $v \in V$ be such that $(\iota_v\gamma_0) \wedge (\iota_v\gamma_0) = 0$. Then

$$0 = (\iota_v\gamma_0) \wedge (\iota_v\gamma_0) = \frac{1}{4}(\iota_v\gamma + \iota_v\bar{\gamma}) \wedge (\iota_v\gamma + \iota_v\bar{\gamma}) = \frac{1}{2}(\iota_v\gamma) \wedge (\iota_v\bar{\gamma}).$$

But $\iota_v\gamma$ is a form of type $(2,0)$ and $\iota_v\bar{\gamma}$ a form of type $(0,2)$. Consequently the last wedge product vanishes if and only if either $\iota_v\gamma = 0$ or $\iota_v\bar{\gamma} = 0$. By virtue of the preceding lemma this implies that $v = 0$.

Lemma 2 shows that the both forms $\gamma_0|V$ and $\gamma_1|V$ are of complex type. As a final result of this type we get the following:

3. Corollary. *Let γ be a three-form on $V^{\mathbb{C}}$ of the type $(3,0)$. Then the real three-forms $(\operatorname{Re}\gamma)|V$ and $(\operatorname{Im}\gamma)|V$ on V are multisymplectic and of complex type.*

Let ω be a three-form on V such that $\Delta(\omega) = \{0\}$. This means that for any $v \in V$, $v \neq 0$ there is $(\iota_v\omega) \wedge (\iota_v\omega) \neq 0$. This implies that $\operatorname{rank} \iota_v\omega \geq 4$. On the other hand obviously $\operatorname{rank} \iota_v\omega \leq 4$. Consequently, for any $v \neq 0$ $\operatorname{rank} \iota_v\omega = 4$. Thus the kernel $K(\iota_v\omega)$ of the 2-form $\iota_v\omega$ has dimension 2. Moreover $v \in K(\iota_v\omega)$. Now we fix a non-zero 6-form on θ on V. For any $v \in V$ there exists a unique vector $Q(v) \in V$ such that

$$(\iota_v\omega) \wedge \omega = \iota_{Q(v)}\theta.$$

The mapping $Q : V \to V$ is obviously a homomorphism. If $v \neq 0$ then $(\iota_v\omega)\wedge\omega \neq 0$, and Q is an automorphism. It is also obvious that if $v \neq 0$, then the vectors v and $Q(v)$ are linearly independent (by applying ι_v to the last equality). We evaluate $\iota_{Q(v)}$ on the last equality and we get

$$\begin{aligned}
(\iota_{Q(v)}\iota_v\omega) \wedge \omega + (\iota_v\omega) \wedge (\iota_{Q(v)}\omega) &= 0,\\
-(\iota_v\iota_{Q(v)}\omega) \wedge \omega + (\iota_v\omega) \wedge (\iota_{Q(v)}\omega) &= 0,\\
-\iota_v[(\iota_{Q(v)}\omega) \wedge \omega] + 2(\iota_v\omega) \wedge (\iota_{Q(v)}\omega) &= 0.
\end{aligned}$$

Now, apply ι_v to the last equality:

$$(\iota_v\omega) \wedge (\iota_v\iota_{Q(v)}\omega) = 0.$$

If the 1-form $\iota_v\iota_{Q(v)}\omega$ were not the zero one then there would exist a 1-form σ such that $\iota_v\omega = \sigma \wedge \iota_v\iota_{Q(v)}\omega$, and we would get

$$(\iota_v\omega) \wedge (\iota_v\omega) = \sigma \wedge \iota_v\iota_{Q(v)}\omega \wedge \sigma \wedge \iota_v\iota_{Q(v)}\omega = 0,$$

which is a contradiction. Thus we have proved the following lemma.

4. Lemma. *For any $v \in V$ there is $\iota_{Q(v)}\iota_v\omega = 0$, i. e. $Q(v) \in K(\iota_v\omega)$.*

This lemma shows that if $v \neq 0$, then $K(\iota_v\omega) = [v, Q(v)]$. Applying $\iota_{Q(v)}$ to the equality $(\iota_v\omega) \wedge \omega = \iota_{Q(v)}\theta$ and using the last lemma we obtain easily the following result.

5. Lemma. *For any $v \in V$ there is $(\iota_v\omega) \wedge (\iota_{Q(v)}\omega) = 0$.*

Lemma 4 shows that $v \in K(\iota_{Q(v)}\omega)$. Because v and $Q(v)$ are linearly independent, we can see that

$$K(\iota_{Q(v)}\omega) = [v, Q(v)] = K(\iota_v\omega).$$

If $v \neq 0$, then $Q^2(v) \in K(\iota_{Q(v)}\omega)$, and consequently there are $a(v), b(v) \in \mathbb{R}$ such that

$$Q^2(v) = a(v)v + b(v)Q(v).$$

For any $v \in V$

$$(\iota_{Q(v)}\omega) \wedge \omega = \iota_{Q^2(v)}\theta.$$

Let us assume that $v \neq 0$. Then

$$(\iota_{Q(v)}\omega) \wedge \omega = a(v)\iota_v\theta + b(v)\iota_{Q(v)}\theta,$$

and applying ι_v we obtain $b(v)\iota_v\iota_{Q(v)}\theta = 0$, which shows that $b(v) = 0$ for any $v \neq 0$. Consequently, $Q^2(v) = a(v)v$ for any $v \neq 0$.

6. Lemma. *Let $A : V \to V$ be an automorphism, and $a : V \backslash \{0\} \to \mathbb{R}$ a function such that*

$$A(v) = a(v)v \quad \text{for any } v \neq 0.$$

Then the function a is constant.

Proof. The condition on A means that every vector v of V is an eigenvector of A with the eigenvalue $a(v)$. But the eigenvalues of two different vectors have to be the same otherwise their sum would not be an eigenvector.

Applying Lemma 6 on Q^2 we get $Q^2 = aI$. If $a > 0$, then $V = V^+ \oplus V^-$, and

$$Qv = \sqrt{a}v \text{ for } v \in V^+, \quad Qv = -\sqrt{a}v \text{ for } v \in V^-.$$

At least one of the subspaces V^+ and V^- is non-trivial. Let us assume for example that $V^+ \neq \{0\}$. Then there is $v \in V^+$, $v \neq 0$, and $Qv = \sqrt{a}v$, which is a contradiction because the vectors v and Qv are linearly independent. This proves that $a < 0$. We can now see that the automorphisms

$$J_+ = \frac{1}{\sqrt{-a}}Q \text{ and } J_- = -\frac{1}{\sqrt{-a}}Q \text{ satisfy } J_+^2 = -I \text{ and } J_-^2 = -I,$$

i. e. they define complex structures on V and $J_- = -J_+$. Setting

$$\theta_+ = \sqrt{-a}\theta, \quad \theta_- = -\sqrt{-a}\theta,$$

we get

$$(\iota_v\omega) \wedge \omega = \iota_{J_+v}\theta_+, \quad (\iota_v\omega) \wedge \omega = \iota_{J_-v}\theta_-.$$

In the sequel we shall denote $J = J_+$. The same results which are valid for J_+ hold also for J_-.

7. Proposition. *There exists a unique (up to the the sign) complex structure J on V such that the form ω satisfies the relation*

$$\omega(Jv_1, v_2, v_3) = \omega(v_1, Jv_2, v_3) = \omega(v_1, v_2, Jv_3) \quad \text{for any } v_1, v_2, v_3 \in V.$$

Proof. We shall prove first that the complex structure J defined above satisfies the relation. By virtue of Lemma 4 for any $v, v' \in V$ $\omega(v, Jv, v') = 0$. Therefore we get

$$\begin{aligned} 0 = \omega(v_1 + v_2, J(v_1 + v_2), v_3) &= \omega(v_1, Jv_2, v_3) + \omega(v_2, Jv_1, v_3) \\ &= -\omega(Jv_1, v_2, v_3) + \omega(v_1, Jv_2, v_3), \end{aligned}$$

which gives

$$\omega(Jv_1, v_2, v_3) = \omega(v_1, Jv_2, v_3).$$

Obviously, the opposite complex structure $-J$ satisfies the same relation. We prove that there is no other complex structure with the same property. Let $\tilde{J}$ be a complex structure on V satisfying the above relation. We set $A = \tilde{J}J^{-1}$. Then we get

$$\begin{aligned} \omega(v_1, Av_2, Av_3) = \omega(v_1, \tilde{J}Jv_2, \tilde{J}Jv_3) &= \omega(v_1, Jv_2, \tilde{J}^2Jv_3) = -\omega(v_1, Jv_2, Jv_3) \\ &= -\,\omega(v_1, v_2, J^2v_3) = \omega(v_1, v_2, v_3). \end{aligned}$$

Any automorphism A satisfying this identity is $\pm I$. Really, the identity means that A is an automorphism of the 2-form $\iota_v\omega$. Consequently, A preserves the kernel $K(\iota_v\omega) = [v, Jv]$. On the other hand it is obvious that any subspace of the form $[v, Jv]$ is the kernel of $\iota_v\omega$. Considering V as a complex vector space with the complex structure J, we can say that every 1-dimensional complex subspace is the kernel of the 2-form $\iota_v\omega$ for some $v \in V$, $v \neq 0$, and consequently is invariant under the automorphism A. Similarly as in Lemma 6 we conclude, that $A = \lambda I$, $\lambda \in \mathbb{C}$. If we write $\lambda = \lambda_0 + i\lambda_1$, then $A = \lambda_0 I + \lambda_1 J$ and

$$\begin{aligned}
&\omega(v_1, v_2, v_3) = \omega(v_1, Av_2, Av_3) \qquad (\lambda_0^2 - \lambda_1^2 - 1)\omega(v_1, v_2, v_3) + 2\lambda_0\lambda_1\omega(v_1, v_2,\\
=&\omega(v_1, \lambda_0 v_2 + \lambda_1 Jv_2, \lambda_0 v_3 + \lambda_1 Jv_3)\\
=&\lambda_0^2\omega(v_1, v_2, v_3) + \lambda_0\lambda_1\omega(v_1, v_2, Jv_3) + \lambda_0\lambda_1\omega(v_1, Jv_2, v_3) + \lambda_1^2\omega(v_1, Jv_2, Jv_3),
\end{aligned}$$

We shall use this last equation together with one obtained by writing Jv_3 instead of v_3. In this way we get the system

$$\begin{aligned}
(\lambda_0^2 - \lambda_1^2 - 1)\omega(v_1, v_2, v_3) + 2\lambda_0\lambda_1\omega(v_1, v_2, Jv_3) &= 0,\\
-2\lambda_0\lambda_1\omega(v_1, v_2, v_3) + (\lambda_0^2 - \lambda_1^2 - 1)\omega(v_1, v_2, Jv_3) &= 0.
\end{aligned}$$

Because it has a non-trivial solution

$$\begin{vmatrix} \lambda_0^2 - \lambda_1^2 - 1 & 2\lambda_0\lambda_1 \\ -2\lambda_0\lambda_1 & \lambda_0^2 - \lambda_1^2 - 1 \end{vmatrix} = 0.$$

It is easy to verify that the solution of the last equation is $\lambda_0 = \pm 1$ and $\lambda_1 = 0$. This finishes the proof.

We shall now consider the vector space V together with a complex structure J, and a three-form ω on V which is pure with respect to this complex structure. Firstly we define a real three-form γ_0 on $V^{\mathbb{C}}$. We set

$$\begin{aligned}
&\gamma_0(v_1, v_2, v_3) = \omega(v_1, v_2, v_3),\\
&\gamma_0(iv_1, v_2, v_3) = \omega(Jv_1, v_2, v_3),\\
&\gamma_0(iv_1, iv_2, v_3) = \omega(Jv_1, Jv_2, v_3),\\
&\gamma_0(iv_1, iv_2, iv_3) = \omega(Jv_1, Jv_2, Jv_3),
\end{aligned}$$

for $v_1, v_2, v_3 \in V$. Then γ_0 extends uniquely to a real three-form on $V^{\mathbb{C}}$. We can easily verify that

$$\gamma_0(iw_1, w_2, w_3) = \gamma_0(w_1, iw_2, w_3) = \gamma_0(w_1, w_2, iw_3)$$

for any $w_1, w_2, w_3 \in V^{\mathbb{C}}$. Further, we set

$$\gamma_1(w_1, w_2, w_3) = -\gamma_0(iw_1, w_2, w_3) \quad \text{for } w_1, w_2, w_3 \in V^{\mathbb{C}}.$$

It is obvious that γ_1 is a real three-form satisfying

$$\gamma_1(iw_1, w_2, w_3) = \gamma_1(w_1, iw_2, w_3) = \gamma_1(w_1, w_2, iw_3),$$

for any $w_1, w_2, w_3 \in V^{\mathbb{C}}$. Now we define

$$\gamma(w_1, w_2, w_3) = \gamma_0(w_1, w_2, w_3) + i\gamma_1(w_1, w_2, w_3) \quad \text{for } w_1, w_2, w_3 \in V^{\mathbb{C}}.$$

It is obvious that γ is skew-symmetric and 3-linear over $\mathbb{R}$ and has complex values. Moreover

$$\begin{aligned} \gamma(iw_1, w_2, w_3) &= \gamma_0(iw_1, w_2, w_3) + i\gamma_1(iw_1, w_2, w_3) \\ = -\gamma_1(w_1, w_2, w_3) - i\gamma_0(i^2w_1, w_2, w_3) &= -\gamma_1(w_1, w_2, w_3) + i\gamma_0(w_1, w_2, w_3) = \\ &= i[\gamma_0(w_1, w_2, w_3) + i\gamma_1(w_1, w_2, w_3)] = i\gamma(w_1, w_2, w_3), \end{aligned}$$

which proves that γ is a complex three-form on $V^{\mathbb{C}}$. Now we prove that γ is a form of type $(3, 0)$. Obviously, it suffices to prove that for $v_1 + iJv_1 \in V^{0,1}$ and $v_2, v_3 \in V$, $\gamma(v_1 + iJv_1, v_2, v_3) = 0$. Indeed,

$$\begin{aligned} &\gamma(v_1 + iJv_1, v_2, v_3) = \gamma(v_1, v_2, v_3) + i\gamma(Jv_1, v_2, v_3) \\ =&\gamma_0(v_1, v_2, v_3) + i\gamma_1(v_1, v_2, v_3) + i\gamma_0(Jv_1, v_2, v_3) - \gamma_1(Jv_1, v_2, v_3) \\ =&\gamma_0(v_1, v_2, v_3) - i\gamma_0(iv_1, v_2, v_3) + i\gamma_0(Jv_1, v_2, v_3) + \gamma_0(iJv_1, v_2, v_3)]. \end{aligned}$$

Now $\gamma_0(iJv_1, v_2, v_3)] = \omega(J^2v_1, v_2, v_3) = -\omega(v_1, v_2, v_3) = -\gamma_0(v_1, v_2, v_3)$ and the real part of the last expression is zero, further $\gamma_0(Jv_1, v_2, v_3) = \omega(Jv_1, v_2, v_3) = \gamma_0(iv_1, v_2, v_3)$ and the complex part of the expression is zero as well. Now we easily obtain the following proposition.

8. Proposition. *Let ω be a real three-form on V satisfying $\Delta(\omega) = \{0\}$, and let J be a complex structure on V (one of the two) such that*

$$\omega(Jv_1, v_2, v_3) = \omega(v_1, Jv_2, v_3) = \omega(v_1, v_2, Jv_3).$$

Then there exists on $V^{\mathbb{C}}$ a unique complex three-form γ of type $(3, 0)$ such that

$$\omega = (\operatorname{Re}\gamma)|V.$$

Remark. The complex structure J on V can be introduced also by means of the Hitchin's invariant λ, as in [H]. Forms of complex type form an open subset U in Λ^3V^*. Hitchin has shown that this manifold also carries an almost complex structure which is integrable. Hitchin uses the following method to introduce an almost complex structure on U. One regards $U \subset \Lambda^3V^*$ as a symplectic manifold (let θ be a fixed element in Λ^6V^*; one defines the symplectic form Θ on Λ^3V^* by the equation $\omega_1 \wedge \omega_2 = \Theta(\omega_1, \omega_2)\theta$). Then the derivative of the Hamiltonian vector field corresponding to the function $\sqrt{-\lambda(\omega)}$ on U gives an integrable almost complex structure on U.

There is another way of introducing the (Hitchin's) almost complex structure on U. Given a three-form $\omega \in U$ we choose the complex structure J_ω on V (one of the two), whose existence is guaranteed by the Proposition 7. Then we define endomorphisms A_{J_ω} and D_{J_ω} of Λ^kV^* by

$$(A_{J_\omega}\Omega)(v_1, \dots, v_k) = \Omega(J_\omega v_1, \dots, J_\omega v_k),$$

$$(D_{J_\omega}\Omega)(v_1, \dots, v_k) = \sum_{i=1}^{k} \Omega(v_1, \dots, v_{i-1}, J_\omega v_i, v_{i+1}, \dots, v_k).$$

Then A_{J_ω} is an automorphism of ΛV^* and D_{J_ω} is a derivation of ΛV^*. If $k = 3$ then the automorphism $-\frac{1}{2}(A_{J_\omega} + D_{J_\omega})$ of Λ^3V^* $(= T_\omega U)$ gives a complex structure on U and coincides with the Hitchin one.

Three-forms of complex type on manifolds

We use facts from the previous section to obtain some global results on three-forms on six-dimensional manifolds. We shall denote by X, Y, Z the real vector fields on a (real) manifold M and by V, W the complex vector fields on M. $\mathfrak{X}(M)$ stands for the set of all (real) vector fields on M, $\mathfrak{X}^{\mathbb{C}}(M)$ means all the complex vector fields on M.

A three-form ω on M is called the form of complex type if for every $x \in M$ there is $\Delta(\omega_x) = \{0\}$. Let ω be a form of complex type on M and let $U \subset M$ be an open orientable submanifold. Then there exists an everywhere nonzero differentiable 6-form on U. In each T_xM, $x \in U$ construct J_- and J_+ as in Proposition 7. The construction is evidently smooth on U. Thus we obtained the following lemma.

9. Lemma. *Let ω be a form of complex type on M and let $U \subset M$ be an orientable open submanifold. Then there exist two differentiable almost complex structures J_+ and J_- on U such that*

(i) $J_+ + J_- = 0$,
(ii) $\omega(J_+X_1, X_2, X_3) = \omega(X_1, J_+X_2, X_3) = \omega(X_1, X_2, J_+X_3)$,
(iii) $\omega(J_-X_1, X_2, X_3) = \omega(X_1, J_-X_2, X_3) = \omega(X_1, X_2, J_-X_3)$,

for any vector fields X_1, X_2, X_3.

At each point $x \in M$ consider a 1-dimensional subspace of the space $\mathcal{T}^1_{1x}(M)$ of tensors of type $(1,1)$ at x generated by the tensors J_{+x} and J_{-x}. The above considerations show that it is a 1-dimensional subbundle $\mathcal{J} \subset \mathcal{T}^1_1(M)$.

10. Lemma. *The 1-dimensional vector bundles $\mathcal{J}$ and $\Lambda^6T^*(M)$ are isomorphic.*

Proof. Let us choose a Riemannian metric g_0 on TM. If $x \in M$ and $v, v' \in T_xM$ we define a riemannian metric g by the formula

$$g(v, v') = g_0(v, v') + g_0(J_+v, J_+v') = g_0(v, v') + g_0(J_-v, J_-v').$$

It is obvious that for any $v, v' \in T_xM$ we have

$$g(J_+v, J_+v') = g(v, v'), \quad g(J_-v, J_-v') = g(v, v').$$

We now define

$$\sigma_+(v, v') = g(J_+v, v'), \quad \sigma_-(v, v') = g(J_-v, v').$$

It is easy to verify that σ_+ and σ_- are nonzero 2-forms on T_xM satisfying $\sigma_+ + \sigma_- = 0$.

We define an isomorphism $h : \mathcal{J} \to \Lambda^6T^*M$. Let $x \in M$ and let $A \in \mathcal{J}_x$. We can write

$$A = aJ_+, \quad A = -aJ_-.$$

Then we set

$$hA = a\sigma_+ \wedge \sigma_+ \wedge \sigma_+ = -a\sigma_- \wedge \sigma_- \wedge \sigma_-.$$

11. Corollary. *There exist two almost complex structures J_+ and J_- on M such that*

(i) $J_+ + J_- = 0$,
(ii) $\omega(J_+X_1, X_2, X_3) = \omega(X_1, J_+X_2, X_3) = \omega(X_1, X_2, J_+X_3)$,
(iii) $\omega(J_-X_1, X_2, X_3) = \omega(X_1, J_-X_2, X_3) = \omega(X_1, X_2, J_-X_3)$,

for any vector fields X_1, X_2, X_3 if and only if the manifold M is orientable.

Hence the assertions in the rest of the article can be simplified correspondingly if M is an orientable manifold.

12. Lemma. *Let J be an almost complex structure on M such that for any vector fields $X_1, X_2, X_3 \in \mathfrak{X}(M)$,*

$$\omega(JX_1, X_2, X_3) = \omega(X_1, JX_2, X_3) = \omega(X_1, X_2, JX_3).$$

If ∇ is a linear connection on M such that $\nabla\omega = 0$, then also $\nabla J = 0$.

Proof. Let $Y \in \mathfrak{X}(M)$, and let us consider the covariant derivative ∇_Y. We get

$$\begin{aligned}
0 = (\nabla_Y\omega)(JX_1, X_2, X_3) &= Y(\omega(JX_1, X_2, X_3) - \omega((\nabla_Y J)X_1, X_2, X_3) \\
&\quad -\omega(J\nabla_Y X_1, X_2, X_2) - \omega(JX_1, \nabla_Y X_2, X_3) - \omega(JX_1, X_2, \nabla_Y X_3), \\
0 = (\nabla_Y\omega)(X_1, JX_2, X_3) &= Y(\omega(JX_1, X_2, X_3) - \omega(\nabla_Y X_1, JX_2, X_3) \\
&\quad -\omega(X_1, (\nabla_Y J)X_2, X_3) - \omega(X_1, J\nabla_Y X_2, X_3) - \omega(X_1, JX_2, \nabla_Y X_3).
\end{aligned}$$

Because the above expressions are equal we find easily that

$$\omega((\nabla_Y J)X_1, X_2, X_3) = \omega(X_1, (\nabla_Y J)X_2, X_3).$$

We denote $A = \nabla_Y J$. Extending in the obvious way the above equality, we get

$$\omega(AX_1, X_2, X_3) = \omega(X_1, AX_2, X_3) = \omega(X_1, X_2, AX_3).$$

Moreover $J^2 = -I$, and applying ∇_Y to this equality, we get

$$AJ + JA = 0.$$

We know that $K(\iota_X\omega) = [X, JX]$. Furthermore

$$\omega(X, AX, X') = \omega(X, X, AX') = 0, \quad \omega(X, AJX, X') = \omega(X, JX, AX') = 0,$$

which shows that A preserves the distribution $[X, JX]$. By the very same arguments as in Proposition 7 we can see that $A = \lambda_0 I + \lambda_1 J$. Consequently

$$\begin{gathered}
(\lambda_0 I + \lambda_1 J)J + J(\lambda_0 I + \lambda_1 J) = 0, \\
-2\lambda_1 I + 2\lambda_0 J = 0,
\end{gathered}$$

which implies $\lambda_0 = \lambda_1 = 0$. Thus $\nabla_Y J = A = 0$.

The statement of the previous lemma can to some extent be reversed, and we get the following theorem.

13. Theorem. *Let ω be a real three-form on a six-dimensional differentiable manifold M satisfying $\Delta(\omega_x) = \{0\}$ for any $x \in M$. Let J be an almost complex structure on M such that for any vector fields $X_1, X_2, X_3 \in \mathfrak{X}(M)$*

$$\omega(JX_1, X_2, X_3) = \omega(X_1, JX_2, X_3) = \omega(X_1, X_2, JX_3).$$

Then there exists a symmetric connection $\tilde{\nabla}$ on M such that $\tilde{\nabla}\omega = 0$ if and only if the following conditions are satisfied

(i) $d\omega = 0$,
(ii) *the almost complex structure J is integrable.*

Proof. First, we prove that the integrability of the structure J and the fact that ω is closed implies the existence of a symmetric connection with respect to which ω is parallel.

For any connection ∇ on M we shall denote by the same symbol its complexification. Namely, we set

$$\nabla_{X_0+iX_1}(Y_0 + iY_1) = (\nabla_{X_0}Y_0 - \nabla_{X_1}Y_1) + i(\nabla_{X_0}Y_1 + \nabla_{X_1}Y_0).$$

Let us assume that there exists a symmetric connection $\overset{\circ}{\nabla}$ such that $\overset{\circ}{\nabla}J = 0$. We shall consider a three-form γ of type $(3, 0)$ such that $(\operatorname{Re}\gamma)|TM = \omega$. Our next aim is to try to find a symmetric connection

$$\nabla_V W = \overset{\circ}{\nabla}_V W + Q(V, W),$$

satisfying $\nabla_V\gamma = 0$. Obviously, the connection ∇ is symmetric if and only if

$$Q(V, W) = Q(W, V).$$

Moreover, $\nabla_V\gamma = 0$ suggests that $\nabla J = 0$.

$$0 = (\nabla_V J)W = \nabla_V(JW) - J\nabla_V W = \overset{\circ}{\nabla}_V(JW) + Q(V, JW) - J\overset{\circ}{\nabla}_V W - JQ(V, W),$$

which shows that we should require

$$Q(JV, W) = Q(V, JW) = JQ(V, W).$$

Because $\overset{\circ}{\nabla}J = 0$, we can immediately see that for any $V \in \mathfrak{X}^{\mathbb{C}}(M)$ the covariant derivative $\overset{\circ}{\nabla}_V\gamma$ is again a form of type $(3, 0)$. Consequently there exists a uniquely determined complex 1-form ρ such that

$$\overset{\circ}{\nabla}_V\gamma = \rho(V)\gamma.$$

Then

$$\begin{aligned}
&(\nabla_V\gamma)(W_1, W_2, W_3)\\
=&V(\gamma(W_1, W_2, W_3)) - \gamma(\nabla_V W_1, W_2, W_3) - \gamma(W_1, \nabla_V W_2, W_3) - \gamma(W_1, W_2, \nabla_V W_3)\\
=&V(\gamma(W_1, W_2, W_3)) - \gamma(\overset{\circ}{\nabla}_V W_1, W_2, W_3) - \gamma(W_1, \overset{\circ}{\nabla}_V W_2, W_3) - \gamma(W_1, W_2, \overset{\circ}{\nabla}_V W_3)\\
&- \gamma(Q(V, W_1), W_2, W_3) - \gamma(W_1, Q(V, W_2), W_3) - \gamma(W_1, W_2, Q(V, W_3))\\
=&\rho(V)\gamma(W_1, W_2, W_3)\\
&- \gamma(Q(V, W_1), W_2, W_3) - \gamma(W_1, Q(V, W_2), W_3) - \gamma(W_1, W_2, Q(V, W_3)).
\end{aligned}$$

In other words $\nabla_V \gamma = 0$ if and only if

$$\begin{aligned}&\rho(V)\gamma(W_1, W_2, W_3)\\ &= \gamma(Q(V, W_1), W_2, W_3) + \gamma(W_1, Q(V, W_2), W_3) + \gamma(W_1, W_2, Q(V, W_3)).\end{aligned}$$

Sublemma. *If $d\gamma = 0$, then ρ is a form of type $(1, 0)$.*

Proof. Let $V_1 \in T^{0,1}(M)$. Because $\overset{\circ}{\nabla}$ is symmetric $d\gamma = -\mathcal{A}(\overset{\circ}{\nabla}\gamma)$, where $\mathcal{A}$ denotes the alternation. We obtain

$$\begin{aligned}0 = -4!(d\gamma)(V_1, V_2, V_3, V_4) &= \sum_{\pi} \operatorname{sign}(\pi)(\overset{\circ}{\nabla}_{V_{\pi 1}}\gamma)(V_{\pi 2}, V_{\pi 3}, V_{\pi 4})\\ + \sum_{\tau} \operatorname{sign}(\tau)(\overset{\circ}{\nabla}_{V_1}\gamma)(V_{\tau 2}, V_{\tau 3}, V_{\tau 4}) &= 3!(\overset{\circ}{\nabla}_{V_1}\gamma)(V_2, V_3, V_4) = 3!\rho(V_1)\gamma(V_2, V_3, V_4).\end{aligned}$$

The first sum is taken over all permutations π satisfying $\pi 1 > 1$, and the second one is taken over all permutations of the set $\{2, 3, 4\}$. The first sum obviously vanishes, and $\rho(V_1) = 0$. This finishes the proof.

We set now

$$Q(V, W) = \frac{1}{8}[\rho(V)W - \rho(JV)JW + \rho(W)V - \rho(JW)JV].$$

It is easy to see that $Q(JV, W) = Q(V, JW) = JQ(V, W)$. For $V, W_1, W_2, W_3 \in T^{1,0}(M)$ we can compute

$$\begin{aligned}&8\gamma(Q(V, W_1), W_2, W_3)\\ =&\gamma(\rho(V)W_1 - \rho(JV)JW_1 + \rho(W_1)V - \rho(JW_1)JV, W_2, W_3)\\ =&\gamma(2\rho(V)W_1 + 2\rho(W_1)V, W_2, W_3) = 2\rho(V)\gamma(W_1, W_2, W_3) + 2\rho(W_1)\gamma(V, W_2, W_3),\end{aligned}$$

where we used for $V \in T^{(1,0)}(M)$ that $\rho(JV) = i\rho(V)$ and $\gamma(JV, V', V'') = i\gamma(V, V', V'')$, since γ is of type $(3, 0)$ and ρ of type $(1, 0)$.

Similarly we can compute $\gamma(W_1, Q(V, W_2), W_3)$ and $\gamma(W_1, W_2, Q(V, W_3))$. Without a loss of generality we can assume that the vector fields W_1, W_2, W_3 are linearly independent (over $\mathbb{C}$). Then we can find uniquely determined complex functions f_1, f_2, f_3 such that

$$V = f_1 W_1 + f_2 W_2 + f_3 W_3.$$

Then we get

$$\begin{aligned}&\rho(W_1)\gamma(V, W_2, W_3) + \rho(W_2)\gamma(W_1, V, W_3) + \rho(W_3)\gamma(W_1, W_2, V)\\ =&f_1\rho(W_1)\gamma(W_1, W_2, W_3) + f_2\rho(W_2)\gamma(W_1, W_2, W_3) + f_3\rho(W_3)\gamma(W_1, W_2, W_3)\\ =&\rho(f_1 W_1 + f_2 W_2 + f_3 W_3)\gamma(W_1, W_2, W_3) = \rho(V)\gamma(W_1, W_2, W_3).\end{aligned}$$

Finally we obtain

$$\begin{aligned}\gamma(Q(V, W_1), W_2, W_3) + &\gamma(W_1, Q(V, W_2), W_3) + \gamma(W_1, W_2, Q(V, W_3))\\ =&\rho(V)\gamma(W_1, W_2, W_3).\end{aligned}$$

which proves $\nabla_V \gamma = 0$.

Let us continue with the main stream of the proof. We shall now use the complex connection ∇. For $X, Y \in TM$ we shall denote $\nabla^0_X Y = \operatorname{Re} \nabla_X Y$ and $\nabla^1_X Y = \operatorname{Im} \nabla_X Y$. This means that we have $\nabla_X Y = \nabla^0_X Y + i\nabla^1_X Y$. For a real function f on M we have

$$\nabla_X(fY) = \nabla^0_X(fY) + i\nabla^1_X(fX),$$

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y = [(Xf)Y + f\nabla^0_X Y] + if\nabla^1_X Y,$$

which implies

$$\nabla^0_X(fY) = (Xf)Y + f\nabla^0_X Y, \quad \nabla^1_X(fY) = f\nabla^1_X Y.$$

This shows that ∇^0 is a real connection while ∇^1 is a real tensor field of type $(1, 2)$. We have also

$$0 = \nabla_X Y - \nabla_Y X - [X, Y] = \nabla^0_X Y + i\nabla^1_X Y - \nabla^0_Y X - i\nabla^1_Y X - [X, Y] =$$

$$= [\nabla^0_X Y - \nabla^0_Y X - [X, Y]] + i[\nabla^1_X Y - \nabla^1_Y X],$$

which shows that

$$\nabla^0_X Y - \nabla^0_Y X - [X, Y] = 0, \quad \nabla^1_X Y - \nabla^1_Y X = 0.$$

These equations show that the connection ∇^0 is symmetric, and that the tensor ∇^1 is also symmetric. Moreover, we have

$$\nabla_X(JY) = \nabla^0_X(JY) + i\nabla^1_X(JY),$$

$$\nabla_X(JY) = J\nabla_X Y = J\nabla^0_X Y + iJ\nabla^1_X Y,$$

which gives

$$\nabla^0_X J = 0, \quad \nabla^1_X(JY) = J\nabla^1_X Y.$$

For the real vectors $X, Y_1, Y_2, Y_3 \in TM$ we can compute

$$0 = (\nabla_X \gamma)(Y_1, Y_2, Y_3) = X(\gamma(Y_1, Y_2, Y_3)) -$$

$$-\gamma(\nabla_X Y_1, Y_2, Y_3) - \gamma(Y_1, \nabla_X Y_2, Y_3) - \gamma(Y_1, Y_2, \nabla_X Y_3) =$$

$$= X(\gamma(Y_1, Y_2, Y_3)) -$$

$$-\gamma(\nabla^0_X Y_1 + i\nabla^1_X Y_1, Y_2, Y_3) - \gamma(Y_1, \nabla^0_X Y_2 + i\nabla^1_X Y_2, Y_3) - \gamma(Y_1, Y_2, \nabla^0_X Y_3 + i\nabla^1_X Y_3) =$$

$$= X(\gamma(Y_1, Y_2, Y_3)) - \gamma(\nabla^0_X Y_1, Y_2, Y_3) - \gamma(Y_1, \nabla^0_X Y_2, Y_3) - \gamma(Y_1, Y_2, \nabla^0_X Y_3) -$$

$$-i[\gamma(\nabla^1_X Y_1, Y_2, Y_3) + \gamma(Y_1, \nabla^1_X Y_2, Y_3) + \gamma(Y_1, Y_2, \nabla^1_X Y_3)] =$$

$$= [X(\gamma_0(Y_1, Y_2, Y_3)) - \gamma_0(\nabla^0_X Y_1, Y_2, Y_3) - \gamma_0(Y_1, \nabla^0_X Y_2, Y_3) - \gamma_0(Y_1, Y_2, \nabla^0_X Y_3) +$$

$$+\gamma_1(\nabla^1_X Y_1, Y_2, Y_3) + \gamma_1(Y_1, \nabla^1_X Y_2, Y_3) + \gamma_1(Y_1, Y_2, \nabla^1_X Y_3)] +$$

$$+i[X(\gamma_1(Y_1, Y_2, Y_3)) - \gamma_1(\nabla^0_X Y_1, Y_2, Y_3) - \gamma_1(Y_1, \nabla^0_X Y_2, Y_3) - \gamma_1(Y_1, Y_2, \nabla^0_X Y_3) -$$

$$-\gamma_0(\nabla^1_X Y_1, Y_2, Y_3) - \gamma_0(Y_1, \nabla^1_X Y_2, Y_3) - \gamma_0(Y_1, Y_2, \nabla^1_X Y_3)].$$

This shows that the real part is zero. The complex part then gives in fact the same identity, and thus it is zero as well. Using the relations between γ_0 and γ_1 we get

$$0 = X(\gamma_0(Y_1, Y_2, Y_3)) - \gamma_0(\nabla^0_X Y_1, Y_2, Y_3) - \gamma_0(Y_1, \nabla^0_X Y_2, Y_3) - \gamma_0(Y_1, Y_2, \nabla^0_X Y_3) -$$

$$-\gamma_0(J\nabla^1_X Y_1, Y_2, Y_3) - \gamma_0(Y_1, J\nabla^1_X Y_2, Y_3) - \gamma_0(Y_1, Y_2, J\nabla^1_X Y_3) =$$

$$= X(\gamma_0(Y_1, Y_2, Y_3)) - \gamma_0(\nabla^0_X Y_1 + J\nabla^1_X Y_1, Y_2, Y_3)$$

$$-\gamma_0(Y_1, \nabla^0_X Y_2 + J\nabla^1_X Y_2, Y_3) - \gamma_0(Y_1, Y_2, \nabla^0_X Y_3 + J\nabla^1_X Y_3).$$

We define now

$$\tilde{\nabla}_X Y = \nabla^0_X Y + J\nabla^1_X Y.$$

It is easy to verify that $\tilde{\nabla}$ is a real connection. Moreover, the previous equation shows that

$$\tilde{\nabla}\gamma_0 = 0.$$

Furthermore, it is very easy to see that the connection $\tilde{\nabla}$ is symmetric.

The inverse implication can also be proved easily.

Let us use the standard definition of integrability of a k-form ω on M, that is every $x \in M$ has a neighbourhood N such that ω has the constant expresion in dx^i, x^i being suitable coordinate functions on N.

14. Corollary. *Let ω be a real three-form on a six-dimensional differentiable manifold M satisfying $\Delta(\omega_x) = \{0\}$ for any $x \in M$. Let J be an almost complex structure on M such that for any vector fields $X_1, X_2, X_3 \in \mathfrak{X}(M)$*

$$\omega(JX_1, X_2, X_3) = \omega(X_1, JX_2, X_3) = \omega(X_1, X_2, JX_3).$$

Then ω is integrable if and only if there exists a symmetric connection ∇ preserving ω, that is $\nabla\omega = 0$.

Proof. Let ∇ be a symmetric connection such that $\nabla\omega = 0$. Then according to the previous proposition $d\omega = 0$ and J is integrable. Then we construct the complex form γ on $T^{\mathbb{C}}M$ of type $(3,0)$ such and $\operatorname{Re}\gamma|T_xM = \omega$, for any $x \in M$ (point by point, according to Proposition 8). Moreover if ω is closed then so is γ. That is $\gamma = f \cdot dz^1 \wedge dz^2 \wedge dz^3$, where z^1, z^2, and z^3 are (complex) coordinate functions on M, dz^1, dz^2, dz^3 are a basis of $\Lambda^{1,0}M$ and f a function on M. Further

$$0 = d\gamma = \partial\gamma + \overline{\partial}\gamma = \partial f \cdot dz^1 \wedge dz^2 \wedge dz^3 + \overline{\partial} f \cdot dz^1 \wedge dz^2 \wedge dz^3.$$

Evidently $\partial\gamma = 0$, which means $\overline{\partial} f = 0$ and f is holomorphic. Now we exploit a standard trick. There exists a holomorphic function $F(z^1, z^2, z^3)$ such that $\frac{\partial F}{\partial z^1} = f$. We introduce new complex coordinates $\tilde{z}^1 = F(z^1, z^2, z^3)$, $\tilde{z}^2 = z^2$, and $\tilde{z}^3 = z^3$. Then $\gamma = f dz^1 \wedge dz^2 \wedge dz^3 = d\tilde{z}^1 \wedge d\tilde{z}^2 \wedge d\tilde{z}^3$. Now write $\tilde{z}^1 = x^1 + ix^4$, $\tilde{z}^2 = x^2 + ix^5$, and $\tilde{z}^3 = x^3 + ix^6$ for real coordinate functions x^1, x^2, x^3, x^4, x^5, and x^6 on M. There is

$$\begin{aligned}\operatorname{Re}\gamma =& \operatorname{Re}(d(x^1+ix^4)\wedge d(x^2+ix^5)\wedge d(x^3+ix^6))\\ =& dx^1\wedge dx^2\wedge dx^3 - dx^1\wedge dx^5\wedge dx^6 + dx^2\wedge dx^4\wedge dx^6 - dx^3\wedge dx^4\wedge dx^5.\end{aligned}$$

And $\omega = (\operatorname{Re}\gamma)|TM$ is an integrable on M.

Conversely, if ω is integrable, then for any $x\in M$ there is a basis $dx_1,\dots,dx_6$ of T^*N in some neighbourhoud $N\subset M$ of x such that ω has a constant expression in all T_xM, $x\in N$. Then the flat connection ∇ given by the coordinate system $x_1,\dots,x_6$ is symmetric and $\nabla\omega = 0$ on N. We use the partition of the unity and extend ∇ over the whole M.

We can reformulate Proposition 13 as 'The Darboux theorem for complex type forms':

15. Corollary. *Let ω be a real three-form on a six-dimensional differentiable manifold M satisfying $\Delta(\omega_x) = \{0\}$ for any $x\in M$. Let J be an almost complex structure on M such that for any vector fields $X_1, X_2, X_3 \in \mathfrak{X}(M)$*

$$\omega(JX_1, X_2, X_3) = \omega(X_1, JX_2, X_3) = \omega(X_1, X_2, JX_3).$$

Then ω is integrable if and only if the following conditions are satisfied

(i) $d\omega = 0$,
(ii) *the almost complex structure J is integrable.*

16. Observation. There is an interesting relation between structures given by a form of complex type on six-dimensional vector spaces and G_2-structures on 7-dimensional ones (G_2 being the exeptional Lie group, the group of automorphisms of the algebra of Caley numbers and also the group of automorphism of the three-form given below), i.e. structures given by a form of the type

$$\begin{aligned}&\alpha_1\wedge\alpha_2\wedge\alpha_3 + \alpha_1\wedge\alpha_4\wedge\alpha_5 - \alpha_1\wedge\alpha_6\wedge\alpha_7 + \alpha_2\wedge\alpha_4\wedge\alpha_6 + \alpha_2\wedge\alpha_5\wedge\alpha_7 +\\ &+\alpha_3\wedge\alpha_4\wedge\alpha_7 - \alpha_3\wedge\alpha_5\wedge\alpha_6,\end{aligned}$$

where $\alpha_1,\dots,\alpha_7$ are the basis of the vector space V. *If we restrict a form of this type to any six-dimensional subspace of V we get a form of complex type.* Thus any G_2 structure on a 7-dimensional manifold gives a structure of complex type on any six-dimensional submanifold. Thus we get a vast variety of examples.

G_2 structures have been well studied and many examples of G_2 structures are known. See for example [J].

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