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Phase transitions with interfacial energy: convexity conditions and the existence of minimizers

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The article presents a variational theory of sharp phase interfaces bearing a deformation dependent energy. The theory involves both the standard and Eshelby stresses. The constitutive theory is outlined including the symmetry considerations and some particular cases. The existence of phase equilibria is proved based on appropriate convexity properties of the interfacial energy. Some generalization of the convexity properties is given and a relationship established to the semiellipticity condition from the theory of parametric integrals over rectifiable currents.

Contents

Introduction

This article presents convexity conditions for the deformation dependent energies of sharp phase interfaces. The interfaces are modeled as two dimensional surfaces separating bulk phases of a phase transforming body. For the bulk phases, the satisfaction or violation of the convexity properties [3–4] has proved to be extremely useful in understanding the behavior of bodies. The latter range from ‘regular’ ones showing the existing equilibrium states with good properties under favorable convexity properties up to the ‘singular’ ones with no equilibrium states and formation of microstructures in case of energies missing these convexity properties [4, 21]. The microstructures of different phases or variants of the body are frequently examined under the approximation that phase interfaces bear no energy, which leads to a picture of infinitely fine microstructures, mathematically described by Young’s measures, whereas observations show limited fineness.

In this article it is assumed that the contribution of the interface to the total energy can be described by a surface density $\mathbb{f}$ which depends on the deformation of the interface via the constitutive equation

$$\mathbb{f}(\mathbf{x}) = \hat{\mathbb{f}}(\mathbb{F}(\mathbf{x}), \mathbb{m}(\mathbf{x})) \tag{1}$$

for each material point $\mathbf{x} \in \mathcal{S}$ of the interface where $\mathbb{F}$ is the surface deformation gradient (see below), $\mathbb{m}$ the normal of $\mathcal{S}$, and $\hat{\mathbb{f}}$ is a response function. The surface energy leads to the standard and configurational stresses acting in the interface. The bulk phases are governed by the standard equations of nonlinear elasticity.

A constitutive theory for the equation (1) and the associated stress relations are given below, including the symmetry considerations and the discussion of the roles and interrelations of the two types of stresses. Next we discuss the formal aspects of the equilibrium of phases with interfaces governed by (1). The equilibrium equations for standard and configurational stresses are obtained as necessary conditions for minimum of the total energy. The resulting equations including the constitutive theory are equivalent to the equilibrium part of the theory developed by Gurtin and collaborators [16, 13], [14; Chapter 21] although the motivation is slightly different.

We examine the existence of two phase equilibrium states. We introduce the interface quasiconvexity of $\hat{\mathbb{f}}$ as the basic convexity property, which is an analog of the quasiconvexity of the bulk energy, but in contrast, it involves a variation of the integration domain. Associated are the interface null lagrangians, functions $\hat{\mathbb{f}}$ such that both $\hat{\mathbb{f}}$ and $-\hat{\mathbb{f}}$ are interface quasiconvex. The interface null lagrangians admit an explicit description. In dimension 3, there are 15 independent interface null lagrangians. Based on interface null lagrangians are interface polyconvex functions, defined as the supremum of some family of interface null lagrangians. The interface convexity properties are related to the continuity properties of the total interface energy under appropriate weak type convergences. The interface polyconvexity of $\hat{\mathbb{f}}$, accompanied by the polyconvexity of the *individual* bulk phases and by the coercivity, leads to the existence of the energy minimizing states of coexistent phases.

The theory described so far deals with integral functionals over varying surfaces of dimension $n-1$ in R^n. One can consider, more generally, integrals over surfaces of dimension r in R^n with $0 \le r \le n$. The interface quasiconvexity easily generalizes to this situation; the resulting notion is termed quasiconvexity of degree r here. (The quasiconvexity of degree n in R^n is the classial quasiconvexity of bulk phases.) One then defines null lagrangians of degree r and polyconvex functions of degree r in R^n. The null lagrangians and polyconvex functions of degree r admit explicit descriptions.

The variable nature of the integration domains of the integral functionals considered here also occurs in parametric integrands in the theory of minimal surfaces. Indeed, there are close relationships. Roughly, each deformed surface of dimension r gives rise to its graph; if the deformation is lipschitzian, then the graph can be intepreted as an r dimensional rectifiable current. In this way, each integral functional gives rise to a degree r parametric integral over rectifiable currents of the type considered in the theory of minimal surfaces. It turns out that the convexity properties of the so related functionals essentially coincide.

Thus the degree r quasiconvexity is implied by the semiellipticity of parametric integrands of degree r introduced by Almgren [1; Section 1] (see also Federer [9; Subsection 5.1.2]). Conversely, each integrand that is degree r quasiconvex satisfies the semiellipticity inequality on rectifiable currents that can be represented as graphs. We mention that the semiellipticity of a nonnegative parametric integrand is equivalent to the lowersemicontinuity of the parametric integral under the flat norm. Thus it plays the same role as the bulk quasiconvexity and our interface quasiconvexity. Pursuing this relationsip further, one can define the semielliptic null lagrangians of degree r as parametric integrands such that the integrand and its negative are semielliptic. Semielliptic null lagrangians admit a simple explicit description giving a one to one correspondence with the null lagrangians of degree r. The degree r polyconvexity then corresponds to the convex parametric integrands. The graph view leads to a noncalculational proof of the structure of the null lagrangians of degree $r \le n-1$ in R^n and substatializes some propositions dicussed hitherto. It is also recalled that the graph view is basic to the approach of the elasticity of the bulk matter by Giaquinta, Modica and Souček [11–12].

Some of the results announced in [27] are proved here.

Chapter 1

Constitutive theory

1.1 Informal description

Consider a deformed body in a state with two coexistent phases separated by a phase interface. We use a fixed reference configuration represented by a bounded open set $\Omega \subset \mathrm{R}^3$ and identify material points of the body with elements $\mathbf{x}$ of Ω. The state of the body is described by a deformation function $\mathbf{y} : \Omega \to \mathrm{R}^3$ and by an open subset E of Ω occupied by the first phase; the region occupied by the second phase is the complement of E in Ω. The phase interface $\mathcal{S}$ is the part of the boundary of E that is contained in Ω (the rest of the boundary of E being a subset of the boundary of Ω, possibly empty). The deformation function $\mathbf{y}$ gives the actual position $\mathbf{y}(\mathbf{x})$ of the material point $\mathbf{x} \in \Omega$, in particular for $\mathbf{x} \in \mathcal{S}$ the value $\mathbb{y}(\mathbf{x}) = \mathbf{y}(\mathbf{x})$ gives the actual position of the interface points. We define the bulk deformation gradient $\mathbf{F}$ by

$$\mathbf{F} = \nabla \mathbf{y}$$

for every $\mathbf{x} \in \Omega \sim \mathcal{S}$ and the surface deformation gradient [16, 13]

$$\mathbb{F} = \mathbb{\nabla} \mathbb{y}, \quad \mathbb{F}\mathbb{m} = \mathbf{0}, \tag{1.1.1}$$

where $\mathbb{\nabla}$ denotes the surface gradient (see Section A.1) and $\mathbb{m}$ is the interface normal.

The density of energy of the bulk phases $\alpha = 1, 2$ is given by the energy functions $\hat{f}_\alpha$, $\alpha = 1, 2$, by

$$f_\alpha(\mathbf{x}) = \hat{f}_\alpha(\mathbf{F}(\mathbf{x})), \quad \mathbf{x} \in E_\alpha, \quad E_1 := E, \quad E_2 := \Omega \sim \operatorname{cl} E$$

where throughout, cl and bd denote the closure and boundary. The surface density of the interface energy is given by the interfacial energy function $\hat{\mathbb{f}}$ by

$$\mathbb{f}(\mathbf{x}) = \hat{\mathbb{f}}(\mathbb{F}(\mathbf{x}), \mathbb{m}(\mathbf{x})), \quad \mathbf{x} \in \mathcal{S}. \tag{1.1.2}$$

We note that the two arguments of $\hat{\mathbb{f}}$ in (1.1.2) are not independent [see (1.1.1)]; this has some consequences on the form of the stress relations.

The total energy of the state $(\mathbf{y}, E)$ is

$$\mathsf{E}(\mathbf{y}, E) = \mathsf{E}_{\mathrm{b}}(\mathbf{y}, E) + \mathsf{E}_{\mathrm{if}}(\mathbf{y}, E)$$

where

$$\mathsf{E}_{\mathrm{b}}(\mathbf{y}, E) = \int_E \hat{f}_1(\mathbf{F})\, d\mathcal{L}^3 + \int_{\Omega \sim E} \hat{f}_2(\mathbf{F})\, d\mathcal{L}^3,$$

$$\mathsf{E}_{\mathrm{if}}(\mathbf{y}, E) = \int_{\mathcal{S}} \hat{\mathbb{f}}(\mathbb{F}, \mathbb{m})\, d\mathcal{H}^2$$

are the bulk and interface energies, with $d\mathcal{L}^3$ the referential volume element and $d\mathcal{H}^2$ the referential area element. Equilibrium states of the system correspond to minimum energy under the constraints imposed by the environment of the system. Here the region E is unknown.

The two bulk energies represent two energy wells of the substance. The functions $\hat{f}_\alpha$ give rise to the bulk standard and configurational stresses $\hat{\mathbf{S}}$ and $\hat{\mathbf{C}}$ given by

$$\hat{\mathbf{S}}_\alpha = \mathrm{D}\hat{f}_\alpha, \quad \hat{\mathbf{C}}_\alpha = \hat{f}_\alpha \mathbf{1} - \mathbf{F}^{\mathrm{T}}\, \mathrm{D}\hat{f}_\alpha. \tag{1.1.3}$$

The interfacial energy $\hat{\mathbb{f}}$ leads to the standard and configurational interfacial stresses $\hat{\mathbb{S}}$ and $\hat{\mathbb{C}}$ given by

$$\hat{\mathbb{S}} = \mathrm{D}_1 \hat{\mathbb{f}}\, \mathbb{P}, \tag{1.1.4}$$

$$\hat{\mathbb{C}} = \hat{\mathbb{f}}\, \mathbb{P} - \mathbb{F}^{\mathrm{T}}\, \mathrm{D}_1 \hat{\mathbb{f}}\, \mathbb{P} + \mathbb{m} \otimes (\mathbb{F}^{\mathrm{T}}\, \mathrm{D}_1 \hat{\mathbb{f}}\, \mathbb{m} - \mathrm{D}_2 \hat{\mathbb{f}}). \tag{1.1.5}$$

Here $\mathrm{D}_1 \hat{\mathbb{f}}$ and $\mathrm{D}_2 \hat{\mathbb{f}}$ denote the 'partial derivatives' of $\hat{\mathbb{f}} = \hat{\mathbb{f}}(\mathbb{F}, \mathbb{m})$ with respect to the first and second variables as defined in Sections 1.2 and A.1, and $\mathbb{P} = \mathbf{1} - \mathbb{m} \otimes \mathbb{m}$ is the projection onto the tangent spave to the interface and the given point. The forms of the stress relations (1.1.3), (1.1.4) and (1.1.5) is dictated by the equilibrium equations for the standard and configurational stresses (see Sections 2.1), which, in turn, are uniquely determined as necessary conditions for the minima of the total energy. Our motivation for the standard and configurational stresses is variational, relating the standard standard stress with outer variations and configurational stress with inner variations, as explained in Section 2.1. Also, the standard and configurational stresses exchange their roles under the exchange of the roles of actual and reference configurations.

The principle of material frame indifference restricts the behavior of the response functions under the multiplications of the deformation gradients from left; the symmetry of the material (such as the isotropy or the crystallographic symmetry) restricts the behavior of the response functions under the action of the symmetry tensors on the arguments from right. The last is related to the change of the reference configuration Ω used to describe the body. These matters are discussed in Section 1.5 where also the main types of the symmetry are mentioned.

1.2 Response functions

We generalize and formalize the picture of Section 1.1 as follows.

We denote by $\mathrm{Lin}(V, W)$ the set of all linear transformations from a vectorspace V into a vectorspace W. Throughout, m, n are positive integers and we write $\mathrm{Lin} :=$

$\mathrm{Lin}(\mathrm{R}^n, \mathrm{R}^m)$ unless stated otherwise. Denote by S^{n-1} the unit sphere in R^n. We model the reference configuration of the body by an open bounded subset Ω of R^n and consider deformations $\mathbf{y} : \Omega \to \mathrm{R}^m$. In applications, $m = n = 3$. In the treatment of the constitutive theory for the interface it is necessary to take into account that the surface energy $\hat{\mathbb{f}} = \hat{\mathbb{f}}(\mathbb{F}, \mathbb{m})$ is defined on pairs satisfying $\mathbb{F}\mathbb{m} = \mathbf{0}$, this set G forms a *submanifold* of the space $\mathrm{Lin} \times \mathrm{R}^n$. Thus the derivatives of $\hat{\mathbb{f}}$ belong to the tangent space of G and hence the "partial derivatives" with respect to $\mathbb{F}$ and $\mathbb{m}$ are not independent. The derivative of a map on a manifold is defined in Section A.1.

The system of forces acting in a phase transforming body consists of standard and configurational forces. The standard forces are represented by the (referential) bulk stress tensor $\mathbf{S}$ and the (referential) interface stress tensor $\mathbb{S}$ acting, respectively, in the bulk matter and in the interface. The configurational forces are described by the (referential) bulk configurational stress tensor $\mathbf{C}$ and by the (referential) interface stress tensor $\mathbb{C}$. The response functions for all these 4 stresses are completely determined by the response functions for the free energy.

Definitions 1.2.1 (Constitutive information and response functions).

(i) The two bulk phases are indexed by $\alpha = 1, 2$, each phase is described by the ***bulk energy*** $\hat{f}_\alpha : U_\alpha \to \mathrm{R}$ where $U_\alpha \subset \mathrm{Lin}$ is an open set and $\hat{f}_\alpha$ are class 2 functions. We define the response functions for the standard and configurational stresses $\hat{\mathbf{S}}_\alpha : U_\alpha \to \mathrm{Lin}(\mathrm{R}^n, \mathrm{R}^m)$, $\hat{\mathbf{C}}_\alpha : U_\alpha \to \mathrm{Lin}(\mathrm{R}^n, \mathrm{R}^n)$ by

$$\hat{\mathbf{S}}_\alpha = \mathrm{D}\hat{f}_\alpha, \quad \hat{\mathbf{C}}_\alpha = \hat{f}_\alpha \mathbf{1} - \mathbf{F}^{\mathrm{T}} \mathrm{D}\hat{f}_\alpha$$

for each $\mathbf{F} \in U_\alpha$, where $\hat{\mathbf{S}}_\alpha$, $\hat{\mathbf{C}}_\alpha$, $\hat{f}_\alpha$ and its derivatives are evaluated at $\mathbf{F}$.

(ii) The interface is described by the ***interfacial energy*** $\hat{\mathbb{f}} : \mathbb{U} \to \mathrm{R}$ where $\mathbb{U}$ is a (relatively) open subset of the class ∞ manifold

$$\mathrm{G} = \{(\mathbb{F}, \mathbb{m}) \in \mathrm{Lin} \times \mathrm{S}^{n-1} : \mathbb{F}\mathbb{m} = \mathbf{0}\}$$

and $\hat{\mathbb{f}}$ is a class 2 function. The derivative of $\hat{\mathbb{f}}$ at $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$ is an element of the tangent space $\mathrm{Tan}(\mathrm{G}, (\mathbb{F}, \mathbb{m}))$ of G at $(\mathbb{F}, \mathbb{m})$ given by

$$\mathrm{Tan}(\mathrm{G}, (\mathbb{F}, \mathbb{m})) = \{(\mathbb{G}, \mathbb{mm}) \in \mathrm{Lin} \times \mathrm{R}^n : \mathbb{G}\mathbb{m} + \mathbb{F}\mathbb{mm} = \mathbf{0},\ \mathbb{mm} \cdot \mathbb{m} = 0\}; \quad (1.2.1)$$

we write $\mathrm{D}\hat{\mathbb{f}} = (\mathrm{D}_1\hat{\mathbb{f}}, \mathrm{D}_2\hat{\mathbb{f}})$ for its components in Lin and R^n, respectively. We define the ***response functions*** for the ***standard*** and ***configurational stresses*** $\hat{\mathbb{S}} : \mathrm{G} \to \mathrm{Lin}(\mathrm{R}^n, \mathrm{R}^m)$ and $\hat{\mathbb{C}} : \mathrm{G} \to \mathrm{Lin}(\mathrm{R}^n, \mathrm{R}^n)$ by

$$\hat{\mathbb{S}} = \mathrm{D}_1\hat{\mathbb{f}}\,\mathbb{P}, \quad (1.2.2)$$

$$\hat{\mathbb{C}} = \hat{\mathbb{f}}\,\mathbb{P} - \mathbb{F}^{\mathrm{T}}\mathrm{D}_1\hat{\mathbb{f}}\,\mathbb{P} + \mathbb{m} \otimes (\mathbb{F}^{\mathrm{T}}\mathrm{D}_1\hat{\mathbb{f}}\,\mathbb{m} - \mathrm{D}_2\hat{\mathbb{f}}) \quad (1.2.3)$$

for every $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$ where $\mathbb{P} = \mathbf{1} - \mathbb{m} \otimes \mathbb{m}$ and $\hat{\mathbb{S}}$, $\hat{\mathbb{C}}$, $\hat{\mathbb{f}}$ and its derivatives are evaluated at $(\mathbb{F}, \mathbb{m})$.

The form of the stress relations (1.2.2) and (1.2.3) is motivated by the variations formulas for the total energy, (2.1.5) and (2.1.6), by the correponding balance equations (2.1.7) and (2.1.8), and by the fact that with the above definitions $\hat{\mathbb{S}}$ and $\hat{\mathbb{C}}$ neatly exchange their roles under the exchange of the actual and reference configurations, Section 1.3 (below). The 'partial derivatives' $\mathrm{D}_1\hat{\mathbb{f}}$ and $\mathrm{D}_2\hat{\mathbb{f}}$ satisfy

$$\mathrm{D}_1 \hat{\mathbb{f}}(\mathbb{F}, \mathbb{m})\mathbb{m} + \mathbb{F}\,\mathrm{D}_2 \hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \mathbf{0}, \quad \mathbb{m} \cdot \mathrm{D}_2 \hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = 0$$

by (1.2.1).

1.3 The exchange of the actual and reference configurations

This section discusses the exchange of the roles of the standard and configurational stresses under the exchange of the actual and reference configurations. We consider the format the energy of Section 1.2 with $m = n$.

Given a state $(\mathbf{y}, E)$ with $\mathbf{y}$ injective, the actual configuration of the body is $\bar{\Omega} := \mathbf{y}(\Omega)$, the actual configuration of the interface is $\bar{\mathcal{S}} := \mathbb{y}(\mathcal{S})$, and the spatial interface normal $\bar{\mathbb{m}} \circ \mathbb{y} = \operatorname{cof} \mathbb{F}\mathbb{m}/|\operatorname{cof} \mathbb{F}\mathbb{m}| = \operatorname{cof} \mathbb{F}\mathbb{m}/|\operatorname{cof} \mathbb{F}|$. The fields of referential energy densities f, $\mathbb{f}$ can be transformed to the actual configuration of the body via the formulas

$$\bar{f} \circ \mathbf{y} = f/J, \quad \bar{\mathbb{f}} \circ \mathbb{y} = \mathbb{f}/\mathbb{J},$$

where

$$J = |\det \mathbf{F}|, \quad \mathbb{J} = |\operatorname{cof} \mathbb{F}|$$

are the bulk and interface jacobians. The deformation $\mathbf{y}$ is replaced by its inverse $\mathbf{y}^{-1}$ and hence the bulk deformation gradient $\mathbf{F}$ is replaced by the inverse $\mathbf{F}^{-1}$, the surface deformation gradient $\mathbb{F}$ by the pseudoinverse $\mathbb{F}^{-1}$ (see Section A.1) and the referential interface normal $\mathbb{m}$ by the spatial normal $\bar{\mathbb{m}}$.

Letting $\hat{f}$ stand for any of the energy functions $\hat{f}_\alpha$, $\alpha = 1, 2$, we thus find that under the exchange of the actual and reference configurations the response functions $\hat{f}$ and $\hat{\mathbb{f}}$ change to the response functions $\hat{f}^\star : U^\star \to \mathrm{R}$ and $\hat{\mathbb{f}}^\star : \mathbb{U}^\star \to \mathrm{R}$ given by

$$\hat{f}^\star(\mathbf{F}) = \det \mathbf{F}\,\hat{f}(\mathbf{F}^{-1}),$$
$$\hat{\mathbb{f}}^\star(\mathbb{F}, \mathbb{m}) = |\operatorname{cof} \mathbb{F}|\hat{\mathbb{f}}(\mathbb{F}^{-1}, \bar{\mathbb{m}})$$

where $\bar{\mathbb{m}} = \operatorname{cof} \mathbb{F}\mathbb{m}/|\operatorname{cof} \mathbb{F}|$, whenever

$$\mathbf{F} \in U^\star := \{\mathbf{F} \in \mathrm{Lin} : \mathbf{F}^{-1} \in U\},$$
$$(\mathbb{F}, \mathbb{m}) \in \mathbb{U}^\star := \{(\mathbb{F}, \mathbb{m}) \in \mathrm{Lin} \times \mathrm{S}^{n-1} : (\mathbb{F}^{-1}, \bar{\mathbb{m}}) \in \mathbb{U}\}.$$

In these definitions, we have denoted by $\mathbf{F}$ and $(\mathbb{F}, \mathbb{m})$ the natural variables of $\hat{f}^\star$ and $\hat{\mathbb{f}}^\star$, i.e. the variables previously denoted by $\mathbf{F}^{-1}$ and $(\mathbb{F}^{-1}, \bar{\mathbb{m}})$. We furthermore let $\hat{\mathbf{S}}$, $\hat{\mathbf{C}}$, $\hat{\mathbb{S}}$ and $\hat{\mathbb{C}}$ denote the response functions for the stresses calculated from $\hat{f}$ and $\hat{\mathbb{f}}$ and the same letters with the superscript $\star$ denote the response functions for the stresses calculated from $\hat{f}^\star$ and $\hat{\mathbb{f}}^\star$ according to Definition 1.2.1.

Proposition 1.3.1. *Under the passage from the response functions from $\hat{f}$ and $\hat{\mathbb{f}}$ to $\hat{f}^\star$ and $\hat{\mathbb{f}}^\star$ the standard and configurational stresses exchange their roles according to*

$$\hat{\mathbf{S}}^\star(\mathbf{F}) = \det \mathbf{F}\,\hat{\mathbf{C}}(\mathbf{F}^{-1})\mathbf{F}^{-\mathrm{T}}, \quad \hat{\mathbf{C}}^\star(\mathbf{F}) = \det \mathbf{F}\,\hat{\mathbf{S}}(\mathbf{F}^{-1})\mathbf{F}^{-\mathrm{T}},$$

for each $\mathbf{F} \in U^\star$ *and*

$$\hat{\mathbb{S}}^\star(\mathbb{F}, \mathbb{m}) = |\operatorname{cof} \mathbb{F}|\,\hat{\mathbb{C}}(\mathbb{F}^{-1}, \bar{\mathbb{m}})\mathbb{F}^{-\mathrm{T}}, \quad \hat{\mathbb{C}}^\star(\mathbb{F}, \mathbb{m}) = |\operatorname{cof} \mathbb{F}|\,\hat{\mathbb{S}}(\mathbb{F}^{-1}, \bar{\mathbb{m}})\mathbb{F}^{-\mathrm{T}}$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathbb{U}^\star$.

See [27; Proposition 4.1] for the proof.

1.4 Frame indifference

Mechanically realistic energy functions must satisfy the principle of material frame indifference. Letting $\hat{f} : U \to \mathrm{R}$ stand for any of the response functions $\hat{f}_\alpha$, $\alpha = 1, 2$, this requires that for every $\mathbf{F} \in U$, $(\mathbb{F}, \mathbb{m}) \in \mathbb{U}$ and $\mathbf{Q} \in SO(n)$ we have

$$\mathbf{QF} \in U, \quad (\mathbf{Q}\mathbb{F}, \mathbb{m}) \in \mathbb{U},$$

$$\hat{f}(\mathbf{QF}) = \hat{f}(\mathbf{F}), \quad \hat{\mathbf{S}}(\mathbf{QF}) = \mathbf{Q}\hat{\mathbf{S}}(\mathbf{F}), \quad \hat{\mathbf{C}}(\mathbf{QF}) = \hat{\mathbf{C}}(\mathbf{F}), \tag{1.4.1}$$

$$\hat{\mathbb{f}}(\mathbf{Q}\mathbb{F}, \mathbb{m}) = \hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}), \quad \hat{\mathbb{S}}(\mathbf{Q}\mathbb{F}, \mathbb{m}) = \mathbf{Q}\hat{\mathbb{S}}(\mathbb{F}, \mathbb{m}), \quad \hat{\mathbb{C}}(\mathbf{Q}\mathbb{F}, \mathbb{m}) = \hat{\mathbb{C}}(\mathbb{F}, \mathbb{m}). \tag{1.4.2}$$

We note that of (1.4.1) and (1.4.2) only $(1.4.1)_1$ and $(1.4.2)_1$ are independent, $(1.4.1)_{2,\,3}$ and $(1.4.2)_{2,\,3}$ follow from the stress relations.

1.5 Change in the reference configuration and the symmetry group

Another restriction of the response functions comes from the symmetry of the material. The latter include, e.g., the isotropy and the crystal symmetries of crystalline materials; the symmetry also distinguishes fluids from solids etc. Roughly, the symmetry group of a material is the set of all changes of the reference configurations that leave the response functions unchanged. There may be different symmetries of the bulk responses of the phases $\alpha = 1, 2$, and yet another symmetry of the interface response. We consider the format the energy of Section 1.2, let $\hat{f} : U \to \mathrm{R}$ stand for any of the functions $\hat{f}_\alpha$, $\alpha = 1, 2$, and assume

$$m = n, \ U = \mathrm{Lin}_+ := \{\mathbf{F} \in \mathrm{Lin} : \det \mathbf{F} > 0\}, \ \mathbb{U} = \{(\mathbb{F}, \mathbb{m}) \in \mathrm{G} : \operatorname{rank} \mathbb{F} = n - 1\}$$

for simplicity.

We first derive the change of the response functions under a change in the reference configuration. We thus consider a passage from the reference configuration Ω to $\bar{\Omega} = \mathbf{H}^{-1}\Omega$ where $\mathbf{H} \in \mathrm{Lin}_+$. Assume that in the reference configuration Ω the global state of the body is described by the pair $(\mathbf{y}, E)$ consisting of the deformation $\mathbf{y} : \Omega \to \mathrm{R}^m$ and the region $E \subset \Omega$ occupied by the phase $\alpha = 1$. In the reference configuration $\bar{\Omega}$, the same state is described by the pair $(\bar{\mathbf{y}}, \bar{E})$ where

$$\bar{\mathbf{y}}(\bar{\mathbf{x}}) = \mathbf{y}(\mathbf{H}\bar{\mathbf{x}}), \quad \bar{E} = \mathbf{H}^{-1}E,$$

$\bar{\mathbf{x}} \in \bar{\Omega}$. The interface is given by $\bar{\mathcal{S}} = \operatorname{bd} \bar{E}$ in the new reference configuration; hence

$$\bar{\mathcal{S}} = \mathbf{H}^{-1}\mathcal{S}$$

and the deformations of the interface in the original and new reference configurations are related by

$$\bar{\mathbb{y}}(\bar{\mathbf{x}}) = \mathbb{y}(\mathbf{H}\bar{\mathbf{x}}),$$

$\bar{\mathbf{x}} \in \bar{\mathcal{S}}$. At the points $\mathbf{x}$ and $\bar{\mathbf{x}}$ related by $\mathbf{x} = \mathbf{H}\bar{\mathbf{x}}$, the deformation gradients $\mathbf{F} = \nabla\mathbf{y}$, $\bar{\mathbf{F}} = \nabla\bar{\mathbf{y}}$, $\mathbb{F} = \mathbb{\nabla}\,\mathbf{y}$, $\bar{\mathbb{F}} = \mathbb{\nabla}\,\bar{\mathbf{y}}$ and the interface normals $\mathbb{m}$ and $\bar{\mathbb{m}}$ to $\mathcal{S}$ and $\bar{\mathcal{S}}$ are related by

$$\bar{\mathbf{F}} = \mathbf{FH}, \quad (\bar{\mathbb{F}}, \bar{\mathbb{m}}) = (\mathbb{F} * \mathbf{H}, \mathbb{m} * \mathbf{H})$$

where we define

$$\mathbb{m} * \mathbf{H} = \mathbf{H}^{\mathrm{T}}\mathbb{m}/|\mathbf{H}^{\mathrm{T}}\mathbb{m}|, \quad \mathbb{F} * \mathbf{H} = \mathbb{F}\mathbf{H}(\mathbf{1} - \mathbb{m} * \mathbf{H} \otimes \mathbb{m} * \mathbf{H})$$

for any $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$ and any $\mathbf{H} \in \mathrm{Lin}_+$. One easily finds that

$$\mathbb{F} * (\mathbf{HK}) = (\mathbb{F} * \mathbf{H}) * \mathbf{K}, \quad \mathbb{m} * (\mathbf{HK}) = (\mathbb{m} * \mathbf{H}) * \mathbf{K}$$

for any $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$ and $\mathbf{H}, \mathbf{K} \in \mathrm{Lin}_+$. Generally, the expressions for $\mathbb{F} * \mathbf{H}$ and $\mathbb{m} * \mathbf{H}$ are nonlinear in $\mathbb{F}$, $\mathbb{m}$ and $\mathbf{H}$; only if $\mathbf{H} = \mathbf{Q}$ is orthogonal, we have $\mathbb{F} * \mathbf{Q} = \mathbb{F}\mathbf{Q}$, $\mathbb{m} * \mathbf{Q} = \mathbf{Q}^{\mathrm{T}}\mathbb{m}$ and thus

$$(\bar{\mathbb{F}}, \bar{\mathbb{m}}) = (\mathbb{F}\mathbf{Q}, \mathbf{Q}^{\mathrm{T}}\mathbb{m}).$$

The change of variables formula for the volume and surface integrals shows that in the reference configuration $\bar{\Omega}$ the material is described by the bulk and surface energies $\bar{\hat{f}} : \bar{U} \to \mathrm{R}$, $\bar{\hat{\mathbb{f}}} : \bar{\mathbb{U}} \to \mathrm{R}$ given by

$$\bar{\hat{f}}(\bar{\mathbf{F}}) = \hat{f}(\bar{\mathbf{F}}\mathbf{H}^{-1})/J, \quad \bar{\hat{\mathbb{f}}}(\bar{\mathbb{F}}, \bar{\mathbb{m}}) = \hat{\mathbb{f}}(\bar{\mathbb{F}} * \mathbf{H}^{-1}, \bar{\mathbb{m}} * \mathbf{H}^{-1})/\mathbb{J}$$

whenever $\bar{\mathbf{F}} \in U$, $(\bar{\mathbb{F}}, \bar{\mathbb{m}}) \in \mathbb{U}$ where

$$J = \det \mathbf{H}^{-1}, \quad \mathbb{J} = |\operatorname{cof} \mathbf{H}^{-1}\bar{\mathbb{m}}|.$$

The stress relations then imply that the standard stress response functions in the reference configuration $\bar{\Omega}$ are given by

$$\bar{\hat{\mathbf{S}}}(\bar{\mathbf{F}}) = \hat{\mathbf{S}}(\bar{\mathbf{F}}\mathbf{H}^{-1})\mathbf{H}^{-\mathrm{T}}/J, \quad \bar{\hat{\mathbf{C}}}(\bar{\mathbf{F}}) = \mathbf{H}^{\mathrm{T}}\hat{\mathbf{C}}(\bar{\mathbf{F}}\mathbf{H}^{-1})\mathbf{H}^{-\mathrm{T}}/J,$$

$$\bar{\hat{\mathbb{S}}}(\bar{\mathbb{F}}, \bar{\mathbb{m}}) = \hat{\mathbb{S}}(\bar{\mathbb{F}} * \mathbf{H}^{-1}, \bar{\mathbb{m}})\mathbf{H}^{-\mathrm{T}}/\mathbb{J}, \quad \bar{\hat{\mathbb{C}}}(\bar{\mathbb{F}}, \bar{\mathbb{m}}) = \mathbf{H}^{\mathrm{T}}\hat{\mathbb{C}}(\bar{\mathbb{F}} * \mathbf{H}^{-1}, \bar{\mathbb{m}})\mathbf{H}^{-\mathrm{T}}/\mathbb{J}.$$

We define symmetry group of the bulk response as the set of all $\mathbf{H} \in \mathrm{Lin}_+$ such that the response functions in the reference configuration $\bar{\Omega} = \mathbf{H}^{-1}\Omega$ coincide with the original ones; in view of the stress relations it suffices to require the invariance of the response functions for the free energy.

Definition 1.5.1. The symmetry group $\mathscr{G}(\hat{f})$ of the bulk response is the set of all $\mathbf{H} \in \mathrm{Lin}_+$ satisfying

$$\hat{f}(\mathbf{FH}) = \hat{f}(\mathbf{F})/J$$

for all $\mathbf{F} \in U$ where $J = \det \mathbf{H}^{-1}$.

Definition 1.5.2. The symmetry group $\mathscr{G}(\hat{\mathbb{f}})$ of the interface response is the set of all $\mathbf{H} \in \mathrm{Lin}_+$ satisfying

$$\hat{\mathbb{f}}(\mathbb{F} * \mathbf{H}, \mathbb{m} * \mathbf{H}) = \hat{\mathbb{f}}(\mathbb{F}, \mathbb{m})/\mathbb{J},$$

for all $(\mathbb{F}, \mathbb{m}) \in \mathbb{U}$ where $\mathbb{J} = |\operatorname{cof} \mathbf{H}^{-1}\,\mathbb{m}|$.

One easily finds that $\mathscr{G}(\hat{f})$ and $\mathscr{G}(\hat{\mathbb{f}})$ are subgroups of Lin_+. Let $\mathscr{G}$ stand for $\mathscr{G}(\hat{f})$ or $\mathscr{G}(\hat{\mathbb{f}})$. The common types of symmetry are

- isotropy: $\mathscr{G} \supset SO(n)$;
- fluidity: $\mathscr{G} \supset \{\mathbf{H} \in \mathrm{Lin}_+ : \det \mathbf{H} = 1\}$;
- crystal symmetry: $\mathscr{G}$ is one of the 32 crystallographic point groups [24; Chapters 3 and 5].

Proposition 1.5.3. *Consider bulk and interface responses $\hat f : U \to \mathrm{R}$ and $\hat{\mathbb{f}} : \mathbb{U} \to \mathrm{R}$ that satisfy the principle of material frame indifference.*

(i) *If the bulk response is isotropic then there exists a function $\tilde f : (0,\infty)^n \to \mathrm{R}$ that is symmetric under the permutation of arguments such that*

$$\hat f(\mathbf{F}) = \tilde f(\lambda_1,\dots,\lambda_n)$$

for each $\mathbf{F} \in U$ where $\lambda_1 \ge \cdots \ge \lambda_n$ are the singular values of $\mathbf{F}$; if the interface response is isotropic then there exists a function $\tilde{\mathbb{f}} : (0,\infty)^{n-1} \to \mathrm{R}$ that is symmetric under the permutation of arguments such that

$$\hat{\mathbb{f}}(\mathbb{F},\mathbb{m}) = \tilde{\mathbb{f}}(\lambda_1,\dots,\lambda_{n-1}) \tag{1.5.1}$$

for each $(\mathbb{F},\mathbb{m}) \in \mathbb{U}$ where $\lambda_1 \ge \cdots \ge \lambda_{n-1} > 0$ are the singular values of $\mathbb{F}$;

(ii) *if the bulk response is of fluid type then there exists a function $\bar f : (0,\infty) \to \mathrm{R}$ such that*

$$\hat f(\mathbf{F}) = \bar f(\det \mathbf{F})$$

for each $\mathbf{F} \in U$; if the interface response is of fluid type then there exists a $\sigma \in \mathrm{R}$ such that

$$\hat{\mathbb{f}}(\mathbb{F},\mathbb{m}) = \sigma|\operatorname{cof}\mathbb{F}| \tag{1.5.2}$$

for each $(\mathbb{F},\mathbb{m}) \in \mathbb{U}$.

Proof (i): We omit the simple and standard proof in case of the bulk response. Consider the case of an isotropic interface response. Let $\tilde f : (0,\infty)^{n-1} \to \mathrm{R}$ be defined by

$$\tilde f(\lambda_1,\dots,\lambda_{n-1}) = \hat{\mathbb{f}}(\mathbb{F}_0,\mathbb{m}_0)$$

for each $\lambda_1,\dots,\lambda_{n-1} \in (0,\infty)^{n-1}$ where

$$\mathbb{F}_0 = \operatorname{diag}(\lambda_1,\dots,\lambda_{n-1},0), \quad \mathbb{m}_0 = (0,\dots,0,1).$$

Elementary considerations show that if $(\mathbb{F},\mathbb{m}) \in \mathrm{G}$ is a pair such that $\mathbb{F}$ has the singular values $(\lambda_1,\dots,\lambda_{n-1},0)$ with $\lambda_1 \ge \cdots \ge \lambda_{n-1} \ge 0$ then there exist $\mathbf{Q}$, $\mathbf{R} \in SO(n)$ such that

$$\mathbf{Q}\mathbb{F}\mathbf{R} = \operatorname{diag}(\lambda_1,\dots,\lambda_{n-1},0), \quad \mathbf{R}^{\mathrm{T}}\mathbb{m} = (0,\dots,0,1).$$

Here the existence of $\mathbf{Q}$, $\mathbf{R} \in O(n)$ follows from the singular value decomposition theorem and the choice of $\mathbf{Q}$, $\mathbf{R} \in SO(n)$ is achieved using $\mathbb{F}\mathbb{m} = \mathbf{0}$. Then the above relations and

$$\hat{\mathbb{f}}(\mathbf{Q}\mathbb{F}\mathbf{R},\mathbf{R}^{\mathrm{T}}\mathbb{m}) = \hat{\mathbb{f}}(\mathbb{F},\mathbb{m})$$

yield (1.5.1).

(ii): We omit the simple and standard proof in case of the bulk response. Consider the case of a fluid like interface response. In particular, the response is isotropic and thus the conclusions of (i) are available. Thus we have (1.5.1). Let $(\lambda_1,\dots,\lambda_{n-1}) \in (0,\infty)^{n-1}$ be fixed, let $\mathbb{F} = \operatorname{diag}((\lambda_1,\dots,\lambda_{n-1},0)$, $\mathbf{H} = \operatorname{diag}(\lambda_1,\dots,\lambda_{n-1},1/\lambda_1\cdots\lambda_{n-1})$, $\mathbb{m} = (0,\dots,0,1)$, and $\mathbb{F}_0 = \operatorname{diag}(1,\dots,1,0)$. Then $\mathbf{H} \in \mathrm{Lin}_+$, $\det\mathbf{H} = 1$, and

$$\mathbb{F}_0 * \mathbf{H} = \mathbb{F}, \quad \mathbb{m} * \mathbf{H} = \mathbb{m}$$

and thus we have

$$\hat{\mathbb{f}}(\mathbb{F},\mathbb{m}) = \hat{\mathbb{f}}(\mathbb{F}_0,\mathbb{m})/\mathbb{J}$$

where $\mathbb{J} = |\operatorname{cof}\mathbf{H}^{-1}\mathbb{m}| = 1/\lambda_1\cdots\lambda_{n-1}$ which reads

$$\tilde{\mathbb{f}}(\lambda_1,\dots,\lambda_{n-1}) = \sigma\lambda_1\cdots\lambda_{n-1} = \sigma|\operatorname{cof}\mathbb{F}|$$

where $\sigma = \mathbb{f}(1,\dots,1)$. Thus (1.5.1) gives (1.5.2). □

1.6 Particular cases of interface response

In this section we consider some particular cases of the interface response: isotropic materials, surface tension, Wulff's energy, and energies depending on $\mathbb{F} \times \mathbb{m}$ and on $\operatorname{cof} \mathbb{F}\, \mathbb{m}$, which are motivated by the polyconvexity condition to be defined in Chapter 2. When the response reduces to surface tension the surface configurational stress vanishes identically; when the free energy depends only on the referential interface normal (the Wulff energy), the standard interface standard stress vanishes identically. We describe the particular cases in Examples 1.6.1–1.6.5 below. In Proposition 1.6.6 we determine all energy functions leading to vanishing configurational stress and show that among them only the surface tension is frame indifferent. We shall also show that the Wulff energy is the only energy with vanishing standard stress. We assume $m = n$.

Example 1.6.1 (Isotropy). Let $\hat{\mathbb{f}} : \mathbb{U} \to \mathrm{R}$ be a frame indifferent isotropic response with

$$\mathbb{U} = \{ (\mathbb{F}, \mathbb{m}) \in \mathrm{G} : \operatorname{rank} \mathbb{F} = n - 1 \} \tag{1.6.1}$$

and with the representation

$$\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \tilde{\mathbb{f}}(\lambda_1, \dots, \lambda_{n-1})$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathbb{U}$ where $\lambda_1 \geq \dots \geq \lambda_{n-1} > 0$ are the singular values of $\mathbb{F}$; where $\tilde{\mathbb{f}}$ is a continuously differentiable function. Let $(\mathbb{F}, \mathbb{m}) \in \mathbb{U}$ be such that

$$\mathbb{F} = \mathbf{Q} \operatorname{diag}(\lambda_1, \dots, \lambda_{n-1}, 0) \mathbf{R}, \quad \mathbb{m} = \mathbf{R}^{\mathrm{T}}(0, \dots, 0, 1)$$

for some $\mathbf{Q}, \mathbf{R} \in SO(n)$. Then

$$\hat{\mathbb{S}}(\mathbb{F}, \mathbb{m}) = \mathbf{Q} \operatorname{diag}(\mathrm{D}_1 \tilde{\mathbb{f}}, \dots, \mathrm{D}_{n-1} \tilde{\mathbb{f}}, 0) \mathbf{R}, \tag{1.6.2}$$

$$\hat{\mathbb{C}}(\mathbb{F}, \mathbb{m}) = \mathbf{R}^{\mathrm{T}} \operatorname{diag}(\tilde{\mathbb{f}} - \lambda_1 \mathrm{D}_1 \tilde{\mathbb{f}}, \dots, \tilde{\mathbb{f}} - \lambda_{n-1} \mathrm{D}_{n-1} \tilde{\mathbb{f}}, 0) \mathbf{R} \tag{1.6.3}$$

where $\tilde{\mathbb{f}}$ and its derivatives are evaluated at $(\lambda_1, \dots, \lambda_{n-1})$; in particular, the configurational stress tensor is symmetric.

Proof Let $\tilde{f} : O \times \mathrm{R}^n \to \mathrm{R}$ be defined by defining O as the set of all second order tensors with at most one singular value 0, and setting

$$\tilde{f}(\mathbf{F}, \mathbf{p}) = \tilde{\mathbb{f}}(\lambda_1, \dots, \lambda_{n-1})$$

for each $(\mathbf{F}, \mathbf{p}) \in O \times \mathrm{R}^n$ with of the form

$$\mathbf{F} = \mathbf{Q} \operatorname{diag}(\lambda_1, \dots, \lambda_n) \mathbf{R}$$

with $\mathbf{Q}, \mathbf{R} \in SO(n)$ and $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$. A well known result on the derivatives of isotropic functions gives that $\hat{f}$ is continuously differentiable and hence

$$\mathrm{D}\hat{f}(\mathbf{F}, \mathbf{p}) = \big(\mathbf{Q} \operatorname{diag}(\mathrm{D}_1 \tilde{\mathbb{f}}, \dots, \mathrm{D}_{n-1} \tilde{\mathbb{f}}, 0) \mathbf{R}, \mathbf{0}\big)$$

where $\tilde{\mathbb{f}}$ and its derivatives are evaluated at $(\lambda_1, \dots, \lambda_{n-1})$. Since the function $\hat{f}$ is an extension of $\hat{\mathbb{f}}$, we have $\mathrm{D}\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \Pi\, \mathrm{D}\tilde{f}(\mathbb{F}, \mathbb{m})$ where $\Pi : \mathrm{Lin} \times \mathrm{R}^n \to \mathrm{Tan}(\mathrm{G}, (\mathbb{F}, \mathbb{m}))$ is the orthogonal projection onto the tangent space of G at $(\mathbb{F}, \mathbb{m})$.

One finds from (1.2.1) that $\mathrm{D}\hat{f}(\mathbf{F},\mathbb{p})$ is already in the tangent space and thus $\Pi\,\mathrm{D}\hat{f}(\mathbb{F},\mathbb{m}) = \mathrm{D}\hat{f}(\mathbb{F},\mathbb{m})$ and

$$\mathrm{D}\,\hat{\mathbb{f}}(\mathbb{F},\mathbb{m}) = \big(\mathbf{Q}\,\mathrm{diag}(\mathrm{D}_1\,\tilde{\mathbb{f}},\dots,\mathrm{D}_{n-1}\,\tilde{\mathbb{f}},0)\mathbf{R},\mathbf{0}\big),$$

i.e.,

$$\mathrm{D}_1\,\hat{\mathbb{f}}(\mathbb{F},\mathbb{m}) = \mathbf{Q}\,\mathrm{diag}(\mathrm{D}_1\,\tilde{\mathbb{f}},\dots,\mathrm{D}_{n-1}\,\tilde{\mathbb{f}},0)\mathbf{R},\quad \mathrm{D}_2\,\hat{\mathbb{f}}(\mathbb{F},\mathbb{m}) = \mathbf{0}.$$

The stress relations then give (1.6.2) and (1.6.3). □

Example 1.6.2 (Surface tension (i.e., fluid–like interface response)). Let $\hat{\mathbb{f}} : \mathbb{U} \to \mathrm{R}$ be a frame indifferent, fluid like inteface response with $\mathbb{U}$ given by (1.6.1) and

$$\hat{\mathbb{f}}(\mathbb{F},\mathbb{m}) = \sigma|\,\mathrm{cof}\,\mathbb{F}\,| \tag{1.6.4}$$

for each $(\mathbb{F},\mathbb{m}) \in \mathbb{U}$ where σ is a constant. Then a particular case of Example 1.6.1 with $\tilde{\mathbb{f}}(\lambda_1,\dots,\lambda_{n-1}) = \lambda_1\cdots\lambda_{n-1}$ gives

$$\hat{\mathbb{S}} = \sigma|\,\mathrm{cof}\,\mathbb{F}\,|\mathbb{F}^{-\mathrm{T}},\quad \hat{\mathbb{C}} = \mathbf{0}$$

and the spatial surface stresses by

$$\hat{\bar{\mathbb{S}}}(\mathbb{F},\theta) = \sigma\bar{\mathbb{P}}$$

for all $(\mathbb{F},\mathbb{m}) \in \mathbb{U}$ where $\bar{\mathbb{P}} = \mathbf{1} - \bar{\mathbb{m}}\otimes\bar{\mathbb{m}}$ and $\bar{\mathbb{m}}$ is the spatial interface normal, i.e., any of the two unit vectors such that $\mathbb{F}^{\mathrm{T}}\bar{\mathbb{m}} = \mathbf{0}$.

Example 1.6.3 (Wulff energy). Let $\hat{\mathbb{f}} : \mathrm{G} \to \mathrm{R}$ be given by

$$\hat{\mathbb{f}}(\mathbb{F},\mathbb{m}) = \upphi(\mathbb{m}) \tag{1.6.5}$$

for each $(\mathbb{F},\mathbb{m}) \in \mathrm{G}$ and $\upphi : \mathrm{S}^{n-1} \to \mathrm{R}$ is a function; we call $\hat{\mathbb{f}}$ the Wulff energy. It is used to model the growth of crystals, with $\upphi$ restricted by the symmetry of the lattice. One finds that $\mathrm{D}_1\,\hat{\mathbb{f}} = \mathbf{0}$, $\mathrm{D}_2\,\hat{\mathbb{f}} = \mathrm{D}\,\upphi$ and hence the standard stress response functions are

$$\hat{\mathbb{S}} = \mathbf{0},\quad \hat{\mathbb{C}} = \upphi(\mathbb{m})\mathbb{P} - \mathbb{m}\otimes\mathrm{D}\,\upphi(\mathbb{m}) \tag{1.6.6}$$

where $\mathbb{P} = \mathbf{1} - \mathbb{m}\otimes\mathbb{m}$. The energy (1.6.5) produces no standard stress, the interface equilibrium is governed solely by the configurational stress. Already the special case

$$\hat{\mathbb{f}}(\mathbb{F},\mathbb{m}) = \tau = \mathrm{const} > 0$$

leads to nontrivial phenomena with $\hat{\mathbb{C}} = \tau\mathbb{P}$. The orientation dependent interface energies $\upphi$ were introduced by Wulff [30].

Proof The derivative of $\hat{\mathbb{f}}$ is an element of the tangent space to G at $(\mathbb{F},\mathbb{m})$ [see (1.2.1)] such that

$$\mathrm{D}_1\,\hat{\mathbb{f}}\cdot\mathbb{G} + \mathrm{D}_2\,\hat{\mathbb{f}}\cdot\mathbb{m}\mathbb{m} = \mathrm{D}\,\upphi\cdot\mathbb{m}\mathbb{m}$$

for all $(\mathbb{G},\mathbb{m}\mathbb{m})$ from the tangent space, i.e., for all $(\mathbb{G},\mathbb{m}\mathbb{m})$ such that

$$\mathbb{G}\mathbb{m} + \mathbb{F}\mathbb{m}\mathbb{m} = \mathbf{0}.$$

The theorem on Lagrange multipliers gives a $\boldsymbol{\lambda} \in \mathrm{R}^n$ such that

$$\mathrm{D}_1\,\hat{\mathbb{f}}\cdot\mathbb{G} + \mathrm{D}_2\,\hat{\mathbb{f}}\cdot\mathbb{m}\mathbb{m} = \mathrm{D}\,\upphi\cdot\mathbb{m}\mathbb{m} + \boldsymbol{\lambda}\cdot\mathbb{G}\mathbb{m} + \boldsymbol{\lambda}\cdot\mathbb{F}\mathbb{m}\mathbb{m}$$

for all $(\mathbb{G},\mathbb{m}\mathbb{m}) \in \mathrm{Lin}\times\mathrm{R}^n$. This gives

$$\mathrm{D}_1\,\hat{\mathbb{f}} = \boldsymbol{\lambda}\otimes\mathbb{m},\quad \mathrm{D}_2\,\hat{\mathbb{f}} = \mathrm{D}\,\upphi + \mathbb{F}^{\mathrm{T}}\boldsymbol{\lambda}.$$

One finds that $\boldsymbol{\lambda} = -(\mathbf{1} + \mathbb{F}\mathbb{F}^{\mathrm{T}})^{-1}\mathbb{F}\,\mathrm{D}\,\upphi$, but this is not needed; indeed the stress relations with undetermined $\boldsymbol{\lambda}$ give (1.6.6). □

Example 1.6.4 ("Self–dual" energy). Let $n = 3$ and let $\hat{\mathbb{f}} : \mathrm{G} \to \mathrm{R}$ be given by

$$\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \Psi(\mathbb{F} \times \mathbb{m}). \tag{1.6.7}$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$ where $\Psi : \mathrm{Lin} \to \mathrm{R}$. The stress relations read

$$\hat{\mathbb{S}} = -\,\mathrm{D}\,\Psi \times \mathbb{m}, \tag{1.6.8}$$

$$\hat{\mathbb{C}} = \Psi\mathbb{P} + \mathbb{F}^{\mathrm{T}}\,\mathrm{D}\,\Psi \times \mathbb{m} + \mathbb{m} \otimes \mathbb{P}(\mathbb{F}^{\mathrm{T}}\,\mathrm{D}\,\Psi)^{\times} \tag{1.6.9}$$

where for any $\mathbf{M} \in \mathrm{Lin}$ we define the axial vector $\mathbf{M}^{\times} \in \mathrm{R}^3$ of a tensor $\mathbf{M} \in \mathrm{Lin}$ by $\mathbf{M}^{\times} \cdot \mathbf{a} = \mathrm{tr}(\mathbf{M} \times \mathbf{a})$ for any $\mathbf{a} \in \mathrm{R}^3$. If Ψ is positively 1 homogeneous function then the function $\hat{\mathbb{f}}^{\star}$ corresponding to the exchange of the actual and reference configurations (Section 1.3) is of the same format as $\hat{\mathbb{f}}$ in (1.6.7), viz.,

$$\hat{\mathbb{f}}^{\star}(\mathbb{F}, \mathbb{m}) = \Psi^{\mathrm{D}}(\mathbb{F} \times \mathbb{m}) \tag{1.6.10}$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$, where $\Psi^{\mathrm{T}} : \mathrm{Lin} \to \mathrm{R}$ is given by

$$\Psi^{\mathrm{D}}(\mathbb{A}) = \Psi(-\mathbb{A}^{\mathrm{T}})$$

for each $\mathbb{A} \in \mathrm{Lin}$. In this sense $\hat{\mathbb{f}}$ is self–dual.

Proof Differentiating the relation (1.6.7) one obtains that the derivatives of $\hat{\mathbb{f}}$ at $(\mathbb{F}, \mathbb{m}) \in \mathbb{U}$ and the derivative of Ψ at $\mathbb{F} \times \mathbb{m}$ satisfy

$$\mathrm{D}_1\hat{\mathbb{f}} \cdot \mathbb{G} + \mathrm{D}_2\hat{\mathbb{f}} \cdot \mathbb{mm} = -(\mathrm{D}\,\Psi \times \mathbb{m}) \cdot \mathbb{G} + (\mathbb{F}^{\mathrm{T}}\,\mathrm{D}\,\Psi)^{\times} \cdot \mathbb{mm} \tag{1.6.11}$$

for every $(\mathbb{G}, \mathbb{mm})$ belonging to the tangent space to G at $(\mathbb{F}, \mathbb{mm})$ [see (1.2.1)]. If $(\mathbb{G}, \mathbb{mm}) = (\mathbf{A}\mathbb{P}, \mathbf{0})$ where $\mathbf{A} \in \mathrm{Lin}$ is arbitrary then the identity (1.6.11) gives

$$\mathrm{D}_1\hat{\mathbb{f}}\,\mathbb{P} = -\,\mathrm{D}\,\Psi \times \mathbb{m}.$$

If $(\mathbb{G}, \mathbb{mm}) = (\mathbb{F}\mathbf{a} \otimes \mathbb{m}, -\mathbb{P}\mathbf{a})$ where $\mathbf{a} \in \mathrm{R}^3$ is arbitrary then the identity (1.6.11) gives

$$\mathbb{F}^{\mathrm{T}}\,\mathrm{D}_1\hat{\mathbb{f}} - \mathrm{D}_2\hat{\mathbb{f}} = \mathbb{P}(\mathbb{F}^{\mathrm{T}}\,\mathrm{D}\,\Psi)^{\times}.$$

The stress relatinos then give (1.6.8) and (1.6.9). Assume now that Ψ is positively 1 homogeneous and prove (1.6.10). To this end, we note that

$$\mathbb{F}^{-1} \times \mathrm{cof}\,\mathbb{F}\mathbb{m} = -(\mathbb{F} \times \mathbb{m})^{\mathrm{T}} \tag{1.6.12}$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$. By the singular value decomposition theorem we have

$$(\mathbb{F}, \mathbb{m}) = (\mathbf{Q}\mathbb{U}\mathbf{R}, \mathbf{R}^{\mathrm{T}}\bar{\mathbb{m}})$$

for some $\mathbf{Q}, \mathbf{R} \in SO(3)$ where $\mathbb{U} = \mathrm{diag}(\lambda_1, \lambda_2, 0)$ and $\mathbb{m} = \bar{\mathbb{m}} = (0, 0, 1)$. Using the formula $\mathbf{A}\mathbf{R} \times \mathbf{R}^{\mathrm{T}}\mathbf{a} = (\mathbf{A} \times \mathbf{a})\mathbf{R}$ one finds that (1.6.12) reduces to

$$\mathbb{U}^{-1} \times \mathrm{cof}\,\mathbb{U}\bar{\mathbb{m}} = -(\mathbb{U} \times \mathbb{m})^{\mathrm{T}}$$

which is easily verified. With (1.6.12), the definition of $\hat{\mathbb{f}}^{\star}$ gives immediately (1.6.10).

□

Example 1.6.5 (Generalized surface tension). Let $\hat{\mathbb{f}} : \mathrm{G} \to \mathrm{R}$ be given by

$$\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \psi(\operatorname{cof} \mathbb{F}\mathbb{m}), \tag{1.6.13}$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$, where $\psi : \mathrm{R}^n \to \mathrm{R}$ is a given function. One obtains

$$\hat{\mathbb{S}} = \big((\mathbf{v} \cdot \mathrm{D}\,\psi)\bar{\mathbb{P}} - \mathbf{v} \otimes \mathrm{D}\,\psi\big)\mathbb{F}^{-\mathrm{T}}, \tag{1.6.14}$$

$$\hat{\mathbb{C}} = \big(\psi - (\mathbf{v} \cdot \mathrm{D}\,\psi)\big)\mathbb{P} \tag{1.6.15}$$

where $\mathbf{v} = \operatorname{cof} \mathbb{F}\mathbb{m}$, $\bar{\mathbb{P}}$ is the projector onto $\mathbf{v}^{\perp}$ and ψ and its derivative are evaluated at $\mathbf{v}$. One finds that if $\hat{\mathbb{f}}$ is of the form (1.6.13) with ψ positively 1 homogeneous then $\hat{\mathbb{f}}^{\star}$ is of the form considered in Example 1.6.3, with $\varphi = \psi$, i.e.,

$$\hat{\mathbb{f}}^{\star}(\mathbb{F}, \mathbb{m}) = \psi(\mathbb{m})$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$. In this sense the present example and Example 1.6.3 are dual to each other. However, unlike Example 1.6.3, in the present example for a realistic model the function ψ in (1.6.13) cannot be arbitrary. Namely, the objectivity requires $\psi(\mathbf{R}\mathbf{v}) = \psi(\mathbf{v})$ for each $\mathbf{v} \in \mathrm{R}^3$, $\mathbf{R} \in SO(3)$ which implies that ψ is a multiple of the euclidean norm,

$$\psi(\mathbf{v}) = \sigma|\mathbf{v}|$$

for each $\mathbf{v} \in \mathrm{R}^3$ where σ is a constant. We thus obtain the surface tension.

Proof Let $\hat{\mathbb{v}} : \mathrm{G} \to \mathrm{R}^n$ be defined by

$$\hat{\mathbb{v}}(\mathbb{F}, \mathbb{m}) = \operatorname{cof} \mathbb{F}\mathbb{m}$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$. Then for any $(\mathbb{G}, \mathbb{mm})$ from the tangent space of G at $(\mathbb{F}, \mathbb{m})$ we have

$$\mathrm{D}\,\hat{\mathbb{v}}(\mathbb{G}, \mathbb{mm}) = \hat{\mathbb{v}}\mathbb{F}^{-\mathrm{T}} \cdot \mathbb{G} - \mathbb{F}^{-\mathrm{T}}\mathbb{G}^{\mathrm{T}}\hat{\mathbb{v}}. \tag{1.6.16}$$

Indeed, differentiating

$$\mathbb{F}^{\mathrm{T}}\hat{\mathbb{v}} = \mathbf{0}, \quad |\hat{\mathbb{v}}| = |\operatorname{cof} \mathbb{F}|$$

and using $\mathrm{D}_{\mathbb{F}}|\operatorname{cof} \mathbb{F}| = |\operatorname{cof} \mathbb{F}|\mathbb{F}^{-\mathrm{T}}$ one obtains

$$\mathbb{F}^{\mathrm{T}}\,\mathrm{D}\,\hat{\mathbb{v}}\,(\mathbb{G}, \mathbb{mm}) + \mathbb{G}^{\mathrm{T}}\,\hat{\mathbb{v}} = \mathbf{0}, \quad \hat{\mathbb{v}} \cdot \mathrm{D}\,\hat{\mathbb{v}}\,(\mathbb{G}, \mathbb{mm}) = |\operatorname{cof} \mathbb{F}|^2\mathbb{F}^{-\mathrm{T}} \cdot \mathbb{G}.$$

The first relation gives

$$\mathrm{D}\,\hat{\mathbb{v}}(\mathbb{G}, \mathbb{mm}) = \lambda\hat{\mathbb{v}} - \mathbb{F}^{-\mathrm{T}}\mathbb{G}^{\mathrm{T}}\hat{\mathbb{v}}$$

for some $\lambda \in \mathrm{R}$ and the second relation $\lambda = \mathbb{F}^{-\mathrm{T}} \cdot \mathbb{G}$, which proves (1.6.16). Hence the differentiation of (1.6.13) gives

$$\mathrm{D}\,\hat{\mathbb{f}} \cdot (\mathbb{G}, \mathbb{mm}) = \hat{\mathbb{v}} \cdot \mathrm{D}\,\psi\,(\mathbb{F}^{-\mathrm{T}} \cdot \mathbb{G}) - \mathrm{D}\,\psi \cdot \mathbb{F}^{-\mathrm{T}}\mathbb{G}^{\mathrm{T}}\hat{\mathbb{v}}$$

for all $(\mathbb{G}, \mathbb{mm})$ from the tangent space of G at $(\mathbb{F}, \mathbb{m})$. As in the proof of Example 1.6.3 we obtain the existence of $\boldsymbol{\lambda} \in \mathrm{R}^n$ such that

$$\mathrm{D}_1\,\hat{\mathbb{f}} = \big((\hat{\mathbb{v}} \cdot \mathrm{D}\,\psi)\bar{\mathbb{P}} - \hat{\mathbb{v}} \otimes \mathrm{D}\,\psi\big)\,\mathbb{F}^{-\mathrm{T}} + \boldsymbol{\lambda} \otimes \mathbb{m}, \quad \mathrm{D}_2\,\hat{\mathbb{f}} = \mathbb{F}^{\mathrm{T}}\boldsymbol{\lambda}.$$

The stress relations then give (1.6.14) and (1.6.15). The rest is immediate. $\square$

Proposition 1.6.6. *Consider an interface described by the free energy function* $\hat{\mathbb{f}} : \mathbb{U} \to \mathrm{R}$ *and by the response functions* $\hat{\mathbb{S}}, \hat{\mathbb{C}} : \mathbb{U} \to \mathrm{Lin}$ *for the standard stress and configurational stress. Then*

(i) *the configurational stress* $\hat{\mathbb{C}}$ *vanishes identically if and only if there exists a class* 1 *function* $\psi : \mathrm{S}^{n-1} \to \mathrm{R}$

$$\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = |\operatorname{cof} \mathbb{F}| \psi(\bar{\mathbb{m}}), \tag{1.6.17}$$

$(\mathbb{F}, \mathbb{m}) \in \mathbb{U}$, *where* $\bar{\mathbb{m}} = \operatorname{cof} \mathbb{F}\mathbb{m} / |\operatorname{cof} \mathbb{F}\mathbb{m}|$*; the response* (1.6.17) *is frame indifferent if and only if there exists a* $\sigma \in \mathrm{R}$ *such that* (1.6.4) *holds for each* $(\mathbb{F}, \mathbb{m}) \in \mathbb{U}$;

(ii) *the standard stress* $\hat{\mathbb{S}}$ *vanishes identically if and only if there exists a class* 1 *function* $\varphi : \mathrm{S}^{n-1} \to \mathrm{R}$ *such that* (1.6.5) *holds for each* $(\mathbb{F}, \mathbb{m}) \in \mathbb{U}$.

The classes of energy functions in (i) and (ii) are dual each to other: in (i) the response depends on the spatial interface normal while in (ii) on the referential interface normal; they are mapped to each other by the exchange of the actual and reference configurations described in Proposition 1.3.1.

Proof (ii): Assume that the inteface standard stress vanishes identically; i.e., $\mathrm{D}_1 \hat{\mathbb{f}}\, \mathbb{P} = \mathbf{0}$; hence $\mathrm{D}_1 \hat{\mathbb{f}} = \mathbb{a}(\mathbb{F}, \mathbb{m}) \otimes \mathbb{m}$ where $\mathbb{a} : \mathbb{U} \to \mathrm{R}^n$ is some function. Let us prove that $\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \hat{\mathbb{f}}(\mathbb{G}, \mathbb{m})$ whenever $(\mathbb{F}, \mathbb{m}), (\mathbb{G}, \mathbb{m}) \in \mathbb{U}$. Given such two tensors, there exists a class 1 curve $\mathbb{H} : [0, 1] \to \mathrm{Lin}$ with endpoints $\mathbb{F}$, $\mathbb{G}$ such that $(\mathbb{H}(t), \mathbb{m}) \in \mathbb{U}$ for each $t \in [0, 1]$. A differentiation gives $\dot{\mathbb{H}}(t)\mathbb{m} = \mathbf{0}$ and hence

$$\frac{d\hat{\mathbb{f}}(\mathbb{H}(t))}{dt} = (\dot{\mathbb{H}}(t)\mathbb{m} \cdot \mathbb{a}(\mathbb{H}(t)) = 0$$

and the integration provides $\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \hat{\mathbb{f}}(\mathbb{G}, \mathbb{m})$. Thus there exists a function $\varphi : \mathrm{S}^{n-1} \to \mathrm{R}$ such that $\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \varphi(\mathbb{m})$ for and $(\mathbb{F}, \mathbb{m}) \in \mathbb{U}$. The proof of the direct implication is complete. Conversely, if $\hat{\mathbb{f}}$ is of the form (1.6.5) then Example 1.6.3 gives that the standard stress vanishes identically.

(i): For the given $\hat{\mathbb{f}}$ let $\hat{\mathbb{f}}^\star, \hat{\mathbb{S}}^\star, \hat{\mathbb{C}}^\star$ be given as in Proposition 1.3.1; then the standard stress tensor of $\hat{\mathbb{f}}$ vanishes identically if and only if the configurational stress of $\hat{\mathbb{f}}^\star$ vanishes identically. By (ii) the last occurs if and only if $\hat{\mathbb{f}}^\star$ is of the form

$$\hat{\mathbb{f}}^\star(\mathbb{F}, \mathbb{m}) = \varphi(\mathbb{m}) \tag{1.6.18}$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathbb{U}$ where $\mathbb{m}$ is any of the two unit vectors satisfying $\mathbb{F}\mathbb{m} = \mathbf{0}$ and where $\varphi : \mathrm{S}^{n-1} \to \mathrm{R}$ is an even function. The defining relation of $\hat{\mathbb{f}}^\star$ gives

$$\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = |\operatorname{cof} \mathbb{F}| \hat{\mathbb{f}}^\star(\mathbb{F}^{-1})$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathbb{U}$ where we have used $|\operatorname{cof} \mathbb{F}^{-1}|^{-1} = |\operatorname{cof} \mathbb{F}|$; a combination with (1.6.18) then gives (1.6.17). This completes the proof of the equivalence asserted in (i). The rest of (i) is immediate. □

Chapter 2

Equilibrium states

To obtain the existence of minimizers of energy, appropriate convexity conditions for the bulk and interface energy have to be assumed. In case of the bulk response the polyconvexity of each of the energy functions $\hat{f}_\alpha$, $\alpha = 1, 2$, appears appropriate. The mutual relation of the minima of $\hat{f}_\alpha$ is arbitrary, so that the geometric incompatibility induced by symmetry can occur and the gross bulk response, given by the energy function $\hat{f}_0$,

$$\hat{f}_0(\mathbf{F}) = \min\{\hat{f}_1(\mathbf{F}), \hat{f}_2(\mathbf{F})\},$$

$\det \mathbf{F} > 0$, exhibits two wells corresponding to the two phases of the material. The existence theory for a *single phase* minimizers with polyconvex energy is well understood [3, 11, 22, 12].

In case of two or more energy wells, in the absence of interfacial energy ($\hat{\mathbb{f}} \equiv 0$) the problem of minimum energy state generally does not have a solution, since in the approach to the least energy, the body exhibits states $(\mathbf{y}^i, E^i)$ with finer and finer microstructure of coexistent phases and with the interfacial area tending to infinity. As the theory does not have any length scale, there is no limit on the fineness of the microstructure, i.e., it is infinitely fine in the limit. The Young measure minimizers represent the idealized limiting states. The least energy is given by the quasiconvex envelope $\hat{f}_0^{\mathrm{qc}}$ (see [6; Section 6.3] for the definition) of the minimum energy $\hat{f}_0$. In particular, under the affine boundary conditions

$$\mathbf{y}(\mathbf{x}) = \mathbf{A}\mathbf{x}, \quad \mathbf{x} \in \mathrm{bd}\,\Omega,$$

where $\mathbf{A}$ is a prescribed constant affine deformation gradient, one has

$$\inf\left\{\mathsf{E}(\mathbf{y}, E) \in \mathscr{G} : \mathbf{y}(\mathbf{x}) = \mathbf{A}\mathbf{x} \text{ on } \mathrm{bd}\,\Omega\right\} = \hat{f}_0^{\mathrm{qc}}(\mathbf{A})$$

where we assume the referential volume of Ω equal to 1 for simplicity; however, the infimum is generally not achieved.

The interface energy has a regularizing effect so that the minimizers of the total energy E can exist. As in case of the bulk response, the interfacial energy has to posses the right convexity properties. These are discussed in Sections 2.2 and 2.3, and, from a more general viewpoint, in Chapter 3.

The interface quasiconvexity of $\hat{\mathbb{f}}$ ensures the stability of a planar homogeneously deformed interface $\mathcal{T}$ against curved inhomogeneously deformed interfaces $\mathcal{S}$ with the same boundary data. An interface null lagrangian is an interfacial energy $\hat{\mathbb{f}}$ such that $\hat{\mathbb{f}}$ and $-\hat{\mathbb{f}}$ are interface quasiconvex. An explicit form is given below [(2.2.2)]. An interface polyconvex surface energy is a convex, positively 1 homogeneous function of interface null lagrangians; it is automatically interface quasiconvex, and our existence result is based on the interface polyconvexity. We note that Parry [23] and Fonseca [10] establish some particular cases of the present notion of interface quasiconvexity as necessary conditions for metastable minima.

2.1 States, minimizers of energy and equilibrium equations

We now use the constitutive information from Definitions 1.2.1 to introduce global states of the phase transforming body Ω, the fields of mechanical/thermodynamic quantities over the bulk phases and on the phase interface, and the total energy. The smoothness assumed in this section allows us to obtain the equilibrium equations for standard and configurational stresses in the classical form. A less restrictive definition of states is needed for the existence theory; that definition is given in Section 2.3 (below).

Definition 2.1.1 (States). We say that $(\mathbf{y}, E)$ is a ***state*** if $\mathbf{y} : \Omega \to \mathrm{R}^m$ is a continuous map and E is an open subset of Ω such that

(i) $\mathcal{S} := \Omega \cap \operatorname{bd} E$ is a class 2 surface of dimension $n-1$ of normal $\mathbb{m} : \mathcal{S} \to \mathrm{S}^{n-1}$;

(ii) with the notation
$$E_1 := E, \quad E_2 = \Omega \sim \operatorname{cl} E$$
the maps $\mathbf{y}_\alpha := \mathbf{y}|E_\alpha$, $\alpha = 1, 2$, and $\mathbb{y} := \mathbf{y}|\mathcal{S}$ are of class 2 with their gradients $\nabla\mathbf{y}_\alpha$, and $\nabla\mathbb{y}$ having continuous extensions $\mathbf{F}_\alpha$ and $\mathbb{F}$ to the closure of their respective domains;

(iii) we have $\operatorname{ran} \mathbf{F}_\alpha \subset U_\alpha$, $\alpha = 1, 2$, and $\operatorname{ran} \mathbb{F} \subset \mathbb{U}$.

Here $f|M$ denotes the restriction of a map f to a subset M of its domain of definition $\operatorname{dom} f$ and $\operatorname{ran} f = \{f(x) : x \in \operatorname{dom} f\}$ denotes the range of f. One has
$$\mathbf{F}_1 = \nabla\mathbf{y} \quad \text{in } E, \qquad \mathbf{F}_2 = \nabla\mathbf{y} \quad \text{in } \Omega \sim \operatorname{cl} E, \tag{2.1.1}$$
and the values of $\mathbf{F}_\alpha$ on $\operatorname{cl} E_\alpha \sim E_\alpha$ are the limits of the gradients in (2.1.1). In particular, $\mathbf{F}_\alpha$ are well defined on $\mathcal{S}$ and we denote by $[\mathbf{F}] := \mathbf{F}_1|\mathcal{S} - \mathbf{F}_2|\mathcal{S}$ the jump of the deformation gradient across the interface. However, let us emphasize that $\mathbf{y}$ is continuous. Also,
$$\mathbb{F} = \nabla\mathbb{y} = \nabla\mathbf{y} \quad \text{on } \mathcal{S} \tag{2.1.2}$$
and $\mathbb{F} : \operatorname{cl}\mathcal{S} \to \mathrm{Lin}$ is the continuous extension of the surface gradient in (2.1.2).

Definition 2.1.2 (Energy and stresses associated with states). Let $(\mathbf{y}, E)$ be a state. We define

(i) the ***energy*** $\mathsf{E}(\mathbf{y}, E)$ of the state by

$$\mathsf{E}(\mathbf{y}, E) = \mathsf{E}_{\mathrm{b}}(\mathbf{y}, E) + \mathsf{E}_{\mathrm{if}}(\mathbf{y}, E) \tag{2.1.3}$$

where

$$\mathsf{E}_{\mathrm{b}}(\mathbf{y}, E) = \int_E \hat{f}_1(\nabla \mathbf{y})\, d\mathcal{L}^n + \int_{\Omega \sim E} \hat{f}_2(\nabla \mathbf{y})\, d\mathcal{L}^n, \tag{2.1.4}$$

$$\mathsf{E}_{\mathrm{if}}(\mathbf{y}, E) = \int_{\mathcal{S}} \hat{\mathbb{f}}(\nabla \mathbf{y}, \mathbb{m})\, d\mathcal{H}^{n-1}$$

are the ***bulk*** and ***interfacial energies***, respectively;

(ii) the ***bulk standard stress*** $\mathbf{S}$, the ***bulk configurational stress*** $\mathbf{C}$ and the ***bulk energy density*** f on $\Omega \sim \mathcal{S}$ by

$$\mathbf{S} = \begin{cases} \hat{\mathbf{S}}_1 \circ \mathbf{F}_1 & \text{on} \quad E, \\ \hat{\mathbf{S}}_2 \circ \mathbf{F}_2 & \text{on} \quad \Omega \sim \operatorname{cl} E, \end{cases}$$

and similarly for $\mathbf{C}$ and f; here and below $\hat{\mathbf{S}}_1 \circ \mathbf{F}_1$ denotes the composition of the maps $\hat{\mathbf{S}}_1$ and $\mathbf{F}_1$, i.e., $(\hat{\mathbf{S}}_1 \circ \mathbf{F}_1)(\mathbf{x}) = \hat{\mathbf{S}}_1(\mathbf{F}_1(\mathbf{x}))$ for each $\mathbf{x} \in \operatorname{dom} \mathbf{F}_1 = \operatorname{cl} E$ and similarly for compositions of general maps;

(iii) the ***jumps*** $[\mathbf{S}]$, $[\mathbf{C}]$ $[f]$ of the bulk stresses on $\mathcal{S}$ and of the bulk energy on $\mathcal{S}$ by

$$[\mathbf{S}] = \hat{\mathbf{S}}_1 \circ \mathbf{F}_1 | \mathcal{S} - \hat{\mathbf{S}}_2 \circ \mathbf{F}_2 | \mathcal{S}$$

and similarly for $[\mathbf{C}]$, $[f]$ and $[\mathbf{S}\mathbb{m} \cdot \mathbf{F}\mathbb{m}]$;

(iv) the ***interfacial standard stress*** $\mathbb{S}$, ***interfacial configurational stress*** $\mathbb{C}$, and the ***interfacial energy density*** $\mathbb{f}$ on $\mathcal{S}$ by

$$\mathbb{S} = \hat{\mathbb{S}} \circ (\mathbb{F}, \mathbb{m})$$

and similarly for $\mathbb{C}$, and $\mathbb{f}$, where we use the notation of Definition 2.1.1.

As in case of the jump of $\mathbf{F}$, the jumps defined in (iii) are the differences of the limits of the corresponding bulk fields from the two sides of the interface. Note that $\mathbb{S}$ and $\mathbb{C}$ are superficial tensors, i.e., $\mathbb{S}\mathbb{m} = \mathbb{C}\mathbb{m} = \mathbf{0}$.

Definition 2.1.3 (Local perturbations and minima).

(i) A state $(\mathbf{z}, F)$ is said to be a ***local perturbation*** of the state $(\mathbf{y}, E)$ if there exists a compact subset K of Ω with

$$\mathbf{z} | (\Omega \sim K) = \mathbf{y} | (\Omega \sim K), \quad F \cap (\Omega \sim K) = E \cap (\Omega \sim K).$$

(ii) The state $(\mathbf{y}, E)$ is said to be a ***local minimizer of energy*** if $\mathsf{E}(\mathbf{y}, E) \le \mathsf{E}(\mathbf{z}, F)$ for each local perturbation $(\mathbf{z}, F)$ of $(\mathbf{y}, E)$.

Thus a local perturbation $(\mathbf{z}, F)$ is identical with the state $(\mathbf{y}, E)$ near the boundary of Ω and (ii) considers the minima of total energy in this class of states. For the considerations below, and in particular for the validity of the interfacial configuration force balance, it is crucial that the interface in the state $(\mathbf{z}, F)$ can be different from that of $(\mathbf{y}, E)$ (apart from the mentioned coincidence near the boundary of Ω). Thus in passing from $(\mathbf{y}, E)$ to $(\mathbf{z}, F)$, part of the phase 1 is transformed into the phase 2 and/or conversely. A stronger notion of minimum is considered in the existence theorems in Section 2.3. The reader is referred to [23] and [10] for different but related notions of minima.

The following proposition [27; Lemma 3.2] clarifies the roles of the standard and configurational force systems by evaluating the variation of total energy under outer and inner variations.

Lemma 2.1.4 (Outer and inner variations). *Let $(\mathbf{y}, E)$ be a state. With the notation of Definitions* 2.1.1 *and* 2.1.2 *we have the following statements, in which $t \in \mathrm{R}$ is a parameter and $\delta > 0$ a number with $|t|$, δ sufficiently small:*

(i) *Let $\boldsymbol{\alpha} \in C_0^\infty(\Omega, \mathrm{R}^m)$ and let $\mathbf{y}_t : \Omega \to \mathrm{R}^m$ be defined by*

$$\mathbf{y}_t = \mathbf{y} + t\boldsymbol{\alpha}.$$

Then $(\mathbf{y}_t, E)$ is a state that is a local perturbation of $(\mathbf{y}, E)$, the function $t \mapsto \mathsf{E}(\mathbf{y}_t, E)$ is continuously differentiable and

$$\left.\frac{d\mathsf{E}(\mathbf{y}_t, E)}{dt}\right|_{t=0} = \int_{\Omega\sim\mathcal{S}} \mathbf{S}\cdot\nabla\boldsymbol{\alpha}\, d\mathcal{L}^n + \int_{\mathcal{S}} \mathbb{S}\cdot\mathbb{\nabla}\,\boldsymbol{\alpha}\, d\mathcal{H}^{n-1}. \tag{2.1.5}$$

*The family $\{(\mathbf{y}_t, E) : |t| < \delta\}$ is said to be an **outer variation** of $(\mathbf{y}, E)$.*

(ii) *Let $\boldsymbol{\beta} \in C_0^\infty(\Omega, \mathrm{R}^n)$ and let $\boldsymbol{\phi}_t : \Omega \to \mathrm{R}^n$ be defined by*

$$\boldsymbol{\phi}_t(\mathbf{x}) = \mathbf{x} + t\boldsymbol{\beta}(\mathbf{x}),$$

$\mathbf{x} \in \Omega$. Then $\boldsymbol{\phi}_t$ maps Ω bijectively onto Ω; if we define

$$\mathbf{y}_t = \mathbf{y}\circ\boldsymbol{\phi}_t^{-1}, \quad E_t = \boldsymbol{\phi}_t(E)$$

then $(\mathbf{y}_t, E_t)$ is a state that is a local perturbation of $(\mathbf{y}, E)$, the function $t \mapsto \mathsf{E}(\mathbf{y}_t, E_t)$ is continuously differentiable and

$$\left.\frac{d\mathsf{E}(\mathbf{y}_t, E_t)}{dt}\right|_{t=0} = \int_{\Omega\sim\mathcal{S}} \mathbf{C}\cdot\nabla\boldsymbol{\beta}\, d\mathcal{L}^n + \int_{\mathcal{S}} \mathbb{C}\cdot\mathbb{\nabla}\,\boldsymbol{\beta}\, d\mathcal{H}^{n-1}. \tag{2.1.6}$$

*The family $\{(\mathbf{y}_t, E_t) : |t| < \delta\}$ is said to be an **inner variation** of $(\mathbf{y}, E)$.*

The forms (2.1.5) and (2.1.6) justifies the particular forms of the interfacial stress relations postulated above. The stress relations continue to hold also in dynamical situations, although the variational arguments do not suffice [14, 26].

The minimum of energy then leads to equilibrium equations.

Proposition 2.1.5. *If $(\mathbf{y}, E)$ is a local minimizer of energy then*

$$\operatorname{div}\mathbf{S} = \mathbf{0}, \qquad \operatorname{div}\mathbf{C} = \mathbf{0} \qquad \text{in } \Omega\sim\mathcal{S}, \tag{2.1.7}$$

$$\mathbb{div}\,\mathbb{S} + [\mathbf{S}]\mathbb{m} = \mathbf{0}, \quad \mathbb{div}\,\mathbb{C} + [\mathbf{C}]\mathbb{m} = \mathbf{0} \qquad \text{on } \mathcal{S}, \tag{2.1.8}$$

where we use the notation of Definitions 2.1.1 *and* 2.1.2. *Equation* $(2.1.7)_2$ *and the tangential component of* $(2.1.8)_2$ *is a consequence of* $(2.1.7)_1$ *and* $(2.1.8)_1$. *Granted* $(2.1.8)_1$, *the normal component of* $(2.1.8)_2$ *is equivalent to*

$$[f - \mathbf{S}\mathbb{m} \cdot \mathbf{F}\mathbb{m}] - (\mathbb{f}\mathbb{P} - \mathbb{F}^{\mathrm{T}}\mathbb{S}) \cdot \mathbb{L} + \mathbb{div}\, \mathbb{t} = 0 \qquad \text{on } \mathcal{S},$$

where $\mathbb{L} = \nabla \mathbb{m}$ *is the curvature tensor and* $\mathbb{t} : \mathcal{S} \to \mathrm{R}^n$ *is given by*

$$\mathbb{t} = \mathbb{F}^{\mathrm{T}} \mathrm{D}_1 \hat{\mathbb{f}} \circ (\mathbb{F}, \mathbb{m})\, \mathbb{m} - \mathrm{D}_2 \hat{\mathbb{f}} \circ (\mathbb{F}, \mathbb{m}).$$

See [7–8, 15, 17, 16, 13–14, 25–26, 28]. See also [14] for further references.

Assuming $m = n$ and the invertibility of the deformation $\mathbf{y}$, we can introduce the spatial configuration $\bar{\Omega} = \mathbf{y}(\Omega)$ of the body and the spatial interface $\bar{\mathcal{S}} = \mathbf{y}(\mathcal{S})$, and the spatial stress tensors

$$\bar{\mathbf{S}} \circ \mathbf{y} = \mathbf{S}\mathbf{F}^{\mathrm{T}}/J, \quad \bar{\mathbf{C}} \circ \mathbf{y} = \mathbf{C}\mathbf{F}^{\mathrm{T}}/J,$$
$$\bar{\mathbb{S}} \circ \mathbb{y} = \mathbb{S}\mathbb{F}^{\mathrm{T}}/\mathbb{J}, \quad \bar{\mathbb{C}} \circ \mathbb{y} = \mathbb{C}\mathbb{F}^{\mathrm{T}}/\mathbb{J},$$

where

$$J = \det \mathbf{F}, \quad \mathbb{J} = |\operatorname{cof} \mathbb{F}|.$$

The equilibrium equations take the forms

$$\operatorname{Div} \bar{\mathbf{S}} = \mathbf{0}, \qquad \operatorname{Div} \bar{\mathbf{C}} = \mathbf{0} \qquad \text{in } \bar{\Omega} \sim \bar{\mathcal{S}},$$
$$\mathbb{Div}\, \bar{\mathbb{S}} + [\bar{\mathbf{S}}]\bar{\mathbb{m}} = \mathbf{0}, \quad \mathbb{Div}\, \bar{\mathbb{C}} + [\bar{\mathbf{C}}]\bar{\mathbb{m}} = \mathbf{0} \qquad \text{on } \bar{\mathcal{S}},$$

where $\operatorname{Div}, \mathbb{Div}$ denote the spatial bulk and surface divergences.

2.2 Interface quasiconvexity, null lagrangians and polyconvexity

Let $n \geq 2$ and put

$$s := \min\{m, n\}, \quad t := \min\{m, n-1\}.$$

For the purpose of the following definition, by an oriented surface $\mathcal{S}$ of normal $\mathbb{m}$ we mean a bounded class ∞ surface in R^n of dimension $n-1$ for which $\mathbb{m}$ is a continuous field of unit normal, such that the boundary $\operatorname{bd} \mathcal{S} := \operatorname{cl} \mathcal{S} \sim \mathcal{S}$ is a class ∞ surface of dimension $n-2$, with the orientation of $\operatorname{bd} \mathcal{S}$ dictated by the Stokes theorem. We say that $\mathcal{T}$ is a planar surface of normal $\mathbb{mm}$ if $\mathcal{T}$ is a subset of some $n-1$ dimensional hyperplane in R^n.

Definitions 2.2.1. Let $\hat{\mathbb{f}} : \mathrm{G} \to \mathrm{R} \cup \{\infty\}$ be a continuous function. We say that $\hat{\mathbb{f}}$ is

(i) ***interface quasiconvex*** if

$$\int_{\mathcal{S}} \hat{\mathbb{f}}(\nabla \mathbb{y}, \mathbb{m})\, d\mathcal{H}^{n-1} \geq \mathcal{H}^{n-1}(\mathcal{T})\hat{\mathbb{f}}(\mathbb{G}, \mathbb{mm}) \tag{2.2.1}$$

for every $(\mathbb{G}, \mathbb{mm}) \in \mathrm{G}$, every planar surface $\mathcal{T}$ of normal $\mathbb{mm}$, every orented surface $\mathcal{S}$ of normal $\mathbb{m}$ and every continuous map $\mathbb{y} : \operatorname{cl} \mathcal{S} \to \mathrm{R}^m$ that is class 1 on $\mathcal{S}$ such that

$$\operatorname{bd} \mathcal{S} = \operatorname{bd} \mathcal{T} \quad \text{and} \quad \mathbb{y}(\mathbf{x}) = \mathbb{G}\mathbf{x} \text{ if } \mathbf{x} \in \operatorname{bd} \mathcal{T};$$

(ii) an ***interface null lagrangian*** if $\hat{\mathbb{f}}$ is finite valued and $\pm\hat{\mathbb{f}}$ are interface quasiconvex [in other words, (2.2.1) holds with the equality sign for each collection of objects listed in (i)];

(iii) ***interface polyconvex*** if $\hat{\mathbb{f}}$ is the supremum of some family of interface null lagrangians.

Remarks 2.2.2.

(i) If $\hat{\mathbb{f}}$ is the supremum of some family of interface quasiconvex functions then $\hat{\mathbb{f}}$ is interface quasiconvex; in particular any interface polyconvex function is interface quasiconvex. Any standard (bulk) polyconvex function is also the supremum of some family of standard (bulk) null lagrangians.

(ii) In Chapter 3 we will introduce generalizations of the notions in Definition 2.2.1 which involve integrations on surfaces of dimension r wirh $0 \le r \le n$ in R^n. Also the surfaces as considered above will be replaced by more general objects, viz., integral currents, etc.

(iii) The main motivation of the interface quasiconvexity comes from the lowersemicontinuity of the surface energy E_{if} with respect to a suitable convergence of states $(\mathbf{y}, E)$ with migrating interface. These matters are counterparts of the corresponding "bulk" assertions [19–20, 3, 5]. We refer to Section 3.6 for a simple result of this type.

We now give a complete description of interface null lagrangians as linear combinations, with constant tensorial coefficients, of the members of the list

$$\wedge_k \mathbb{F} \wedge \mathbb{m}, \quad k = 0, \dots, t$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$. The reader is referred to Section A.2 (below) for the notation. In particular,

$$\wedge_0 \mathbb{F} \wedge \mathbb{m} = \mathbb{m},$$

and if $m = n$ then

$$\wedge_{n-1} \mathbb{F} \wedge \mathbb{m} = *(\operatorname{cof} \mathbb{F} \mathbb{m}).$$

Theorem 2.2.3. *A function* $\hat{\mathbb{f}} : \mathrm{G} \to \mathrm{R}$ *is an interface null lagrangian if and only if it is of the form*

$$\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \sum_{k=0}^{t} \mathbf{\Omega}_k \cdot (\wedge_k \mathbb{F} \wedge \mathbb{m})$$

for all $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$ *where*

$$\mathbf{\Omega}_k \in \mathrm{Lin}(\wedge_{k+1} \mathrm{R}^n, \wedge_k \mathrm{R}^m)$$

are constants for all $k = 0, \dots, t$. *If* $m = n = 3$ *then a general form of an interface null lagrangian is*

$$\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \mathbf{c} \cdot \mathbb{m} + \mathbf{\Omega} \cdot (\mathbb{F} \times \mathbb{m}) + \mathbf{a} \cdot \operatorname{cof} \mathbb{F} \mathbb{m} \tag{2.2.2}$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$ *where* $\mathbf{c}, \mathbf{a} \in \mathrm{R}^3$ *and* $\mathbf{\Omega} \in \mathrm{Lin}(\mathrm{R}^3, \mathrm{R}^3)$ *are constants.*

Here $\mathbb{F} \times \mathbb{m}$ is a second order tensor defined by

$$(\mathbb{F} \times \mathbb{m})\mathbf{t} = \mathbb{F}(\mathbb{m} \times \mathbf{t})$$

for any vector $\mathbf{t}$; in components,

$$(\mathbb{F} \times \mathbb{m})_{iA} = \varepsilon_{ABC} \mathbb{F}_{iB} \mathbb{m}_C$$

where ε_{ABC} is the permutation symbol, summation convention applies, and $\mathbb{F}_{iB}$, $\mathbb{m}_C$ are the components of $\mathbb{F}$ and $\mathbb{m}$ with $i = 1, 2, 3$ the spatial indices and $A, B, C = 1, 2, 3$ the referential indices. Since $\mathbb{F}\mathbb{m} = \mathbf{0}$ we have

$$\mathbb{F} = -(\mathbb{F} \times \mathbb{m}) \times \mathbb{m};$$

thus $\mathbb{F} \times \mathbb{m}$ carries the same information as $\mathbb{F}$; however, it is $\mathbb{F} \times \mathbb{m}$, and not $\mathbb{F}$, that enters the interface null lagrangians. The reader is referred to Proposition 3.5.2 (below) for a general form of Theorem 2.2.3.

Theorem 2.2.4. *A function* $\hat{\mathbb{f}} : \mathrm{G} \to \mathrm{R} \cup \{\infty\}$ *is interface polyconvex if and only if there exists a positively* 1 *homogeneous function* $\Psi : \mathrm{Y} \to \mathrm{R} \cup \{\infty\}$ *defined on*

$$\mathrm{Y} := \prod_{k=0}^{t} \mathrm{Lin}(\wedge_{k+1}\mathrm{R}^n, \wedge_k \mathrm{R}^m) \tag{2.2.3}$$

such that

$$\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \Psi(\wedge_0 \mathbb{F} \wedge \mathbb{m}, \wedge_1 \mathbb{F} \wedge \mathbb{m}, \dots, \wedge_t \mathbb{F} \wedge \mathbb{m})$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$. *If* $m = n = 3$ *then* $\hat{\mathbb{f}}$ *is interface polyconvex if and only if there exists a positively* 1 *homogeneous convex function* Φ *on the space* X *such that*

$$\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \Phi(\mathbb{m}, \mathbb{F} \times \mathbb{m}, \operatorname{cof} \mathbb{F}\mathbb{m})$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$.

The reader is referred to Proposition 3.5.3 (below) for a general form of this result.

We now consider some particular cases.

Proposition 2.2.5. *Let* $m = n$ *and let* $\hat{\mathbb{f}} : \mathrm{G} \to \mathrm{R}$ *be an interface energy function.*

(i) *Let* $n = 3$ *and let* $g : [0, \infty)^3 \times \mathrm{R}^3 \to \mathrm{R}$ *be a positively* 1 *homogeneous convex function such that*

- *for each* $s \geq 0$, $\mathbf{p} \in \mathrm{R}^3$ *the function* $g(\cdot, \cdot, s, \mathbf{p})$ *is symmetric under the exchange of its two arguments,*
- *for each* $\mathbf{p} \in \mathrm{R}^3$ *the function* $g(\cdot, \cdot, \cdot, \mathbf{p})$ *is nondecreasing.*

If

$$\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = g(\lambda_1, \lambda_2, \lambda_1\lambda_2, \mathbb{m})$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$ *where* $\lambda_1, \lambda_2, 0$ *are the singular numbers of* $\mathbb{F}$, *then* $\hat{\mathbb{f}}$ *is interface polyconvex; in particular, if* g *is independent of* $\mathbb{m}$ *then* $\hat{\mathbb{f}}$ *is an isotropic interface polyconvex function.*

(ii) *Let*

$$\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \sigma |\operatorname{cof} \mathbb{F}|$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathbb{U}$ *where* σ *is a constant. Then* $\hat{\mathbb{f}}$ *is interface polyconvex if and only if* $\sigma \geq 0$.

(iii) *Let*

$$\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \varphi(\mathbb{m})$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$ *where* $\varphi : \mathrm{S}^{n-1} \to \mathrm{R}$ *is a positively* 1 *homogeneous function; then* $\hat{\mathbb{f}}$ *is interface polyconvex if and only if* φ *is convex.*

(iv) *Let* $n = 3$ *and*

$$\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \Psi(\mathbb{F} \times \mathbb{m})$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$ *where* $\Psi : \mathrm{Lin} \to \mathrm{R}$ *is a positively* 1 *homogeneous function. Then* $\hat{\mathbb{f}}$ *is interface polyconvex if and only if* Ψ *is convex.*

(v) *Let*

$$\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \psi(\operatorname{cof} \mathbb{F}\mathbb{m}),$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$*, where* $\psi : \mathrm{R}^n \to \mathrm{R}$ *is a given positively* 1 *homogeneous function. Then* $\hat{\mathbb{f}}$ *is interface polyconvex if and only if* ψ *is convex.*

Proof (i): Let $\Psi : \mathrm{R}^3 \times \mathrm{Lin} \times \mathrm{R}^3 \to \mathrm{R}$ be defined by

$$\Phi(\mathbf{p}, \mathbf{H}, \mathbf{q}) = g(\lambda_1, \lambda_2, |\mathbf{q}|, \mathbf{p})$$

for each $\mathbf{p}, \mathbf{q} \in \mathrm{R}^3$ and each $\mathbf{H} \in \mathrm{Lin}$ where $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge 0$ are the singular values of $\mathbf{H}$. If $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$ and $\mathbb{F}$ has the singular values $\lambda_1 \ge \lambda_2 \ge 0$ then $\mathbb{F} \times \mathbb{m}$ also has the singular values $\lambda_1 \ge \lambda_2 \ge 0$ and $\mathrm{cof}\, \mathbb{F}\mathbb{m}| = \lambda_1 \lambda_2$. Hence

$$\hat{\mathbb{f}}(\mathbb{F}, \mathbb{m}) = \Phi(\mathbb{m}, \mathbb{F} \times \mathbb{m}, \mathrm{cof}\, \mathbb{F}\mathbb{m})$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$. The symmetry and convexity of g imply that $\mathrm{D}_1 g \ge \mathrm{D}_2 g$ and the nondecreasing character that $\mathrm{D}_2 g \ge 0$ and $\mathrm{D}_3 g \ge 0$. The von Neumann and Cauchy inequalities give that Φ is a convex function and thus $\hat{\mathbb{f}}$ is polyconvex.

(ii): The sufficiency of $\sigma \ge 0$ is a particular case of (i). The necessity is immediate.

(iii): If φ is convex and positively 1 homogeneous (by hypothesis) then the definition immediately gives that $\hat{\mathbb{f}}$ is interface polyconvex. Conversely assume that $\hat{\mathbb{f}}$ is interface polyconvex so that there exists a positively 1 homogeneous convex function Φ such that

$$\varphi(\mathbb{m}) = \Phi(\mathbb{m}, \wedge_1 \mathbb{F} \wedge \mathbb{m}, \dots, \wedge_{n-1} \mathbb{F} \wedge \mathbb{m})$$

for each $(\mathbb{F}, \mathbb{m}) \in \mathrm{G}$. For $(\mathbf{0}, \mathbb{m}) \in \mathrm{G}$ this gives

$$\varphi(\mathbb{m}) = \Phi(\mathbb{m}, \mathbf{0}, \dots, \mathbf{0})$$

which must hold for each unit vector $\mathbb{m}$. The positive 1 homogeneity of φ and Φ then implies that

$$\varphi(\mathbf{v}) = \Phi(\mathbf{v}, \mathbf{0}, \dots, \mathbf{0})$$

for each $\mathbf{v} \in \mathrm{R}^n$ and the convexity of Φ implies that φ is convex.

(iv) and (v) are proved analogously. □

2.3 The existence of equilibrium states

This section outlines the existence theory for the minimizers of energy. We first enlarge the state space in Definition 2.3.1. The reader is referred to Section A.2 for the necessary notions of multilinear algebra (in particular the notion of interior derivative of a k vectorfield occurring in (2.3.1) below); furthermore, $\mathcal{M}(\Omega, V)$ denotes the set of all measures on an open set $\Omega \subset \mathrm{R}^n$ with values in a finite dimensional vectorspace V. Throughout the section we assume $m = n$ and consider the constitutive data of Definition 1.2.1.

Definitions 2.3.1 (State space for the existence theory). Let $\Omega \subset \mathrm{R}^n$ be a bounded open set with lipschitzian boundary and $n - 1 \le p \le \infty$, $1 \le q \le \infty$. We denote by $\mathcal{G}^{p,q}(\Omega, \mathrm{R}^n)$ the set of all pairs $(\mathbf{y}, E)$ such that

(i) $\mathbf{y} \in W^{1,p}(\Omega, \mathrm{R}^n)$,

$$\wedge_{n-1} \nabla \mathbf{y} \in L^q(\Omega, \mathrm{Lin}(\wedge_{n-1} \mathrm{R}^n, \wedge_{n-1} \mathrm{R}^n)),$$

(ii) E is an $\mathcal{L}^n$ measurable subset of Ω;

(iii) for each k with $0 \le k \le n-1$ there exists a measure $\mathbb{B}_k$ in the space $\mathcal{M}(\Omega, \mathrm{Lin}(\wedge_{k+1}\mathrm{R}^n, \wedge_k \mathrm{R}^n))$ satisfying

$$\int_E \wedge_k \nabla \mathbf{y}\, \partial \boldsymbol{\xi}\, d\mathcal{L}^n = (-1)^{k+1} \int_\Omega d\mathbb{B}_k\, \boldsymbol{\xi} \tag{2.3.1}$$

for each $\boldsymbol{\xi} \in \mathcal{D}_{k+1}(\Omega)$ [in the integral on the right hand side of (2.3.1) the integration measure $\mathbb{B}_k$ precedes the integrand $\boldsymbol{\xi}$ for algebraic reasons].

We call the elements $(\mathbf{y}, E) \in \mathcal{G}^{p,q}(\Omega, \mathrm{R}^n)$ ***states.*** We call the measure $\mathbb{B}_k$ the interface null lagrangian of order k corresponding to $(\mathbf{y}, E)$. We write $\mathbb{B}_k = \mathbb{B}_k(\mathbf{y}, E)$ to indicate the dependence on $(\mathbf{y}, E)$; we abbreviate $\mathbb{L}(\mathbf{y}, E) = (\mathbb{B}_0, \dots, \mathbb{B}_{n-1})$. Underlying the definition of the interface null lagrangians is the vanishing of the exterior derivative of the bulk jacobian minors $\wedge_k \nabla \mathbf{y}$. Namely, if $\mathbf{y} \in W^{1,p}(\Omega, \mathrm{R}^n)$ with $p \ge k$ then

$$\int_\Omega \wedge_k \nabla \mathbf{y}\, \partial \boldsymbol{\xi}\, d\mathcal{L}^n = \mathbf{0} \tag{2.3.2}$$

for each $\boldsymbol{\xi} \in \mathcal{D}_{k+1}(\Omega)$; we here recall that the interior derivative ∂ is dual (formal adjoint) of the exterior derivative. The reader is referred to [12; Corollary 2, Subsection 3.2.3] for a coordinate version of (2.3.2). Since (2.3.1) involves the bulk integral over E, one expects that the integration by parts will result in an object $\mathbb{B}_k$ concentrated on the boundary of E. This is indeed the case, as we shall show now.

Remark 2.3.2. *If $(\mathbf{y}, E) \in \mathcal{G}^{p,q}(\Omega, \mathrm{R}^n)$ with $p \ge n-1$, $q \ge 1$ then E is a set of finite perimeter and*

$$\mathbb{B}_0 = \mathbb{m}\, \mathcal{H}^{n-1} \llcorner\ \mathrm{bd}_*(E, \Omega) \tag{2.3.3}$$

where $\mathrm{bd}_*(E, \Omega) := \Omega \cap \mathrm{bd}_* E$ and $\mathrm{bd}_* E$ is the measure theoretic boundary of E. Moreover, if $1 \le k \le n-1$ then

$$\mathrm{spt}\, \mathbb{B}_k \subset \mathrm{cl}\, \mathrm{bd}_*(E, \Omega).$$

Proof For $k = 0$ Equation (2.3.1) reads

$$\int_E \mathrm{div}\, \mathbf{v}\, d\mathcal{L}^n = -\int_\Omega d\mathbb{B}_0 \mathbf{v}$$

for each $\mathbf{v} \in C_0^\infty(\Omega, \mathrm{R}^n)$ where $\mathbb{B}_0 \in \mathcal{M}(\Omega, \mathrm{Lin}(\mathrm{R}^n, \mathrm{R}))$. This shows that the perimeter of E in Ω is finite and as the rest of the perimeter of E can be only a subset of $\mathrm{bd}\, \Omega$, the lipschitzian character of $\mathrm{bd}\, \Omega$ implies that E is a set of finite perimeter. Equation (2.3.3) is then a consequence. If $1 \le k \le n-1$ and $\boldsymbol{\xi} \in \mathcal{D}^{k+1}(\Omega)$ satisfies $\mathrm{spt}\, \boldsymbol{\xi} \cap \mathrm{cl}\, \mathrm{bd}_* E = \emptyset$ then $\boldsymbol{\xi} = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2$ where $\mathrm{spt}\, \boldsymbol{\xi}_1 \subset E$ and $\mathrm{spt}\, \boldsymbol{\xi}_2 \subset \Omega \sim E$ and thus

$$\int_E \wedge_k \nabla \mathbf{y}\, \partial \boldsymbol{\xi}\, d\mathcal{L}^n = \int_E \wedge_k \nabla \mathbf{y}\, \partial \boldsymbol{\xi}_1\, d\mathcal{L}^n = \int_{\mathrm{R}^n} \wedge_k \nabla \mathbf{y}\, \partial \boldsymbol{\xi}_1\, d\mathcal{L}^n = 0$$

by (2.3.2). □

Remark 2.3.3. *If $(\mathbf{y}, E)$ is a pair where $\mathbf{y} : \Omega \to \mathrm{R}^n$ is lipschitzian and $E \subset \Omega$ is a set of finite perimeter then $(\mathbf{y}, E) \in \mathcal{G}^{p,q}(\Omega, \mathrm{R}^n)$ for all $p \ge n-1$ and $q \ge 1$; the measures $\mathbb{B}_k$ are given by*

$$\mathbb{B}_k = \wedge_k \nabla \mathbf{y} \wedge \mathbb{m}\, \mathcal{H}^{n-1} \llcorner\ \mathrm{bd}_*(E, \Omega) \tag{2.3.4}$$

were $\mathbb{y} = \mathbf{y} \,|\, \mathrm{bd}_*(E,\Omega)$ *and* $\nabla\mathbb{y}$ *is the approximate surface gradient of the lipschitzian map* $\mathbb{y}$ *on the* $\mathcal{H}^{n-1}$ *rectifiable set* $\mathrm{bd}_*(E,\Omega)$.

Thus $\mathbb{B}_k$ are the measure theoretic generalizations of the interface null lagrangians. See Section 3.6 (below) for the proof of (2.3.4).

Definition 2.3.4 (Energy functional for the existence theory). Let $\hat{f}_\alpha : \mathrm{Lin} \to [0,\infty]$, $\alpha = 1,2$, be functions of the forms

$$\hat{f}_\alpha(\mathbf{F}) = \Phi_\alpha(\wedge_1\mathbf{F},\dots,\wedge_s\mathbf{F}) \tag{2.3.5}$$

for all $\alpha = 1,2$ and all $\mathbf{F} \in \mathrm{Lin}$, where $\Phi_\alpha : \mathrm{Z} \to [0,\infty]$ are continuous convex functions on

$$\mathrm{Z} = \prod_{k=1}^{s} \mathrm{Lin}(\wedge_k\mathrm{R}^n, \wedge_k\mathrm{R}^n).$$

Let $\hat{\mathbb{f}} : \mathrm{G} \to [0,\infty)$ be a function of the form

$$\hat{\mathbb{f}}(\mathbb{F},\mathbb{m}) = \Phi(\wedge_0\mathbb{F}\wedge\mathbb{m},\dots,\wedge_{n-1}\mathbb{F}\wedge\mathbb{m}) \tag{2.3.6}$$

for each $(\mathbb{F},\mathbb{m}) \in \mathrm{G}$ where $\Phi : \mathrm{Y} \to [0,\infty)$ is a positively 1 homogeneous convex function. If $n-1 \le p \le \infty$, we define the ***total energy*** $\mathsf{E} : \mathcal{G}^{p,q}(\Omega,\mathrm{R}^n) \to [0,\infty]$ by (2.1.3) for each $(\mathbf{y},E) \in \mathcal{G}^{p,q}(\Omega,\mathrm{R}^n)$ where E_b is given by (2.1.4) and

$$\mathsf{E}_\mathrm{if}(\mathbf{y},E) = \int_{\mathrm{R}^n} \Phi(\mathbb{A})\, d\,|\mathbb{L}(\mathbf{y},E)| \tag{2.3.7}$$

where $|\mathbb{L}(\mathbf{y},E)|$ is the total variation of $\mathbb{L}(\mathbf{y},E)$ and $\mathbb{A} : \Omega \to \mathrm{Y}$ satisfies $\mathbb{L}(\mathbf{y},E) = \mathbb{A}\,|\mathbb{L}(\mathbf{y},E)|$; cf. [2; Corollary 1.29 and Section 2.6].
The definition (2.3.7) reduces to

$$\mathsf{E}_\mathrm{if}(\mathbf{y},E) = \int_{\mathrm{bd}_*(E,\Omega)} \hat{\mathbb{f}}(\nabla\mathbf{y},\mathbb{m})\, d\mathcal{H}^{n-1}$$

if $(\mathbf{y},E)$ consists of a lipschitzian map $\mathbf{y}$ and a set of finite perimeter $E \subset \Omega$.

Theorem 2.3.5. *Let* $n-1 \le p < \infty$, $n/(n-1) \le q < \infty$ *and assume that*

(i) $\hat{f}_\alpha$, $\alpha = 1,2$, *are polyconvex in the sense of* (2.3.5) *where* Φ_α *are continuous convex* $[0,\infty]$ *valued functions,* $\hat{\mathbb{f}}$ *is interface polyconvex in the sense of* (2.3.6) *where* Φ *is a positively* 1 *homogeneous convex* $[0,\infty)$ *valued function,*

(ii) *for all* $\alpha = 1,2$, *all* $\mathbf{F} \in \mathrm{Lin}$, *all* $\mathrm{A} \in \mathrm{Y}$, *some* $c > 0$ *and some* $d \in \mathrm{R}$ *we have*

$$\hat{f}_\alpha(\mathbf{F}) \ge c(|\mathbf{F}|^p + |\operatorname{cof}\mathbf{F}|^q) + d, \quad \Phi(\mathrm{A}) \ge c|\mathrm{A}|,$$

(iii) $\hat{f}_\alpha(\mathbf{F}) = \infty$ *if* $\det\mathbf{F} \le 0$.

Given $\mathbf{z}_0 \in W^{1,p}(\Omega,\mathrm{R}^n)$, *consider the Dirichlet class*

$$\mathcal{A}(\mathbf{z}_0) = \{(\mathbf{y},E) \in \mathcal{G}^{p,q}(\Omega,\mathrm{R}^n) : \mathbf{y} = \mathbf{z}_0 \ \text{ on } \ \mathrm{bd}\,\Omega\}$$

and let E *be given by Definition* 2.3.4. *If* E *is finite at some element of* $\mathcal{A}(\mathbf{z}_0)$ *then there exists an* $(\mathbf{y},E) \in \mathcal{A}(\mathbf{z}_0)$ *such that*

$$\mathsf{E}(\mathbf{y},E) \le \mathsf{E}(\mathbf{z},F)$$

for all $(\mathbf{z},F) \in \mathcal{A}(\mathbf{z}_0)$. *Each solution* $(\mathbf{y},E)$ *of the problem satisfies*

$$\det \nabla \mathbf{y} > 0 \; \textit{for} \; \mathcal{L}^n \; \textit{a.e. point of} \; \Omega.$$

Proof Let $\mathbf{M}(\mu)$ denote the mass of the measure $\mu \in \mathcal{M}(\Omega, V)$, i.e., $\mathbf{M}(\mu) = |\mu|(\Omega)$ where $|\mu|$ denotes the total variation of μ. Let $(\mathbf{y}^i, E^i) \in \mathcal{A}(\mathbf{z}_0)$ be a minimizing sequence. By the coercivity assumptions on $\hat{f}_\alpha$ and Φ the sequences $|\nabla \mathbf{y}^i|_{L^p}$ and $\mathbf{M}(\mathbb{L}(\mathbf{y}^i, E^i))$ are bounded. Combining the boundedness of $|\nabla \mathbf{y}^i|_{L^p}$ with the Dirichlet boundary data, one obtains the boundedness of $|\mathbf{y}^i|_{W^{1,p}}$. Standard compactness theorems for Sobolev space and for the spaces of measures give that for some subsequence of $(\mathbf{y}^i, E^i)$, denoted again $(\mathbf{y}^i, E^i)$, and some $\mathbf{y} \in W^{1,p}(\Omega, \mathrm{R}^n)$, $\boldsymbol{\Delta} \in \mathcal{M}(\Omega, \mathrm{Y})$ we have the following facts:

$$\mathbf{y}^i \rightharpoonup \mathbf{y} \quad \text{in } W^{1,p}(\Omega, \mathrm{R}^n), \tag{2.3.8}$$

$$\wedge_{n-1} \nabla \mathbf{y} \simeq \operatorname{cof} \nabla \mathbf{y} \quad \text{bounded in } L^q(\Omega, \operatorname{Lin}(\wedge_{n-1}\mathrm{R}^n, \wedge_{n-1}\mathrm{R}^n))$$

$$\mathbb{L}(\mathbf{y}^i, E^i) \rightharpoonup^* \boldsymbol{\Delta} \quad \text{in } \mathcal{M}(\Omega, \mathrm{Y}) \tag{2.3.9}$$

where Y is defined by (2.2.3). From $\mathbb{B}_0(\mathbf{y}^i, E^i) = \mathbb{m}^{E^i} \mathcal{H}^{n-1} \llcorner \operatorname{bd}_*(E^i, \Omega)$ we deduce that the sequence $\mathrm{D}\,1_{E^i}$ is bounded in $\mathcal{M}(\Omega, \mathrm{R}^n)$. The imbedding theorem from *BV* functions (e.g., [2; Corollary 3.49, Chapter 3] implies

$$1_{E^i} \to 1_E \quad \text{in } L^1(\Omega). \tag{2.3.10}$$

for some set $E \subset \Omega$ of finite perimeter, i.e.,

$$\mathcal{L}^n(\Delta(E^i, E)) \to 0, \tag{2.3.11}$$

where $\Delta(E^i, E)$ is the symmetric difference of E^i and E. The inequality $p \ge n - 1$, Equation (2.3.8) and the weak sequential continuity of minors (e.g., [21; Theorem 2.3(ii)]) gives

$$\wedge_k \nabla \mathbf{y}^i \rightharpoonup \wedge_k \nabla \mathbf{y} \quad \text{in } L^{p/(n-1)}(\Omega, \operatorname{Lin}(\wedge_k \mathrm{R}^n, \wedge_k \mathrm{R}^n)), \quad 0 \le k < n-1. \tag{2.3.12}$$

The condition $\mathsf{E}(\mathbf{y}^i, E^i) < \infty$ for each i and Hypothesis (iii) imply that $\det \nabla \mathbf{y}^i > 0$ for every i and $\mathcal{L}^n$ a.e. point of Ω. By [22; Lemma 4.1] then

$$\wedge_{n-1} \nabla \mathbf{y}^i \rightharpoonup \wedge_{n-1} \nabla \mathbf{y} \quad \text{in } L^q(\Omega, \operatorname{Lin}(\wedge_{n-1}\mathrm{R}^n, \wedge_{n-1}\mathrm{R}^n)), \tag{2.3.13}$$

$$\det \nabla \mathbf{y}^i \rightharpoonup \det \nabla \mathbf{y} \quad \text{in } L^1(K, \mathrm{R})$$

for each compact subset K of Ω.

The equiintegrability of the sequence $\wedge_k \nabla \mathbf{y}^i$ and (2.3.11) yield

$$1_{E^i} \wedge_k \nabla \mathbf{y}^i \rightharpoonup 1_E \wedge_k \nabla \mathbf{y} \quad \text{in } L^1(\Omega, \operatorname{Lin}(\wedge_k \mathrm{R}^n, \wedge_k \mathrm{R}^n)), \quad 0 \le k \le n-1$$

and in particular,

$$\int_{E^i} \wedge_k \nabla \mathbf{y}^i \, \partial \boldsymbol{\xi} \, d\mathcal{L}^n \to \int_E \wedge_k \nabla \mathbf{y} \, \partial \boldsymbol{\xi} \, d\mathcal{L}^n$$

for each $\boldsymbol{\xi} \in \mathcal{D}_{k+1}(\mathrm{R}^n)$, which can be rewritten as

$$\int_\Omega d\,\mathbb{B}_k(\mathbf{y}^i, E^i)\boldsymbol{\xi} \to \int_E \wedge_k \nabla \mathbf{y} \, \partial \boldsymbol{\xi} \, d\mathcal{L}^n. \tag{2.3.14}$$

Hence (2.3.9) yields

$$\int_E \wedge_k \nabla \mathbf{y} \, \partial \boldsymbol{\xi} \, d\mathcal{L}^n = \int_\Omega d\boldsymbol{\Delta}_k \, \boldsymbol{\xi}$$

where we write $\mathbf{\Delta} = (\mathbf{\Delta}_0, \dots, \mathbf{\Delta}_s)$ for the components of $\mathbf{\Delta}$. Thus $(\mathbf{y}, E) \in \mathcal{G}^{p,q}(\Omega, \mathrm{R}^n)$ and $\mathbb{B}_k(\mathbf{y}, E) = \mathbf{\Delta}_k$. Equations (2.3.14) and (2.3.13) reduce to

$$\mathbb{L}(\mathbf{y}^i, E^i) \rightharpoonup^* \mathbb{L}(\mathbf{y}, E) \quad \text{in } \mathcal{M}(\Omega, \mathrm{Y}). \tag{2.3.15}$$

Let $\Xi : \mathrm{R} \times \mathrm{Z} \to [0, \infty]$ be defined by

$$\Xi(\tau, \mathrm{M}) = |\tau| \Phi_1(\mathrm{M})$$

for each $\tau \in \mathrm{R}$ and $\mathrm{M} \in \mathrm{Y}$; note that the function $\Xi(\tau, \cdot)$ is convex for each $\tau \in \mathrm{R}$. For each compact subset K of Ω we have

$$\int_{E^i \cap K} \hat{f}_1(\nabla \mathbf{y}^i)\, d\mathcal{L}^n = \int_K \Xi(1_{E^i}, \wedge_1 \nabla \mathbf{y}^i, \dots, \wedge_s \nabla \mathbf{y}^i)\, d\mathcal{L}^n.$$

The Ioffe lowersemicontinuity theorem [2; Theorem 5.8, Chapter 5] and (2.3.10) and (2.3.12) then give

$$\liminf_{i \to \infty} \int_K \Xi(1_{E^i}, \wedge_0 \nabla \mathbf{y}^i, \dots, \wedge_s \nabla \mathbf{y}^i)\, d\mathcal{L}^n \ge \int_K \Xi(1_E, \wedge_0 \nabla \mathbf{y}, \dots, \wedge_s \nabla \mathbf{y})\, d\mathcal{L}^n.$$

Thus the nonnegativity of $\hat{f}_\alpha$ gives

$$\liminf_{i \to \infty} \int_{E^i} \hat{f}_1(\nabla \mathbf{y}^i)\, d\mathcal{L}^n \ge \int_{E \cap K} \hat{f}_1(\nabla \mathbf{y})\, d\mathcal{L}^n,$$

$$\liminf_{i \to \infty} \int_{\Omega \sim E^i} \hat{f}_2(\nabla \mathbf{y}^i)\, d\mathcal{L}^n \ge \int_{K \sim E} \hat{f}_2(\nabla \mathbf{y})\, d\mathcal{L}^n$$

where the last relation is obtained analogously. The arbitrariness of K then gives

$$\liminf_{i \to \infty} \int_{E^i} \hat{f}_1(\nabla \mathbf{y}^i)\, d\mathcal{L}^n \ge \int_E \hat{f}_1(\nabla \mathbf{y})\, d\mathcal{L}^n, \tag{2.3.16}$$

$$\liminf_{i \to \infty} \int_{\Omega \sim E^i} \hat{f}_2(\nabla \mathbf{y}^i)\, d\mathcal{L}^n \ge \int_{\Omega \sim E} \hat{f}_2(\nabla \mathbf{y})\, d\mathcal{L}^n. \tag{2.3.17}$$

Using (2.3.15) and the Reshetnyak lowersemicontinuity theorem (e.g., [2; Theorem 2.38, Chapter 2]), one obtains

$$\liminf_{i \to \infty} \mathsf{E}_{\mathrm{if}}(\mathbf{y}^i, E^i) \ge \mathsf{E}_{\mathrm{if}}(\mathbf{y}, E). \tag{2.3.18}$$

Thus (2.3.16), (2.3.17) and (2.3.18) provide

$$\liminf_{i \to \infty} \mathsf{E}(\mathbf{y}^i, E^i) \ge \mathsf{E}(\mathbf{y}, E).$$

Clearly, $(\mathbf{y}, E) \in \mathcal{A}(\mathbf{z}_0)$. □

Chapter 3

Graphs, currents, and quasiconvexity of degree r

This chapter deals with the convexity properties of integral functionals of the form

$$F(M, \boldsymbol{\theta}, \boldsymbol{\alpha}) = \int_M \hat{\mathbb{f}}(\nabla \boldsymbol{\theta}, \boldsymbol{\alpha})\, d\mathcal{H}^r \tag{3.0.1}$$

where M is an r dimensional oriented surface in R^n, with $0 \le r \le n$, $\boldsymbol{\theta}$ is a lipschitzian map from a bounded open subset Ω of R^n to R^m, $\nabla \boldsymbol{\theta}$ is the surface gradient of $\boldsymbol{\theta}$ relative to M and $\boldsymbol{\alpha}$ is a vectorfield on M with values in r vectors on R^n giving the surface M an orientation. The surface M is varying, it is an independent variable of the functional F. The function $\hat{\mathbb{f}}$ is called a standard integrand here. Quasiconvex and polyconvex standard integrands will be defined and also standard integrands that are null lagrangians will be introduced. The interface quasiconvexity and polyconvexity and interface null lagrangians defined above in Section 2.2 will become particular cases of the present notions corresponding to $r = n-1$ and the standard “bulk” notions of quasiconvexity, polyconvexity, and null lagrangians correspond to $r = n$.

The general form of the null lagrangians and of polyconvex functions will be established and a lowersemicontinuity theorem for sequences with varying surfaces M_i, $\boldsymbol{\alpha}_i$ and $\boldsymbol{\theta}_i$ will be established. The proof of the structure of null lagrangians is essentially noncomputational, but is based on a lifting of the picture of the map $\boldsymbol{\theta}$ on M to the graph of $\boldsymbol{\theta}$ on M. The integral (3.0.1) can be expressed as

$$\int_S \Phi(\boldsymbol{\gamma})\, d\mathcal{H}^r \tag{3.0.2}$$

where $S := \mathrm{graph}(M, \boldsymbol{\theta})$ is interpreted as a surface of dimension r in R^{m+n}, $\boldsymbol{\gamma}$ is an r vectorfield giving $\mathrm{graph}(M, \boldsymbol{\theta})$ an orientation and Φ is some integrand that is uniquely determined by $\hat{\mathbb{f}}$. Following Federer [9; Subsection 5.1.2] we call Φ a parametric integrand of degree r. The correspondence between the standard integrands $\hat{\mathbb{f}}$ and the parametric integrands is essentially bijective and a question arises how the convexity properties of standard integrands translate into the language of parametric integrands. The central convexity notion for the parametric integrands is the semiellipticity by Almgren [1; Section 1] (also Federer [9; Subsection 5.1.2]), which is equivalent to the semiconvexity of the functionals (3.0.2) on surfaces under the flat norm. The relationship between the semiellipticity of Φ and quasiconvexity of $\hat{\mathbb{f}}$ is rather direct: the semiellipticity of Φ implies the quasiconvexity of $\hat{\mathbb{f}}$ and conversely the quasiconvexity of $\hat{\mathbb{f}}$ implies the semiellipticity inequality on surfaces that can be represented as graphs. In case of null lagrangians and of polyconvex functions the relationship is even simpler: $\hat{\mathbb{f}}$ is a null lagrangian if and only if both Φ and $-\Phi$ are semielliptic; $\hat{\mathbb{f}}$ is polyconvex if and only if Φ has a convex extension onto the convex hull of $\operatorname{dom}\Phi$.

3.1 Rectifiable currents

Our model of a surface of dimension r in R^d is a rectifiable r dimensional current. These will be our integration domains. Let U be an open subset of R^d.

Definition 3.1.1 (Cf. [9]). An ***r dimensional current*** T in U $(0 \le r \le d)$ is an $\wedge_r \mathrm{R}^d$ valued distribution on U, i.e., a continuous linear function T on the space $\mathscr{D}^r(U)$ of infinitely differentiable r forms on R^d with compact support which is contained in U, endowed with the Schwartz topology. We denote the value of T on $\boldsymbol{\omega} \in \mathscr{D}^r(U)$ by $\langle T, \boldsymbol{\omega} \rangle$. We define the boundary ∂T of an r dimensional current T in U as an $r-1$ dimensional current defined by

$$\langle \partial T, \boldsymbol{\sigma} \rangle = \langle T, \mathrm{D}\,\boldsymbol{\sigma} \rangle$$

for each $\boldsymbol{\sigma} \in \mathscr{D}^{r-1}(U)$ where $\mathrm{D}\,\boldsymbol{\sigma} \in \mathscr{D}^r(U)$ is the exterior derivative of $\boldsymbol{\sigma}$ and we put $\mathscr{D}^s(U) = \{0\}$ if $s \le 0$.

We now introduce special classes of currents which will play leading roles in the developments below. We say that a subset M of R^d is $\mathscr{H}^r$ rectifiable if M is $\mathscr{H}^r$ measurable, $\mathscr{H}^r(M) < \infty$ and $\mathscr{H}^r$ almost all of M is contained in the set

$$\bigcup_{i=1}^{\infty} \boldsymbol{\phi}_i(\mathrm{R}^r)$$

where $\boldsymbol{\phi}_i : \mathrm{R}^r \to \mathrm{R}^d$ are lipschitzian maps. It follows that the approximate tangent space $\operatorname{Tan}^r(M, \mathbf{x})$ is an r dimensional linear subspace of R^d for $\mathscr{H}^r$ a.e. $\mathbf{x} \in M$. We say that an r vectorfield $\boldsymbol{\alpha} : M \to \wedge_r \mathrm{R}^d$ on a $\mathscr{H}^r$ rectifiable set $M \subset \mathrm{R}^d$ is tangential if $\boldsymbol{\alpha}(\mathbf{x})$ is the product of r vectors from $\operatorname{Tan}^r(M, \mathbf{x})$ for $\mathscr{H}^r$ a.e. $\mathbf{x} \in M$. Let N denote the set of all positive integers.

Definition 3.1.2 (Cf. [9]). An r dimensional current T in U is said to be

(i) ***rectifiable*** if

$$\langle T, \boldsymbol{\omega} \rangle = \int_M \langle \boldsymbol{\alpha}, \boldsymbol{\omega} \rangle \, d\mathscr{H}^r \tag{3.1.1}$$

for each $\boldsymbol{\omega} \in \mathscr{D}^r(U)$ where

$$\left.\begin{array}{c} M \text{ is a } \mathscr{H}^r \text{ rectifiable subset of } U, \\ \boldsymbol{\alpha} : M \to \wedge_r \mathrm{R}^d \text{ is a } \mathscr{H}^r \text{ integrable,} \\ \boldsymbol{\alpha} \text{ is tangential to } M, \\ \operatorname{ran}|\boldsymbol{\alpha}| \subset \mathrm{N}; \end{array}\right\} \tag{3.1.2}$$

we write

$$T = \boldsymbol{\alpha}\,\mathcal{H}^r \llcorner M; \tag{3.1.3}$$

(ii) ***integral*** if T and ∂T are rectifiable.

We note that the objects in (3.1.2) and the formula (3.1.1) defines a rectifiable current in any open subset of R^d containing M; the currents corresponding to different choices of U then differ by their domain of definition $\mathcal{D}^r(U)$. The integrality of the current T depends strongly on the choice of U since the boundary ∂T depends on U. Indeed, if $T_0 : \mathcal{D}^r(\mathrm{R}^d) \to \mathrm{R}$ is a current on R^d defined by

$$\langle T_0, \boldsymbol{\omega} \rangle = \int_M \langle \boldsymbol{\alpha}, \boldsymbol{\omega} \rangle \, d\mathcal{H}^r$$

for each $\boldsymbol{\omega} \in \mathcal{D}^r(\mathrm{R}^d)$ while if $T : \mathcal{D}^r(U) \to \mathrm{R}$ is given by (3.1.1) for every $\boldsymbol{\omega} \in \mathcal{D}^r(U)$ then $\partial T : \mathcal{D}^{r-1}(U) \to \mathrm{R}$ is the restriction of $\partial T_0 : \mathcal{D}^{r-1}(\mathrm{R}^d) \to \mathrm{R}$ to the set $\mathcal{D}^{r-1}(U)$. One may say, informally, that ∂T is the part of the boundary of T_0 that is contained in U.

Definition 3.1.3. We say that an r dimensional current S ***represents a planar region*** if there exists an r dimensional plane $P \subset \mathrm{R}^d$ (i.e., and r dimensional affine subspace of U) and a bounded $\mathcal{H}^r$ measurable subset N of P such that

$$S = \boldsymbol{\beta}\,\mathcal{H}^r \llcorner N$$

where $\boldsymbol{\beta}$ is constant and equal to any of the two unit vectors associated with P.

Each r dimensional rectifiable current T is an r dimensional flat chain as defined in [29, 9] and thus T can be put into the duality pairing $\langle T, \boldsymbol{\omega} \rangle$ with flat r dimensional forms $\boldsymbol{\omega}$, i.e., essentially bounded $\mathcal{L}^d$ measurable forms on R^d such that the weak exterior derivative of $\boldsymbol{\omega}$ is essentially bounded $\mathcal{L}^d$ measurable. In the particular case the flat form $\boldsymbol{\omega}$ is continuous, the duality pairing reduces to the integration

$$\langle T, \boldsymbol{\omega} \rangle = \int_M \langle \boldsymbol{\alpha}, \boldsymbol{\omega} \rangle \, d\mathcal{H}^r$$

where $\langle \boldsymbol{\alpha}, \boldsymbol{\omega} \rangle$ denotes the duality pairing between r vectors and r covectors. If $\boldsymbol{\phi} : \mathrm{R}^d \to \mathrm{R}^e$ is a lipschitzian map and $\boldsymbol{\sigma}$ a flat r form on R^e we define the pullback $\boldsymbol{\phi}^{\#}\boldsymbol{\sigma}$ as an r form on R^d by

$$\boldsymbol{\phi}^{\#}\boldsymbol{\sigma} = \wedge_r(\nabla\boldsymbol{\phi})^* \; \boldsymbol{\sigma} \circ \boldsymbol{\phi}$$

for $\mathcal{L}^d$ a.e. $\mathbf{x} \in \mathrm{R}^d$, where $\nabla\boldsymbol{\phi}$ is the derivative of $\boldsymbol{\phi}$, which exists for $\mathcal{L}^d$ a.e. point of R^d, $(\nabla\boldsymbol{\phi})^*(\mathbf{x}) : \wedge_1\mathrm{R}^e \to \wedge_1\mathrm{R}^d$ denotes the adjoint of $\nabla\boldsymbol{\phi}(\mathbf{x})$, so that $\wedge_r(\nabla\boldsymbol{\phi})^*(\mathbf{x}) : \wedge_r\mathrm{R}^e \to \wedge_r\mathrm{R}^d$. It turns out that $\boldsymbol{\phi}^{\#}\boldsymbol{\sigma}$ is a flat form and the fundamental formula

$$\mathrm{D}(\boldsymbol{\phi}^{\#}\boldsymbol{\sigma}) = \boldsymbol{\phi}^{\#}\,\mathrm{D}\,\boldsymbol{\sigma}$$

relating the weak exterior derivatives holds. Dualizing, one defines the pushforward $\boldsymbol{\phi}_{\#}T$ of a flat r dimensional chain T on R^d by a lipschitzian map $\boldsymbol{\phi} : \mathrm{R}^d \to \mathrm{R}^e$ as a current that satisfies

$$\langle \boldsymbol{\phi}_{\#}T, \boldsymbol{\sigma} \rangle = \langle T, \boldsymbol{\phi}^{\#}\boldsymbol{\sigma} \rangle$$

for each $\boldsymbol{\sigma} \in \mathcal{D}^r(\mathrm{R}^e)$. It is possible to prove that $\boldsymbol{\phi}_{\#}T$ is a flat chain. The following proposition describes the pushforward in the particular case of rectifiable currents.

Proposition 3.1.4 (Cf. [9; Subsection 4.1.30]). *Let T be an r dimensional rectifiable current in U of the form* (3.1.3) *where M and $\boldsymbol{\alpha}$ satisfy* (3.1.2). *Let $\boldsymbol{\phi} : \mathrm{R}^d \to \mathrm{R}^e$ be a lipschitzian map and put $\boldsymbol{\eta} := \boldsymbol{\phi}|M$. Then $S := \boldsymbol{\phi}_\# T$ is an r dimensional rectifiable current on R^e of the form*

$$S = \boldsymbol{\beta}\, \mathcal{H}^r \llcorner \boldsymbol{\eta}(M)$$

where for $\mathcal{H}^r$ point $\mathbf{y}$ of $\boldsymbol{\eta}(M)$,

$$\boldsymbol{\beta}(\mathbf{y}) = \sum_{\mathbf{x} \in \boldsymbol{\eta}^{-1}(\{\mathbf{y}\})} [\wedge_r \nabla\, \boldsymbol{\eta}(\mathbf{x})]\, \boldsymbol{\alpha}(\mathbf{x}) / J_r \boldsymbol{\eta}(\mathbf{x})$$

is a simple r vector tangential to $\boldsymbol{\eta}(M)$ at $\mathbf{x}$ and

$$[\wedge_r \nabla\, \boldsymbol{\eta}(\mathbf{x})]\, \boldsymbol{\alpha}(\mathbf{x}) / J_r \boldsymbol{\eta}(\mathbf{x}) = \pm |\boldsymbol{\alpha}(\mathbf{x})| \boldsymbol{\beta}(\mathbf{y}) / |\boldsymbol{\beta}(\mathbf{y})|$$

for every $\mathbf{x} \in \boldsymbol{\eta}^{-1}(\{\mathbf{y}\})$.

Here and below $\nabla\, \boldsymbol{\eta}(\mathbf{x})$ is the approximate surface gradient of the lipschitzian map $\boldsymbol{\eta}$ relative to the $\mathcal{H}^r$ rectifiable set M (see Section A.1, below) and $J_r \boldsymbol{\eta}(\mathbf{x}) = |\wedge_r \nabla\, \boldsymbol{\eta}(\mathbf{x})|$ is the jacobian. We see that $S = \boldsymbol{\phi}_\# T$ is completely determined by $\boldsymbol{\eta} := \boldsymbol{\phi}|M$ and we write $S = \boldsymbol{\eta}_\# T$.

3.2 Parametric integrands and semiellipticity

Let $\mathrm{si} \wedge_r \mathrm{R}^d$ denote the cone of all simple vectors from $\wedge_r \mathrm{R}^d$.

Definitions 3.2.1 (Cf. [9; Subsection 5.1.1]).

(i) We say that Φ is a ***parametric integrand of degree*** r in R^d if Φ is a positively 1 homogeneous borelian function with values in $\mathrm{R} \cup \{\infty\}$ defined on some cone contained in $\mathrm{si} \wedge_r \mathrm{R}^d$ and the negative part of Φ is locally bounded.

(ii) If Φ is a parametric integral of degree r in R^d, we denote by $\mathrm{dom}\,\langle \Phi, \cdot \rangle$ the set of all rectifiable currents T in R^d with compact support, of the form (3.1.3) where $M, \boldsymbol{\alpha}$ satisfy (3.1.2) and $\mathrm{ran}\, \boldsymbol{\alpha} \subset \mathrm{dom}\, \Phi$; for any such a T we define

$$\langle \Phi, T \rangle = \int \Phi \circ \boldsymbol{\alpha}\, d\mathcal{H}^r$$

and call $\langle \Phi, T \rangle$ the ***parametric integral*** of the current T; we note that $\langle \Phi, T \rangle \in \mathrm{R} \cup \{\infty\}$.

Definitions 3.2.2. Let Φ be a parametric integrand of degree r on R^d. The integrand Φ is said to be

(i) ***elliptic*** if there exists a $c > 0$ such that

$$\langle \Phi, T \rangle - \langle \Phi, S \rangle \ge c\big(\mathbf{M}(T) - \mathbf{M}(S)\big) \tag{3.2.1}$$

whenever $T, S \in \mathrm{dom}\,\langle \Phi, \cdot \rangle$ satisfy $\partial T = \partial S$ and S represents a planar region. Here $\mathbf{M}(T)$ is the mass (total variation) of the measure T;

(ii) ***semielliptic*** if

$$\langle \Phi, T \rangle \ge \langle \Phi, S \rangle$$

for any T, S as above;

(iii) a ***semielliptic null lagrangian*** if $\operatorname{dom}\Phi = \operatorname{si}\wedge_r \mathrm{R}^d$, $\operatorname{ran}\Phi \subset \mathrm{R}$, Φ is continuous and both Φ and $-\Phi$ are semielliptic;

(iv) ***semielliptic polyconvex*** if $\operatorname{dom}\Phi = \operatorname{si}\wedge_r \mathrm{R}^d$ and Φ is the supremum of some family of semielliptic null lagrangians.

Proposition 3.2.3. *Let Φ be a semielliptic integrand of degree r on R^d with $r < d$. Let $\mathbf{a}_0, \dots, \mathbf{a}_r \in \mathrm{R}^d$ be linearly independent and define*

$$\hat{\mathbf{a}}^i := (-1)^i \bigwedge_{k=0,\, k\neq i}^{r} \mathbf{a}_k, \tag{3.2.2}$$

$0 \le i \le r$. Then

$$\Phi\Big(\sum_{i=0}^{r} \hat{\mathbf{a}}^i\Big) \le \sum_{i=0}^{r} \Phi(\hat{\mathbf{a}}^i) \tag{3.2.3}$$

provided $\hat{\mathbf{a}}^i$ and $\sum_{i=0}^{r} \hat{\mathbf{a}}^i$ belong to $\operatorname{dom}\Phi$. If Φ is continuous and $\operatorname{dom}\Phi = \operatorname{si}\wedge_r \mathrm{R}^d$ then the requirement that $\mathbf{a}_1, \dots, \mathbf{a}_r$ be linearly independent can be relaxed.

Proof We have

$$\sum_{i=0}^{r} \hat{\mathbf{a}}^i = (-1)^{k+1} \bigwedge_{i=0,\, i\neq k}^{r} (\mathbf{a}_i - \mathbf{a}_k)$$

for each $k = 0, \dots, r$; thus $\boldsymbol{\gamma} := \sum_{i=0}^{r} \hat{\mathbf{a}}^i$ is a simple vector. If Δ is an $r+1$ simplex in R^d with vertices $\mathbf{0}, \mathbf{a}_0, \dots, \mathbf{a}_r$ then the r vector $-\hat{\mathbf{a}}^i$ is tangent to the face F_i opposite to the vertex $\mathbf{a}_i$ and $\sum_{i=0}^{r} \hat{\mathbf{a}}^i$ is tangent to the face M opposite to $\mathbf{0}$. Putting

$$T_i = \hat{\mathbf{a}}_i \mathcal{H}^r \llcorner F_i, \quad i = 0, \dots, r, \quad T := \sum_{i=0}^{r} T_i, \quad S = \boldsymbol{\gamma} \mathcal{H}^r \llcorner M,$$

we observe that the current represeting the boundary of Δ is $S - T$ and hence $\partial(S - T) = \mathbf{0}$, i.e., $\partial T = \partial S$. Applying the semiellipticity condition to the testcurrents T and S, we obtain (3.2.3). Finally, if Φ is continuous and $\operatorname{dom}\Phi = \operatorname{si}\wedge_r \mathrm{R}^d$ then since $r < d$, one approximates a given $r+1$ tuple of vectors by an $r+1$ tuple of linearly independent vectors and uses the continuity. □

Proposition 3.2.4 ([29; Theorem 9A, Chapter 5]). *Let Φ be a real valued function defined on $\operatorname{si}\wedge_r \mathrm{R}^d$ such that*

$$\Phi(t\boldsymbol{\alpha}) = t\Phi(\boldsymbol{\alpha}) \tag{3.2.4}$$

for any $t \in \mathrm{R}$ and any $\boldsymbol{\alpha} \in \operatorname{si}\wedge_r \mathrm{R}^d$ and

$$\sum_{i=0}^{r} \Phi(\hat{\mathbf{a}}^i) = \Phi\Big(\sum_{i=0}^{r} \hat{\mathbf{a}}^i\Big) \tag{3.2.5}$$

for any $r+1$ tuple of vectors $\mathbf{a}_0, \dots, \mathbf{a}_r \in \mathrm{R}^d$, where $\hat{\mathbf{a}}^i$ are given by (3.2.2). Then there exists a unique $\boldsymbol{\omega} \in \wedge^r \mathrm{R}^d$ such that $\Phi(\boldsymbol{\alpha}) = \langle \boldsymbol{\alpha}, \boldsymbol{\omega} \rangle$ for any $\boldsymbol{\alpha} \in \operatorname{si}\wedge_r \mathrm{R}^d$.

Proof The uniqueness is clear. To prove the existence, set

$$\boldsymbol{\omega}(\mathbf{v}_1, \dots, \mathbf{v}_r) := \Phi(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r)$$

for any $\mathbf{v}_1, \dots, \mathbf{v}_r \in \mathrm{R}^d$, and show that $\boldsymbol{\omega}$ is an r covector. Equation (3.2.4) implies that $\boldsymbol{\omega}$ is antisymmetric with respect to premutations of $\mathbf{v}_1, \dots, \mathbf{v}_r$ and homogeneous in each argument. Thus to show that $\boldsymbol{\omega}$ is an r form we have to prove that $\boldsymbol{\omega}$ is additive in each argument. It suffices to prove that $\boldsymbol{\omega}$ is additive in the first argument. Hence let $\mathbf{v}_2, \dots, \mathbf{v}_r$ be given and show that

$$\boldsymbol{\omega}(\mathbf{v}_1 + \mathbf{v}_1', \mathbf{v}_2, \dots, \mathbf{v}_r) = \boldsymbol{\omega}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r) + \boldsymbol{\omega}(\mathbf{v}_1', \mathbf{v}_2, \dots, \mathbf{v}_r) \tag{3.2.6}$$

for any $\mathbf{v}_1, \mathbf{v}_1' \in \mathrm{R}^d$. Let an $r+1$ tuple of vectors $\mathbf{a}_0, \dots, \mathbf{a}_r \in \mathrm{R}^d$ be defined by $\mathbf{a}_0 := \mathbf{v}_1$, $\mathbf{a}_1 = \mathbf{v}_1'$ and $\mathbf{a}_i = \mathbf{v}_i$ for $2 \le i \le r$. One finds that

$$\hat{\mathbf{a}}^0 = \mathbf{v}_1' \wedge \cdots \wedge \mathbf{v}_r, \quad \hat{\mathbf{a}}^1 = -\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_r$$

and thus (3.2.5) reads

$$\Phi(\mathbf{v}_1' \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_r) - \Phi(\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_r) + \sum_{r=2}^{r} \Phi(\hat{\mathbf{a}}^i) = \Phi(\boldsymbol{\gamma}) \tag{3.2.7}$$

where

$$\boldsymbol{\gamma} = \mathbf{v}_1' \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_r - \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_r + \sum_{r=2}^{r} \hat{\mathbf{a}}^i.$$

Let an $r+1$ tuple of vectors $\mathbf{b}_0, \dots, \mathbf{b}_r \in \mathrm{R}^d$ be defined by $\mathbf{b}_0 := \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_1')$, $\mathbf{b}_1 = \mathbf{v}_1'$ and $\mathbf{b}_i = \mathbf{v}_i$ for $2 \le i \le r$. One finds that

$$\hat{\mathbf{b}}^0 = \mathbf{v}_1' \wedge \cdots \wedge \mathbf{v}_r, \quad \hat{\mathbf{b}}^1 = -\tfrac{1}{2}(\mathbf{v}_1 + \mathbf{v}_1') \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_r,$$

$$\hat{\mathbf{b}}^i = \tfrac{1}{2}\hat{\mathbf{a}}^i \quad \text{if } 2 \le i \le r, \qquad \sum_{i=0}^{r} \hat{\mathbf{b}}^i = \tfrac{1}{2}\boldsymbol{\gamma}.$$

Thus (3.2.5) reads

$$\Phi(\mathbf{v}_1' \wedge \cdots \wedge \mathbf{v}_r) - \tfrac{1}{2}\Phi((\mathbf{v}_1 + \mathbf{v}_1') \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_r) + \tfrac{1}{2} \sum_{r=2}^{r} \Phi(\hat{\mathbf{a}}^i) = \tfrac{1}{2}\Phi(\boldsymbol{\gamma})$$

Multiplying by 2 and subtracting from (3.2.7) one obtains (3.2.6). □

Proposition 3.2.5. *Let* Φ *be a parametric integrand of degree* r *with* $\operatorname{dom} \Phi = \operatorname{si} \wedge_r \mathrm{R}^d$. *Then*

(i) Φ *is a semielliptic null lagrangian if and only if*

$$\Phi(\boldsymbol{\alpha}) = \langle \boldsymbol{\alpha}, \boldsymbol{\omega} \rangle$$

for any $\boldsymbol{\alpha} \in \operatorname{si} \wedge_r \mathrm{R}^d$ *and some* $\boldsymbol{\omega} \in \wedge^r \mathrm{R}^d$.

(ii) Φ *is semielliptic polyconvex if and only if* Φ *has a convex extension to* $\wedge_r \mathrm{R}^d$.

Proof (i): If Φ is a semielliptic null lagrangian, then the assumed continuity and $\operatorname{dom} \Phi = \operatorname{si} \wedge_r \mathrm{R}^d$ imply that (3.2.3) holds with the equality sign for any $r+1$ tupe of vectors $\mathbf{a}_0, \dots, \mathbf{a}_r \in \mathrm{R}^d$. Assuming thst $\mathbf{a}_0 = -\mathbf{a}_1$ we then obtain

$$\Phi(\mathbf{0}) = \Phi(\boldsymbol{\alpha}) + \Phi(-\boldsymbol{\alpha})$$

where $\boldsymbol{\alpha} = \prod_{i=1}^{r} \mathbf{a}_i$. It follows that $\Phi(-\boldsymbol{\alpha}) = -\Phi(\boldsymbol{\alpha})$ for each simple r vector $\boldsymbol{\alpha}$. Thus $\Phi(t\boldsymbol{\alpha}) = t\Phi(\boldsymbol{\alpha})$ for each simple r vector $\boldsymbol{\alpha}$ and each $t \in \mathrm{R}$. Furthermore, we have (3.2.3) with the equality sign and thus the restriction of Φ to the set of all simple r vectors satisfies the hypotheses of Proposition 3.2.4 and (i) follows. (ii): Follows immediately from (i). □

3.3 Standard integrands and degree r quasiconvexity

Throughout this section, let m, n, r be integers with m, n positive and $0 \le r \le n$. We denote by $\mathrm{H}(m,n,r)$ the set of all pairs $(\mathbb{F}, \boldsymbol{\alpha})$ where $\boldsymbol{\alpha}$ is a simple unit r vector in R^n and $\mathbb{F} \in \mathrm{Lin}(\mathrm{R}^n, \mathrm{R}^m)$ is such that $\mathbb{F}\mathbb{m} = \mathbf{0}$ for each $\mathbb{m} \in \mathrm{R}^n$ that is orthogonal to $\boldsymbol{\alpha}$ (the last means $\boldsymbol{\alpha} \llcorner \mathbb{m} = \mathbf{0}$).

Definitions 3.3.1.

(i) An R^m ***valued map on an r current*** is any pair $(T, \mathbf{g})$ consisting of an integral r current T in R^n with compact support and a lipschitzian map $\mathbf{g}$ with $\operatorname{spt} T \subset \operatorname{dom} \mathbf{g} \subset \mathrm{R}^n$ and $\operatorname{ran} \mathbf{g} \subset \mathrm{R}^m$.

(ii) A R^m valued map on an r current $(S, \mathbf{h})$ is said to be ***affine*** if S represents a planar region and $\mathbf{h}$ has an affine extension to a map from R^n to R^m.

(iii) We say that two R^m valued maps on r currents $(T, \mathbf{g})$ and $(S, \mathbf{h})$ ***have matching boundaries*** if

$$\partial T = \partial S,$$
$$\mathbf{g} = \mathbf{h} \quad \text{on} \quad \operatorname{spt} \partial S.$$

Definition 3.3.2.

(i) A function $\hat{\mathbb{f}} : \mathrm{H}(m,n,r) \to \mathrm{R} \cup \{\infty\}$ is said to be a ***standard integrand*** of degree r if $\hat{\mathbb{f}}$ is borelian and locally bounded from below.

(ii) If $(T, \mathbf{g})$ be a R^m valued map on an r current and $\hat{\mathbb{f}} : \mathrm{H}(m,n,r) \to \mathrm{R} \cup \{\infty\}$ a standard integrand we put

$$I(\hat{\mathbb{f}}; T, \mathbf{g}) = \int_M \hat{\mathbb{f}}(\nabla \mathbf{g}, \boldsymbol{\alpha}/|\boldsymbol{\alpha}|)|\boldsymbol{\alpha}| \, d\mathcal{H}^r \tag{3.3.1}$$

where we assume that T has the representation (3.1.3) with M, $\boldsymbol{\alpha}$ as in (3.1.2) and $\nabla \mathbf{g}(\mathbf{x})$ is the approximate derivative of $\mathbf{g}$ which exists at $\mathcal{H}^r$ a.e. $\mathbf{x} \in M$. The value (3.3.1) is in $\mathrm{R} \cup \{\infty\}$ and is independent of the particular representation of T.

Definitions 3.3.3. A standard integrand $\hat{\mathbb{f}} : \mathrm{H}(m,n,r) \to \mathrm{R} \cup \{\infty\}$ is said to be

(i) ***degree r quasiconvex*** if

$$I(\hat{\mathbb{f}}; T, \mathbf{g}) \ge I(\hat{\mathbb{f}}; S, \mathbf{h})$$

whenever $(T, \mathbf{g})$ and $(S, \mathbf{h})$ are R^m valued maps on r currents with matching boundaries and $(S, \mathbf{h})$ is affine;

(ii) a ***degree r null lagrangian*** if $\operatorname{dom} \hat{\mathbb{f}} = \mathrm{H}(m,n,r)$, $\operatorname{ran} \hat{\mathbb{f}} \subset \mathrm{R}$, $\hat{\mathbb{f}}$ is continuous, and $\hat{\mathbb{f}}$ and $-\hat{\mathbb{f}}$ are degree r quasiconvex;

(iii) ***degree r polyconvex*** if it is the supremum of some family of degree r null lagrangians.

The degree r quasiconvexity involves a variation of the domain of integration except the case $r = n$. In the particular case $r = n$ the domain remains the same and in fact the degree n quasiconvexity, degree n null lagrangians and degree n polyconvex functions are related to the standard counterparts of these notions, which we first define and then explain the relationships.

Definitions 3.3.4. A borelian function $f : \mathrm{Lin}(\mathrm{R}^n, \mathrm{R}^m) \to \mathrm{R} \cup \{\infty\}$ which is locally bounded from below is said to be

(i) ***quasiconvex in the standard sense*** if

$$\int_E f(\nabla \mathbf{y})\, d\mathcal{L}^n \geq \mathcal{L}^n(E) f(\mathbf{F})$$

for every $\mathbf{F} \in \mathrm{Lin}(\mathrm{R}^n, \mathrm{R}^m)$, for every bounded open subset E of R^n with $\mathcal{L}^n(\mathrm{bd}\, E) = 0$, and for every $\mathbf{y} \in W^{1,\infty}(E, \mathrm{R}^m)$ such that $\mathbf{y}(\mathbf{x}) = \mathbf{F}\mathbf{x}$ on $\mathrm{bd}\, E$;

(ii) ***null lagrangian in the standard sense*** if f is continuous, its range is R and both f and $-f$ are quasiconvex in the standard sense;

(iii) ***polyconvex in the standard sense*** if it is the supremum of some family of null lagrangian in the standard sense.

Remark 3.3.5 (Degree n quasiconvexity). Since there are only two unit n vectors in R^n, viz., $\boldsymbol{\alpha} = \mathbf{A}_n$ and $\boldsymbol{\alpha} = -\mathbf{A}_n$ where

$$\mathbf{A}_n := \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n \tag{3.3.2}$$

is the standard orientation of R^n, we have

$$\mathrm{H}(m,n,n) = \{(\mathbf{F}, \mathbf{A}_n) : \mathbf{F} \in \mathrm{Lin}(\mathrm{R}^n, \mathrm{R}^m)\} \cup \{(\mathbf{F}, -\mathbf{A}_n) : \mathbf{F} \in \mathrm{Lin}(\mathrm{R}^n, \mathrm{R}^m)\}.$$

Thus each function $\hat{\mathbb{f}} : \mathrm{H}(m,n,n) \to \mathrm{R} \cup \{\infty\}$ can be identified with the pair $(\hat{f}_+, \hat{f}_-)$ where $\hat{f}_\pm : \mathrm{Lin}(\mathrm{R}^n, \mathrm{R}^m) \to \mathrm{R} \cup \{\infty\}$ via

$$\hat{\mathbb{f}}(\mathbf{F}, \pm \mathbf{A}_n) = \hat{f}_\pm(\mathbf{F}), \tag{3.3.3}$$

$\mathbf{F} \in \mathrm{Lin}(\mathrm{R}^n, \mathrm{R}^m)$. *We have the following assertions:*

(i) *If $\hat{\mathbb{f}}$ is degree n quasiconvex then the functions $\hat{f}_\pm$ are quasiconvex in the standard sense; if $\hat{\mathbb{f}}$ is finite valued, then also the converse is true;*

(ii) *$\hat{\mathbb{f}}$ is a degree n null lagrangian if and only if $\hat{f}_\pm$ are null lagrangians in the standard sense;*

(iii) *$\hat{\mathbb{f}}$ is degree n polyconvex if and only if the functions $\hat{f}_\pm$ are polyconvex in the standard sense.*

Proof (i): Each integral n current S in R^n representing a planar region is of the form

$$T = \mathbf{A}_n \mathcal{L}^n \llcorner E \qquad \text{or} \qquad T = -\mathbf{A}_n \mathcal{L}^n \llcorner E$$

where $E \subset \mathrm{R}^n$ is a bounded set of finite perimeter, and necessarily if T is an integral n current with $\partial T = \partial S$ then $T = S$. This follows from the fact that $T = m\mathbf{A}_n \mathcal{L}^n$ where m in a Z valued integrable function, and the condition $\partial T = \partial S$ means that $\nabla m = \nabla 1_E$ in the weak sense and hence $m - 1_E$ is constant, and as m has to be integrable, necessarily $m = 1_E$. If $(T, \mathbf{g})$ and $(S, \mathbf{h})$ are R^m valued maps on these n currents, we have $\mathbf{g} : E \to \mathrm{R}^m$, $\mathbf{h} : E \to \mathrm{R}^m$. Assuming that $(S, \mathbf{h})$ is affine, we have $\nabla \mathbf{h} = \mathbf{F}$ constant on E and thus the degree n quasiconvexity condition gives that if $(T, \mathbf{g})$ and $(S, \mathbf{h})$ have matching boundaries, then

$$\int_E f_\pm(\nabla \mathbf{g})\, d\mathcal{L}^n \geq \mathcal{L}^n(E) f_\pm(\mathbf{F}). \tag{3.3.4}$$

This has to hold for every set E of finite perimeter, every lipschitzian $\mathbf{g} : E \to \mathrm{R}^m$, and every $\mathbf{F}$ such that $\mathbf{g}(\mathbf{x}) - \mathbf{F}\mathbf{x}$ is constant on $\operatorname{spt} \partial S$, and the latter set contains $\operatorname{bd}_* E$. In particular, this must hold for each ball. An argument of Ball & Murat [5] then shows that the equality actually holds for all $\mathbf{F}, E, \mathbf{y}$ as in the definition of the standard quasiconvexity condition.

To prove the converse statement, we assume that $\hat{\mathbb{f}}$ is finite valued and that $f_\pm$ are quasiconvex in the standard sense. Let E be a bounded set of finite perimeter in R^n, let $W = \operatorname{spt} \mathbf{A}_n \mathcal{L}^n \llcorner E$ and let $\mathbf{g} : W \to \mathrm{R}^m$ be a lipschitzian map such that $\mathbf{g}(\mathbf{x}) = \mathbf{F}\mathbf{x}$ on $\operatorname{spt} \partial \mathbf{A}_n \mathcal{L}^n \llcorner E \subset W$ for some $\mathbf{F}$. The goal is to prove (3.3.4). Let $\tilde{\mathbf{f}} : \mathrm{R}^n \to \mathrm{R}^m$ be defined by

$$\tilde{\mathbf{f}}(\mathbf{x}) = \begin{cases} \mathbf{g}(\mathbf{x}) & \text{if } \mathbf{x} \in W, \\ \mathbf{F}\mathbf{x} & \text{otherwise,} \end{cases}$$

$\mathbf{x} \in \mathrm{R}^n$. This is a lipschitzian continuous function and if B is any open ball in R^n which contains W then the standard quasiconvexity of $f_\pm$ gives

$$\int_B f_\pm(\nabla \tilde{\mathbf{f}})\, d\mathcal{L}^n \geq \mathcal{L}^n(B) f_\pm(\mathbf{F}). \tag{3.3.5}$$

For $\mathcal{L}^n$ a.e. point of $B \sim W$ we have $\nabla \tilde{\mathbf{f}} = \mathbf{F}$ and for $\mathcal{L}^n$ a.e. point of W we have $\nabla \tilde{\mathbf{f}} = \nabla \mathbf{g}$ and thus

$$\int_B f_\pm(\nabla \tilde{\mathbf{f}})\, d\mathcal{L}^n = \mathcal{L}^n(B \sim W) f_\pm(\mathbf{F}) + \int_W f_\pm(\nabla \mathbf{g})\, d\mathcal{L}^n. \tag{3.3.6}$$

Using $f_\pm(\mathbf{F}) \in \mathrm{R}$ we deduce that (3.3.5) and (3.3.6) give (3.3.4).

(ii): Since the degree n null lagrangians are finite valued by definition, we see that (i) immediately gives the assertion.

(iii): This follows from (ii) and the definition of degree n polyconvex and standard polyconvex functions. □

Proposition 3.3.6. *If $\hat{\mathbb{f}}$ is a standard integrand which is the supremum of some family of degree r quasiconvex functions then $\hat{\mathbb{f}}$ is degree r quasiconvex. In particular, each degree r polyconvex function is degree r quasiconvex.*

3.4 Graphs of maps on rectifiable currents

Throughout this section, let m, n, r be integers with m, n positive and $0 \leq r \leq n$. Recall that the graph of a map $f : M \to N$ is the set

$$\operatorname{graph} f = \{(x, f(x)) : x \in M\} \subset M \times N$$

and the graph map of f is $(\mathrm{id} \times f)|M : M \to M \times N$ given by

$$(\mathrm{id} \times f)|M(x) = (x, f(x)),$$

$x \in M$.

Introduce the linear maps $\mathbf{C}$, $\mathbf{D}$, $\mathbf{P}$, $\mathbf{Q}$ by

$$\mathbf{C}\mathbf{x} = (\mathbf{x}, \mathbf{0}), \quad \mathbf{D}\mathbf{y} = (\mathbf{0}, \mathbf{y}), \quad \mathbf{P}(\mathbf{x}, \mathbf{y}) = \mathbf{x}, \quad \mathbf{Q}(\mathbf{x}, \mathbf{y}) = \mathbf{y}$$

for all $\mathbf{x} \in \mathrm{R}^n$ and all $\mathbf{y} \in \mathrm{R}^m$. We note that for each r,

$$\wedge_r \mathrm{R}^{m+n} = \bigoplus_{k=0}^{s} \operatorname{ran} \wedge_{r-k}\mathbf{C} \wedge \operatorname{ran} \wedge_k \mathbf{D}$$

where

$$s = \min\{m, r\}.$$

The vector $\boldsymbol{\gamma} \in \wedge_r \mathrm{R}^{m+n}$ is said to be ***vertical*** if $\wedge_r \mathbf{P}\,\boldsymbol{\gamma} = \mathbf{0}$.

Proposition 3.4.1. *Let* $\mathfrak{L}_r : \mathrm{H}(m,n,r) \to \wedge_r \mathrm{R}^{m+n}$ *be defined by*

$$\mathfrak{L}_r(\mathbb{F}, \boldsymbol{\alpha}) = \wedge_r(\mathbf{C} + \mathbf{D}\mathbb{F})\boldsymbol{\alpha} / |\wedge_r (\mathbf{C} + \mathbf{D}\mathbb{F})\boldsymbol{\alpha}|$$

for each $(\mathbb{F}, \boldsymbol{\alpha}) \in \mathrm{H}(m,n,r)$; *it will be proved that the denominator is different from* 0. *Then*

(i) $\mathfrak{L}_r$ *maps* $\mathrm{H}(m,n,r)$ *bijectively onto the set*

$$\mathfrak{H}_r = \{\boldsymbol{\gamma} \in \wedge_r \mathrm{R}^{m+n} : \boldsymbol{\gamma} \text{ is simple, nonvertical, and of unit length}\};$$

(ii) *if we put*

$$\mathfrak{P}_r = \{\wedge_r(\mathbf{C} + \mathbf{D}\mathbb{F})\boldsymbol{\alpha} : (\mathbb{F}, \boldsymbol{\alpha}) \in \mathrm{H}(m,n,r)\},$$
$$\mathfrak{B}_r := \operatorname{co}\{\boldsymbol{\alpha} \in \operatorname{si} \wedge_r \mathrm{R}^n : |\boldsymbol{\alpha}| \le 1\}$$

then

$$\operatorname{span} \mathfrak{P}_r = \wedge_r \mathrm{R}^{m+n}, \tag{3.4.1}$$

$$\operatorname{cl}\operatorname{co} \mathfrak{P}_r = \{\boldsymbol{\gamma} \in \wedge_r \mathrm{R}^{m+n} : \wedge_r \mathbf{P}\,\boldsymbol{\gamma} \in \mathfrak{B}_r\}; \tag{3.4.2}$$

(iii) *if we put*

$$Z_r = \prod_{k=0}^{s} \operatorname{Lin}(\wedge^{r-k}\mathrm{R}^n, \wedge_k \mathrm{R}^m)$$

then there exists a unique linear map $\mathfrak{M}_r : Z_r \to \wedge_r \mathrm{R}^{m+n}$ *such that*

$$\mathfrak{M}_r(\wedge_0 \mathbb{F} \,\lrcorner\, \boldsymbol{\alpha}, \dots, \wedge_r \mathbb{F} \,\lrcorner\, \boldsymbol{\alpha}) = \wedge_r(\mathbf{C} + \mathbf{D}\mathbb{F})\boldsymbol{\alpha} \tag{3.4.3}$$

for each $(\mathbb{F}, \boldsymbol{\alpha}) \in \mathrm{H}(m,n,r)$; *the map* $\mathfrak{M}_r$ *maps* Z_r *bijectively onto* $\wedge_r \mathrm{R}^{m+n}$.

Given $(\mathbb{F}, \boldsymbol{\alpha}) \in \mathrm{H}(m,n,r)$, the simple r vector $\mathfrak{L}_r(\mathbb{F}, \boldsymbol{\alpha})$ is the unit r vector tangent to the r dimensional graph of $\mathbb{F}|U$ in R^{m+n} where $U \subset \mathrm{R}^n$ is the r dimensional subspace tangent to $\boldsymbol{\alpha}$.

Proof (i): From $\wedge_r \mathbf{P} \wedge_r (\mathbf{C} + \mathbf{D}\mathbb{F})\boldsymbol{\alpha} = \boldsymbol{\alpha}$ one deduces that $\wedge_r(\mathbf{C} + \mathbf{D}\mathbb{F})\boldsymbol{\alpha}$ is nonzero, nonvertical, and simple. Thus $\operatorname{ran} \mathfrak{L}_r \subset \mathfrak{H}_r$.

Conversely, let $\boldsymbol{\gamma} \in \mathfrak{H}_r$ and let V be the r dimensional subspace of R^{m+n} associated with $\boldsymbol{\gamma}$. Since $\boldsymbol{\gamma}$ is nonvertical, V is the graph of some linear map $\mathbb{F}_0 : U \to \mathrm{R}^m$ where $\mathbb{F}_0$ and U are uniquely determined by V. Extending $\mathbb{F}_0$ to a linear map $\mathbb{F} : \mathrm{R}^n \to \mathrm{R}^m$ such that $\mathbb{F}$ vanishes on $U^\perp$ and denoting $\boldsymbol{\alpha} := \wedge_r \mathbf{P}\,\boldsymbol{\gamma} / |\wedge_r \mathbf{P}\,\boldsymbol{\gamma}|$ we find that $(\mathbb{F}, \boldsymbol{\alpha}) \in \mathrm{H}(m,n,r)$ and $\mathfrak{L}_r(\mathbb{F}, \boldsymbol{\alpha}) = \boldsymbol{\gamma}$.

To prove the injectivity of $\mathfrak{L}_r$, let $\mathfrak{L}_r(\mathbb{F}, \boldsymbol{\alpha}) = \mathfrak{L}_r(\mathbb{G}, \boldsymbol{\beta})$. Letting T and U be the r dimensional subspaces of R^n tangent to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively, we deduce from the above intepretation that the maps $\mathbb{F}|T$ and $\mathbb{G}|U$ have the same graphs. It follows that $\mathbb{F}|T = \mathbb{G}|U$, $T = U$. Since $\mathbb{F}$ and $\mathbb{G}$ vanish on $T^\perp$ and $U^\perp$, respectively, we deduce that $\mathbb{F} = \mathbb{G}$. From $T = U$ then $\boldsymbol{\alpha} = \pm\boldsymbol{\beta}$ and returning to the original equality $\mathfrak{L}_r(\mathbb{F}, \boldsymbol{\alpha}) = \mathfrak{L}_r(\mathbb{G}, \boldsymbol{\beta})$ with $\mathbb{F} = \mathbb{G}$ we obtain $\boldsymbol{\alpha} = \boldsymbol{\beta}$. Thus $\mathfrak{L}_r$ is injective and (i) is proved.

(ii): Equation (3.4.1): Let us first show that any simple vertical vector belongs to span $\mathfrak{P}_r$. Let $\boldsymbol{\gamma} = \prod_{i=1}^r \mathbf{c}_i$ be a simple vertival vector so that the space

$$V = \operatorname{span}\left\{\mathbf{P}\mathbf{c}_i : i = 1, \dots, r\right\}$$

has dimension $d < r$. Assume that the vectors are enumerated so that the system $\mathfrak{S} := \left\{\mathbf{P}\mathbf{c}_i : i = 1, \dots, d\right\}$ is a basis of V. Let furthermore $\mathfrak{T} := \left\{\mathbf{v}_{d+1}, \dots, \mathbf{v}_r\right\} \subset \mathrm{R}^n$ be a system such that $\mathfrak{S} \cup \mathfrak{T}$ is a basis of V. Consider the system of vectors

$$\boldsymbol{\gamma}_\omega := \prod_{i=1}^{d} \mathbf{c}_i \wedge \prod_{i=d+1}^{r} (\mathbf{c}_i + \omega_i \mathbf{C}\mathbf{v}_i)$$

where $\omega = (\omega_{d+1}, \dots, \omega_r)$ and $\omega_i \in \{-1, 1\}$. Letting $\mathfrak{U}$ denote the system of all ω just described, we see that each $\boldsymbol{\gamma}_\omega$ is a nonvertical vector and

$$\boldsymbol{\gamma} = 2^{d-r} \sum_{\omega \in \mathfrak{U}} \boldsymbol{\gamma}_\omega.$$

By (i), each $\boldsymbol{\gamma}_\omega$ is in span $\mathfrak{P}_r$ and hence also $\boldsymbol{\gamma}$ is in span $\mathfrak{P}_r$, which then contains all simple r vectors. Since each r vector is a linear combination of simple r vectors, the proof of (3.4.1) is complete.

Equation (3.4.2): Let f be a linear function on $\wedge_r \mathrm{R}^{m+n}$ and let c be a constant such that $f(\boldsymbol{\gamma}) \le c$ for all $\boldsymbol{\gamma} \in \mathfrak{P}_r$. Using the definition of $\mathfrak{L}_r$ this means

$$f(\wedge_r(\mathbf{C} + \mathbf{D}\mathbb{F})\boldsymbol{\alpha}) \le c \tag{3.4.4}$$

for each $(\mathbb{F}, \boldsymbol{\alpha}) \in \mathrm{H}(m, n, r)$. Expanding by the binomial theorem (A.2.2) we obtain

$$\sum_{i=0}^{r} f(\wedge_i \mathbf{C} \wedge \wedge_{r-i}(\mathbf{D}\mathbb{F})\boldsymbol{\alpha}) \le c. \tag{3.4.5}$$

We now replace $\mathbb{F}$ by $t\mathbb{F}$, divide the last inequality by t^r and let $t \to \infty$ to obtain

$$f(\wedge_r(\mathbf{D}\mathbb{F})\boldsymbol{\alpha}) \le 0; \tag{3.4.6}$$

replacing $\boldsymbol{\alpha}$ by $-\boldsymbol{\alpha}$ we obtain the opposite inequality; thus we have the equality sign in (3.4.6) and (3.4.5) reduces to

$$\sum_{i=0}^{r-1} f(\wedge_i \mathbf{C} \wedge \wedge_{r-i}(\mathbf{D}\mathbb{F})\boldsymbol{\alpha}) \le c.$$

Proceeding by induction, we obtain

$$f(\wedge_i \mathbf{C} \wedge \wedge_{r-i}(\mathbf{D}\mathbb{F})\boldsymbol{\alpha}) = 0 \quad i = 0, \dots, r-1, \quad f(\wedge_r \mathbf{C}\boldsymbol{\alpha}) \le c.$$

This gives also

$$f(\boldsymbol{\gamma}) = f(\wedge_r \mathbf{C} \wedge_r \mathbf{P}\,\boldsymbol{\gamma}) \tag{3.4.7}$$

for every r vector $\boldsymbol{\gamma}$ of the form $\boldsymbol{\gamma} = \wedge_r(\mathbf{C} + \mathbf{D}\mathbb{F})\boldsymbol{\alpha}$ where $(\mathbb{F}, \boldsymbol{\alpha}) \in \mathrm{H}(m, n, r)$. By linearity (3.4.7) must hold for every $\boldsymbol{\delta}$ from the span of the vectors $\boldsymbol{\gamma}$, which by (3.4.1) gives that (3.4.7) holds for every $\boldsymbol{\gamma} \in \wedge_r \mathrm{R}^{m+n}$.

Thus each linear function f satisfying (3.4.4) for some c and all $(\mathbb{F}, \boldsymbol{\alpha}) \in \mathrm{H}(m, n, r)$ satisfies (3.4.7) and hence vanishes on all $\boldsymbol{\gamma} \in \wedge_r \mathrm{R}^{m+n}$ with $\wedge_r \mathbf{P}\,\boldsymbol{\gamma} = \mathbf{0}$.

Since $\operatorname{cl}\operatorname{co}\mathfrak{P}_r$ is the intersection of all closed halfspaces containing $\mathfrak{P}_r$, we deduce that

$$\operatorname{cl}\operatorname{co}\mathfrak{P}_r = \{\boldsymbol{\gamma} \in \wedge_r \mathrm{R}^{m+n} : \wedge_r \mathbf{P}\,\boldsymbol{\gamma} \in \mathfrak{C}\} \tag{3.4.8}$$

for some subset $\mathfrak{C}$ of $\wedge_r \mathrm{R}^n$. One easily finds that $\mathfrak{C}$ must contain $\mathfrak{B}_r$ and as the choice $\mathfrak{C} = \mathfrak{B}_r$ already gives a convex set on the right hand side of (3.4.8), we have (3.4.2).

(iii): Define $\mathfrak{M}_r : Z_r \to \wedge_r \mathrm{R}^{m+n}$ by

$$\mathfrak{M}_r(\boldsymbol{\Theta}_0, \dots, \boldsymbol{\Theta}_{n-1}) = \sum_{k=0}^{s} (-1)^{(r-k)k} \sum_{I \in \mathcal{I}} \wedge_{r-k}\mathbf{C}\mathbf{e}_I \wedge \wedge_k \mathbf{D}\,\boldsymbol{\Theta}_k\,\mathbf{e}^I \tag{3.4.9}$$

for each $(\boldsymbol{\Theta}_0, \dots, \boldsymbol{\Theta}_s) \in Z_r$ where $\{\mathbf{e}_I : I \in \mathcal{I}\}$ is any orthonormal basis in $\wedge_{r-k}\mathrm{R}^n$ and $\{\mathbf{e}^I : I \in \mathcal{I}\}$ is the dual basis in $\wedge^{r-k}\mathrm{R}^n$. Using the binomial theorem (A.2.2) one finds that (3.4.3) is satisfied, which proves the existence of $\mathfrak{M}_r$. The uniqueness follows from the existence and (3.4.9). Equation (3.4.9) also easily yields the bijectivity. □

Definition 3.4.2. If T is a rectifiable r dimensional current of the form (3.1.3) with M, $\boldsymbol{\alpha}$ as in (3.1.2) and $\boldsymbol{\theta} : \operatorname{dom}\boldsymbol{\theta} \to \mathrm{R}^m$ is a lipschitzian map with $\operatorname{spt} T \subset \operatorname{dom}\boldsymbol{\theta} \subset \mathrm{R}^n$, then by Proposition 3.1.4,

$$\mathfrak{graph}(T, \boldsymbol{\theta}) := (\mathrm{id} \times \boldsymbol{\theta})|M_{\#}\, T$$

is a rectifiable r dimensional current in R^{m+n}; we call $\mathfrak{graph}(T, \boldsymbol{\theta})$ the ***graph of*** T ***under*** $\boldsymbol{\theta}$. If T is an integral current then the general formula $\partial \boldsymbol{\phi}_{\#} T = \boldsymbol{\phi}_{\#}\partial T$ gives

$$\partial\,\mathfrak{graph}(T, \boldsymbol{\theta}) = \mathfrak{graph}(\partial T, \boldsymbol{\theta}). \tag{3.4.10}$$

One finds that if $\boldsymbol{\omega} \in \mathcal{D}^r(\Omega \times \mathrm{R}^m)$ is of the form

$$\boldsymbol{\omega} = \varphi \mathbf{P}^{\#}\boldsymbol{\sigma} \wedge \mathbf{Q}^{\#}\boldsymbol{\tau} \tag{3.4.11}$$

where $\varphi \in C_0^\infty(\Omega \times \mathrm{R}^m, \mathrm{R})$, $0 \le k \le r$, $\boldsymbol{\sigma} \in \wedge^{r-k}\mathrm{R}^n$, and $\boldsymbol{\tau} \in \wedge^k \mathrm{R}^m$ then

$$\langle \mathfrak{graph}(T, \boldsymbol{\theta}), \boldsymbol{\omega} \rangle = (-1)^{k(r-k)} \int_M \varphi \circ (\mathrm{id} \times \boldsymbol{\theta})|M \, \langle \wedge_k \nabla \boldsymbol{\theta} \,\lrcorner\, \boldsymbol{\alpha}\,\boldsymbol{\sigma}, \boldsymbol{\tau} \rangle \, d\mathcal{H}^r. \tag{3.4.12}$$

We now establish a correspondence between standard integrands $\hat{\mathbb{f}}$ and parametric integrands Φ. If $\hat{\mathbb{f}} : \mathrm{H}(m, n, r) \to \mathrm{R} \cup \{\infty\}$ is a standard integrand and $\Phi : \operatorname{dom}\Phi \to \mathrm{R} \cup \{\infty\}$ a parametric integrand we say that $\hat{\mathbb{f}}$ and Φ are related to each other if $\operatorname{dom}\Phi \supset \tilde{\mathfrak{H}}_r := \{t\,\boldsymbol{\gamma} : t \ge 0,\ \boldsymbol{\gamma} \in \mathfrak{H}_r\}$ and

$$I(\hat{\mathbb{f}}; T, \boldsymbol{\theta}) = \langle \mathfrak{graph}(T, \boldsymbol{\theta}), \Phi \rangle \tag{3.4.13}$$

whenever $\boldsymbol{\theta}$ is a R^m valued map on an r current T in R^n. It turns out that $\hat{\mathbb{f}}$ and Φ are related to each other if and only if $\operatorname{dom}\Phi \supset \tilde{\mathfrak{H}}_r$ and

$$\Phi(t \wedge_r (\mathbf{C} + \mathbf{D}\,\mathbb{F})\boldsymbol{\alpha}) = t\,\hat{\mathbb{f}}(\mathbb{F}, \boldsymbol{\alpha}) \tag{3.4.14}$$

for every $(\mathbb{F}, \boldsymbol{\alpha}) \in \mathrm{H}(m, n, r)$ and $t \ge 0$. Indeed, in view of the positive homogeneity of Φ we obtain that (3.4.14) with arbitrary $t > 0$ is equivalent to (3.4.14) with $t = 1/|\wedge_r (\mathbf{C} + \mathbf{D}\,\mathbb{F})\boldsymbol{\alpha}|$ and the necessity and sufficiency of the last special case of (3.4.14) for the validity of (3.4.13) follows from the substitution formulas

$$\mathfrak{graph}(T, \boldsymbol{\theta}) = \boldsymbol{\gamma}\, \mathcal{H}^r \llcorner \operatorname{graph} \boldsymbol{\theta}$$

where

$$\boldsymbol{\gamma} \circ \boldsymbol{\phi} = \wedge_r (\mathbf{C} + \mathbf{D}\, \nabla \boldsymbol{\theta}) \boldsymbol{\alpha} / | \wedge_r (\mathbf{C} + \mathbf{D}\, \nabla \boldsymbol{\theta})|,$$

$$\boldsymbol{\phi} = (\mathrm{id} \times \boldsymbol{\theta})|M,$$

and

$$\langle \mathfrak{graph}(T, \boldsymbol{\theta}), \Phi \rangle = \int_{\operatorname{graph} \boldsymbol{\theta}} \Phi(\boldsymbol{\gamma})\, d\mathcal{H}^r = \int_M \Phi(\boldsymbol{\gamma} \circ \boldsymbol{\phi}) | \wedge_r (\mathbf{C} + \mathbf{D}\, \nabla \boldsymbol{\theta})|\, d\mathcal{H}^r.$$

From (3.4.14) we see that there is a one to one correspondence between standard integrands $\hat{\mathfrak{f}}$ and parametric integrands Φ *with* $\operatorname{dom} \Phi = \tilde{\mathfrak{S}}_r$, i.e., with parametric integrands defined only on nonvertical vectors. The values of Φ on vertical vectors is undetermined, but it must be born in mind that the set of nonvertical vectors is an open dense set in si $\wedge_r \mathrm{R}^{m+n}$.

We have the following relations between the convexity properties of standard and parametric integrands.

Proposition 3.4.3. *Let $\hat{\mathfrak{f}}$ and Φ be a standard and a parametric integrand, respectively, and assume that $\hat{\mathfrak{f}}$ and Φ are related to each other. Then*

(i) *If Φ is semielliptic then $\hat{\mathfrak{f}}$ is degree r quasiconvex;*

(ii) *Φ is a semielliptic null lagrangian if and only if $\hat{\mathfrak{f}}$ is degree r null lagrangian;*

(iii) *Φ is semielliptic polyconvex if and only if $\hat{\mathfrak{f}}$ is degree r is degree r polyconvex.*

Proof In this section we prove only (i); the proof of (ii) and (iii) is given in Section 3.5 (below). Thus assume that Φ is semielliptic, let $(T, \mathbf{g})$ and $(S, \mathbf{h})$ are R^m valued maps on r currents with matching boundaries and $(S, \mathbf{h})$ is affine. It then follows that $\partial\, \mathfrak{graph}(T, \mathbf{g}) = \partial\, \mathfrak{graph}(S, \mathbf{h})$ and $\mathfrak{graph}(S, \mathbf{h})$ represents a planar region. Thus the semiellipticity gives

$$\langle \mathfrak{graph}(T, \mathbf{g}), \Phi \rangle \ge \langle \mathfrak{graph}(S, \mathbf{h}), \Phi \rangle$$

which by (3.4.13) reads

$$I(\hat{\mathfrak{f}}; T, \mathbf{g}) \ge I(\hat{\mathfrak{f}}; S, \mathbf{h}). \quad \square$$

3.5 Degree r null lagrangians and degree r polyconvexity

In this section we prove the form of the degree r null lagrangians. The proofs are based on the very simple form of the semielliptic null lagrangians stated in Section 3.2 and on the fact that when lifted to graphs, a degree r null lagrangian becomes a semielliptic null lagrangian.

We first show that nondegenerate simplexes with nonvertical tangent vectors in R^{m+n} are graphs of affine maps on nondegenerate simplexes of the same dimension in R^n. We say that an p current T in R^d represents a nondegenerate p simplex if $\operatorname{spt} T$ is a nondegenerate p simplex and $T = \boldsymbol{\gamma} \mathcal{H}^p \llcorner \operatorname{spt} T$ where $\boldsymbol{\gamma}$ is any of the two unit p vectors tangential to $\operatorname{spt} T$.

Lemma 3.5.1. *Let G be an $r+1$ current in* R^{m+n} *representing a nondegenerate* $r+1$ *simplex* Γ *in* R^{m+n} *with tangent vector in* $\mathfrak{H}_{r+1}$. *Then* $G = \mathfrak{graph}(B, \mathbf{g})$ *where* $B = \mathbf{P}_{\#} G$ *and* $\mathbf{g}$ *is a unique affine map on* B.

Proof Let $C \subset \mathrm{R}^{m+n}$ be the $r+1$ dimensional affine subspace of R^{m+n} spanned by the simplex Γ and let $D = \mathbf{P}C$. Since the tangent vector $\boldsymbol{\xi}$ to C is in $\mathfrak{H}_{r+1}$, i.e., $\wedge_{r+1}\mathbf{P}\,\boldsymbol{\xi} \neq \mathbf{0}$, the map $\mathbf{P}|C$ is injective and maps C onto D. Since $\mathbf{P}$ is a projection, the inverse $(\mathbf{P}|C)^{-1}$ is necessarily of the form $(\mathbf{P}|C)^{-1}(\mathbf{x}) = (\mathbf{x}, \mathbf{g}(\mathbf{x}))$ for each $\mathbf{x} \in D$ where $\mathbf{g} : D \to \mathrm{R}^m$ is an affine map. We put $B := \mathbf{P}_{\#} G = (\mathbf{P}|C)_{\#} G$ and hence $(\mathbf{P}|C)_{\#}^{-1} B = A$. On the other hand, denoting by M the support of B, we have $\mathfrak{graph}(B, \mathbf{g}) = (\mathrm{id} \times \mathbf{g})|M_{\#} B = (\mathbf{P}|C)_{\#}^{-1} B$. □

Proposition 3.5.2. *A function* $\hat{\mathbb{f}} : \mathrm{H}(m, n, r) \to \mathrm{R}$ *is a degree* r *null lagrangian if and only if one of the following two conditions holds:*

(i) $r = n$ *and*

$$\hat{\mathbb{f}}(\mathbf{F}, \pm\mathbf{A}_n) = \sum_{k=0}^{\min\{m,n\}} \boldsymbol{\Omega}_k^{\pm} \cdot \wedge_k \mathbf{F} \tag{3.5.1}$$

for all $\mathbf{F} \in \mathrm{Lin}(\mathrm{R}^n, \mathrm{R}^m)$ *and some* $\boldsymbol{\Omega}_k^{\pm} \in \mathrm{Lin}(\wedge_k \mathrm{R}^n, \wedge_k \mathrm{R}^m)$; *here* $\mathbf{A}_n$ *is the standard orientation* (3.3.2)*;*

(ii) $r \le n-1$ *and*

$$\hat{\mathbb{f}}(\mathbb{F}, \boldsymbol{\alpha}) = \sum_{k=0}^{s} \boldsymbol{\Omega}_k \cdot (\wedge_k \mathbb{F} \,\lrcorner\, \boldsymbol{\alpha}) \tag{3.5.2}$$

for every $(\mathbb{F}, \boldsymbol{\alpha}) \in \mathrm{H}(m, n, r)$ *and some* $\boldsymbol{\Omega}_k \in \mathrm{Lin}(\wedge^{r-k}\mathrm{R}^n, \wedge_k \mathrm{R}^m)$.

Thus $\hat{\mathbb{f}}(\mathbb{F}, -\boldsymbol{\alpha}) = -\hat{\mathbb{f}}(\mathbb{F}, \boldsymbol{\alpha})$ if $r \le n-1$ whereas this is not generally true if $r = n$. This difference and more generally the difference in the forms of (3.5.1) and (3.5.2) is related to the fact that unlike that case $r \le n-1$, in the case $r = n$ the quasiconvexity inequality does not involve the variation of the domain of integration, as explained in the proof Remark 3.3.5. These facts also reproduce in the different forms of degree r polyconvex function is Proposition 3.5.3, below.

Proof (i): If $r = n$ then in view of Remark 3.3.5, $\hat{\mathbb{f}}$ can be represented by a pair $\hat{f}_{\pm}$ of functions on $\mathrm{Lin}(\mathrm{R}^n, \mathrm{R}^m)$ via (3.3.3). Then $\hat{\mathbb{f}}$ in a degree n null lagrangian if and only if $\hat{f}_{\pm}$ are standard null lagrangians; the well known result says that the last occurs if and only if $\hat{f}_{\pm}$ are given by the two expressions in (3.5.1).

(ii): Let $r \le n-1$. Assume that $\hat{\mathbb{f}}$ is a degree r null lagrangian and let Φ be the unique parametric integrand with $\mathrm{dom}\,\Phi = \tilde{\mathfrak{H}}_r$ such that $\hat{\mathbb{f}}$ and Φ are related to each other.

Let $\mathbf{a}_0, \dots, \mathbf{a}_r \in \mathrm{R}^{m+n}$ be linearly independent vectors such that $\boldsymbol{\gamma} = \mathbf{a}_0 \wedge \cdots \wedge \mathbf{a}_r$ satisfies $\mathbf{P}_{r+1}\boldsymbol{\gamma} \neq \mathbf{0}$, let $\hat{\mathbf{a}}_i$ be given by (3.2.2) for $i = 0, \dots, r$, and let Γ be the $r+1$ simplex in R^{m+n} with vertices $\mathbf{0}, \mathbf{a}_0, \dots, \mathbf{a}_r$. Then the r vector $-\hat{\mathbf{a}}^i$ is tangent to the face F_i opposite to the vertex $\mathbf{a}_i$ with $\mathcal{H}^r(F_i) = |\hat{\mathbf{a}}^i|$ and the vector $\boldsymbol{\beta} := \sum_{i=0}^r \hat{\mathbf{a}}^i$ is tangent to the face M opposite to $\mathbf{0}$ with $\mathcal{H}^r(M) = |\boldsymbol{\beta}|$. Let $G = \boldsymbol{\gamma}\,\mathcal{H}^{r+1} \llcorner \Gamma$; then

$$\partial G = S' - T' \qquad \text{where} \qquad T' = \sum_{i=0}^{r} T_i', \quad S' = \boldsymbol{\beta}/|\boldsymbol{\beta}|\,\mathcal{H}^r \llcorner M$$

with

$$T_i' = \hat{\mathbf{a}}_i/|\hat{\mathbf{a}}_i|\, \mathcal{H}^r \llcorner F_i, \quad i = 0, \dots, r.$$

The relation $\partial^2 = 0$ gives $\partial T' = \partial S'$. By Lemma 3.5.1 we have $G = \mathfrak{graph}(B, \mathbf{g})$ for an $r+1$ current $B = \mathbf{P}_\# G$ in R^n representing the nondegenerate $r+1$ simplex Δ with vertices $\mathbf{0}, \mathbf{Pa}_0, \dots, \mathbf{Pa}_r$ and for some affine map $\mathbf{g} : \Delta \to \mathrm{R}^m$. Let $T = \mathbf{P}_\# T'$, $S = \mathbf{P}_\# S'$ and observe that $T' = \mathfrak{graph}(T, \mathbf{g})$, $S' = \mathrm{graph}(S, \mathbf{g})$. From $\partial T' = \partial S'$ one finds that the R^m valued maps $(T, \mathbf{g})$, $(S, \mathbf{g})$ have matching boundaries; furthermore, $(S, \mathbf{g})$ is affine. Thus, since $\hat{\mathbb{f}}$ is a degree r null lagrangian, we have $I(\hat{\mathbb{f}}; T, \mathbf{g}) = \langle \hat{\mathbb{f}}, S, \mathbf{g} \rangle$, which can be rewritten as

$$\langle \Phi, T' \rangle = \langle \Phi, S' \rangle$$

by (3.4.13). The above description of the faces of Γ shows that the last relation reads as

$$\Phi\Big(\sum_{i=0}^{r} \hat{\mathbf{a}}^i\Big) = \sum_{i=0}^{r} \Phi(\hat{\mathbf{a}}^i), \tag{3.5.3}$$

which must hold for every $r{+}1$ tuple $\mathbf{a}_0, \dots, \mathbf{a}_r \in \mathrm{R}^{m+n}$ such that $\wedge_{r+1}\mathbf{P}\, \mathbf{a}_0 \wedge \cdots \wedge \mathbf{a}_r \neq \mathbf{0}$.

Let H be the Haar measure on the group $\mathcal{G} := \mathrm{SO}(m+n)$ of rotations in R^{m+n}. Observe that if $\boldsymbol{\gamma} \in \mathrm{si} \wedge_p \mathrm{R}^{m+n}$, $\boldsymbol{\gamma} \neq \mathbf{0}$, with $p \le n$, then

$$H(\{\mathbf{R} \in \mathcal{G} : \wedge_p \mathbf{P} \wedge_p \mathbf{R}\, \boldsymbol{\gamma} = \mathbf{0}\}) = 0. \tag{3.5.4}$$

It suffices to consider the case $|\boldsymbol{\gamma}| = 1$; write $\boldsymbol{\gamma} = \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_p$ where $\mathbf{a}_1, \dots, \mathbf{a}_p$ is an orthonormal system in R^{m+n}. The goal is to prove that the map $m : \mathcal{G} \to \wedge_p \mathrm{R}^n$ given by $m(\mathbf{R}) = \wedge_p \mathbf{P} \wedge_p \mathbf{R}\boldsymbol{\gamma}$ for each $\mathbf{R} \in \mathcal{G}$ is different from $\mathbf{0}$ for H a.e. $\mathbf{R} \in \mathcal{G}$. Since m is an analytic map on an analytic manifold, the assumption that m vanishes on a set of positive measure would lead to m vanishing identically. However, m does not vanish identically: since $p \le n$, there exists an $\mathbf{R} \in \mathcal{G}$ such that $\mathbf{Ra}_i$, $i = 1, \dots, p$, is contained in the subspace $\mathrm{R}^n \times \{\mathbf{0}\}$ of R^{m+n}.

Let $f_\rho : \mathcal{G} \to [0, \infty)$, $\rho > 0$, be a family of continuous functions with $\int_{\mathcal{G}} f_\rho \, dH = 1$ such that the support of f_ρ is contained in the ball in $\mathcal{G}$ of radius ρ and let Φ_ρ : $\mathrm{si} \wedge_r \mathrm{R}^{m+n} \to \mathrm{R}$ be defined by

$$\Phi_\rho(\boldsymbol{\gamma}) = \int_{\mathcal{G}} \Phi(\wedge_r \mathbf{R}\, \boldsymbol{\gamma}) f_\rho(\mathbf{R})\, dH(\mathbf{R})$$

for each $\boldsymbol{\gamma} \in \mathrm{si} \wedge_r \mathrm{R}^{m+n}$; here we use fact that $\wedge_r \mathbf{R}\, \boldsymbol{\gamma} \in D$ for H a.e. $\mathbf{R}$, which follows from (3.5.4). The function Φ_ρ is a parametric integral of degree r in R^{m+n} with $\mathrm{dom}\, \Phi_\rho = \mathrm{si} \wedge_r \mathrm{R}^{m+n}$. If $\mathbf{a}_0, \dots, \mathbf{a}_r \in \mathrm{R}^{m+n}$ are linearly independent vectors in R^{m+n} then for H a.e. $\mathbf{R} \in \mathcal{G}$ we have $\wedge_{r+1}\mathbf{P} \wedge_{r+1} \mathbf{R}\, \mathbf{a}_0 \wedge \cdots \wedge \mathbf{a}_r \neq \mathbf{0}$ by (3.5.4). Hence (3.5.3) gives

$$\Phi\Big(\sum_{i=0}^{r} \wedge_r \mathbf{R}\, \hat{\mathbf{a}}^i\Big) = \sum_{i=0}^{r} \Phi(\wedge_r \mathbf{R}\, \hat{\mathbf{a}}^i).$$

Multiplying this relation by $f_\rho(\mathbf{R})$ and integrating with respect to $H(\mathbf{R})$ we obtain

$$\Phi_\rho\Big(\sum_{i=0}^{r} \hat{\mathbf{a}}^i\Big) = \sum_{i=0}^{r} \Phi_\rho(\hat{\mathbf{a}}^i) \tag{3.5.5}$$

for every $r+1$ tuple $\mathbf{a}_0, \dots, \mathbf{a}_r \in \mathrm{R}^{m+n}$ of linearly independent vectors. If $\mathbf{a}_0, \dots, \mathbf{a}_r \in \mathrm{R}^{m+n}$ are not linearly independent, then (3.5.5) still holds, since then the $r+1$ tuple

$\mathbf{a}_0, \dots, \mathbf{a}_r \in \mathrm{R}^{m+n}$ can be approximated by a sequence of $r+1$ tuples of linearly independent vectors and the limit using the continuity of Φ_ρ then gives (3.5.5).

If $\boldsymbol{\gamma} = \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_r \in \mathrm{si} \wedge_r \mathrm{R}^{m+n}$ then the application of (3.5.5) to the $r+1$ tuple gives $\Phi_\rho(-\boldsymbol{\gamma}) = -\Phi_\rho(\boldsymbol{\gamma})$. Thus the function Φ_ρ satisfies the hypothesis of Proposition 3.2.4 and then for each $\rho > 0$ there exists an r form $\boldsymbol{\omega}_\rho \in \wedge^r \mathrm{R}^{m+n}$ such that

$$\Phi_\rho(\boldsymbol{\gamma}) = \langle \boldsymbol{\omega}_\rho, \boldsymbol{\gamma} \rangle$$

for each $\boldsymbol{\gamma} \in \mathrm{si} \wedge_r \mathrm{R}^{m+n}$. The properties of the family f_ρ and the continuity of Φ imply that $\Phi_\rho(\boldsymbol{\gamma}) \to \Phi(\boldsymbol{\gamma})$ for each $\boldsymbol{\gamma} \in D$. Thus $\langle \boldsymbol{\omega}_\rho, \boldsymbol{\gamma} \rangle \to \Phi(\boldsymbol{\gamma})$ for every $\boldsymbol{\gamma} \in D$. It follows from the linearity that the limit $\lim_{\rho \to 0} \langle \boldsymbol{\omega}_\rho, \boldsymbol{\gamma} \rangle$ exists for every $\boldsymbol{\gamma}$ from $\operatorname{span} D$, which is $\wedge_r \mathrm{R}^{m+n}$ by Proposition 3.4.1(iii). The limit defines a $\boldsymbol{\omega} \in \wedge^r \mathrm{R}^{m+n}$ and thus

$$\Phi(\boldsymbol{\gamma}) = \langle \boldsymbol{\omega}, \boldsymbol{\gamma} \rangle \tag{3.5.6}$$

for each $\boldsymbol{\gamma} \in D$. Let $\boldsymbol{\Omega}_k$ be defined by

$$\langle \boldsymbol{\omega}, \mathfrak{M}_r(\boldsymbol{\Theta}_0, \dots, \boldsymbol{\Theta}_s) \rangle = \sum_{k=0}^{s} \boldsymbol{\Omega}_k \cdot \boldsymbol{\Theta}_k \tag{3.5.7}$$

for each $(\boldsymbol{\Theta}_0, \dots, \boldsymbol{\Theta}_s) \in Z_r$; then (3.5.6) gives (3.5.2). Thus each degree r null lagrangian is of the form (3.5.2).

To prove the converse implication, we let $(T, \mathbf{g})$ and $(S, \mathbf{h})$ be R^m valued maps on r currents in R^n with mathching boundaries, in view of (3.5.2), the integrals $I(\hat{\mathbb{f}}; T, \mathbf{g})$ and $I(\hat{\mathbb{f}}; S, \mathbf{h})$ can be converted to the integrals over the boundaries via (3.6.2) (below); since the boundaries of $(T, \mathbf{g})$ and $(S, \mathbf{h})$ match, the boundary integrals are the same for $I(\hat{\mathbb{f}}; T, \mathbf{g})$ and $I(\hat{\mathbb{f}}; S, \mathbf{h})$ and thus these integral agree. □

Proposition 3.5.3. *A function* $\hat{\mathbb{f}} : \mathrm{H}(m, n, r) \to \mathrm{R} \cup \{\infty\}$ *is degree r polyconvex if and only if*

$$\hat{\mathbb{f}}(\mathbb{F}, \boldsymbol{\alpha}) = \Psi(\wedge_0 \mathbb{F} \lrcorner \boldsymbol{\alpha}, \dots, \wedge_r \mathbb{F} \lrcorner \boldsymbol{\alpha}) \tag{3.5.8}$$

for every $(\mathbb{F}, \boldsymbol{\alpha}) \in \mathrm{H}(m, n, r)$ *and some convex function* $\Psi : Z \to \mathrm{R} \cup \{\infty\}$ *which is additionally positively* 1 *homogeneous if* $r \le n - 1$.

Proof (i): This follows from the fact that degree n null lagrangians are affine functions of $(\wedge_0 \mathbf{F}, \dots, \wedge_n \mathbf{F})$; thus the supremum of any family of degree n null lagrangians induces a convex function of $(\wedge_0 \mathbf{F}, \dots, \wedge_n \mathbf{F})$ and conversely.

(ii): This follows from the fact that degree r null lagrangians are linear functions of $(\wedge_0 \mathbb{F} \lrcorner \boldsymbol{\alpha}, \dots, \wedge_r \mathbb{F} \lrcorner \boldsymbol{\alpha})$; thus the supremum of any family of degree r null lagrangians induces a convex 1 positively homogeneous function of $(\wedge_0 \mathbb{F} \lrcorner \boldsymbol{\alpha}, \dots, \wedge_r \mathbb{F} \lrcorner \boldsymbol{\alpha})$ and conversely. □

Remark 3.5.4. *A function* $\hat{\mathbb{f}} : \mathrm{H}(m, n, n) \to \mathrm{R} \cup \{\infty\}$ *is degree n polyconvex if and only if*

$$\hat{\mathbb{f}}(\mathbf{F}, \pm \mathbf{A}_n) = \Psi^{\pm}(\wedge_0 \mathbf{F}, \dots, \wedge_n \mathbf{F}) \tag{3.5.9}$$

for all $\mathbf{F} \in \mathrm{Lin}(\mathrm{R}^n, \mathrm{R}^m)$ *and some convex functions* $\Psi^{\pm} : Z_n^* \to \mathrm{R} \cup \{\infty\}$ *where*

$$Z_n^* = \prod_{k=0}^{s} \mathrm{Lin}(\wedge_k \mathrm{R}^n, \wedge_k \mathrm{R}^m).$$

Proof of Proposition* 3.4.3 *(completion) We note that the direct implications in (ii) and (iii) follow from the direct implication in (i).

To prove the converse implication in (ii), we note that if $\hat{\mathbb{f}}$ is a degree r null lagrangian then it is of the form asserted in Proposition 3.5.2(ii). Given the tensors $\mathbf{\Omega}_i$, there exists a unique $\boldsymbol{\omega} \in \wedge^r \mathrm{R}^{m+n}$ such that (3.5.7) holds for every $(\mathbf{\Theta}_0, \dots, \mathbf{\Theta}_r) \in Z_r$. If one defines a parametric integrand Φ of degree r by

$$\Phi(\boldsymbol{\gamma}) = \langle \boldsymbol{\omega}, \boldsymbol{\gamma} \rangle$$

for each $\boldsymbol{\gamma} \in \mathrm{si} \wedge_r \mathrm{R}^{m+n}$ then Φ is a semielliptic null lagrangian and $\hat{\mathbb{f}}$ and Φ are related to each other.

To prove the converse implication in (iii), we assume that $\hat{\mathbb{f}}$ is polyconvex of degree r so that we have (3.5.8) with some positively 1 homogeneous convex function $\Psi : Z_r \to \mathrm{R} \cup \{\infty\}$. Let $\tilde{\Phi} := \Psi \circ \mathfrak{M}_r^{-1}$ and Φ let be the restriction of $\tilde{\Phi}$ to $\mathrm{si} \wedge_r \mathrm{R}^{m+n}$. Then Φ is semielliptic polyconvex and $\hat{\mathbb{f}}$ and Φ are relatred to each other. □

3.6 Convergence of graphs

We now discuss the convergence of graphs of varying R^m valued maps on varying r dimensional currents in R^n. We view the domain currents and the graphs as measures. We use the Reshetnyak lowersemicontinuity theorem to establish a lowersemicontinuity result for integral functionals in this context. Throughout this section, let Ω be an open subset of R^n.

Proposition 3.6.1. *Let T be an r dimensional integral current in Ω and write*

$$T = \boldsymbol{\alpha}\, \mathcal{H}^r \llcorner M, \quad \partial T = \boldsymbol{\beta}\, \mathcal{H}^{r-1} \llcorner N,$$

where ∂T is the boundary of T in Ω, with M, $\boldsymbol{\alpha}$ and N, $\boldsymbol{\beta}$ satisfying, respectively,

$$\left.\begin{array}{c} M,\ N \text{ are } \mathcal{H}^r \text{ rectifiable and } \mathcal{H}^{r-1} \text{ rectifiable subsets of } \Omega, \\ \boldsymbol{\alpha} : M \to \wedge_r \mathrm{R}^n \text{ and } \boldsymbol{\beta} : N \to \wedge_{r-1} \mathrm{R}^n \text{ are } \mathcal{H}^r \text{ and } \mathcal{H}^{r-1} \text{ integrable}, \\ \boldsymbol{\alpha} \text{ and } \boldsymbol{\beta} \text{ are tangential to } M \text{ and } N, \\ \operatorname{ran} |\boldsymbol{\alpha}|, \operatorname{ran} |\boldsymbol{\beta}| \subset \mathrm{N}. \end{array}\right\} \tag{3.6.1}$$

Let $\boldsymbol{\theta} : \Omega \to \mathrm{R}^m$ be lipschitzian, put $\mathbf{g} = \boldsymbol{\theta}|M$, $\mathbf{h} = \boldsymbol{\theta}|N$, and let

$$\mathbf{M}_k = \wedge_k \nabla \mathbf{g} \lrcorner \boldsymbol{\alpha}, \quad \mathbf{N}_k = \wedge_k \nabla \mathbf{h} \lrcorner \boldsymbol{\beta}.$$

Then we have the identities

$$\int_M \mathbf{M}_k \, \mathrm{D} \boldsymbol{\tau} \, d\mathcal{H}^r = (-1)^k \int_N \mathbf{N}_k \, \boldsymbol{\tau} \, d\mathcal{H}^{r-1}, \tag{3.6.2}$$

$$\int_M (\mathbf{M}_{k+1} \boldsymbol{\tau}) \llcorner \mathbf{r} + \langle \mathbf{r}, \mathbf{g} \rangle \mathbf{M}_k \, \mathrm{D} \boldsymbol{\tau} \, d\mathcal{H}^r = (-1)^k \int_N \langle \mathbf{r}, \boldsymbol{\theta} \rangle \mathbf{N}_k \boldsymbol{\tau} \, d\mathcal{H}^{r-1} \tag{3.6.3}$$

for any $k \le r-1$, any $\boldsymbol{\tau} \in \mathcal{D}^{r-k-1}(\Omega)$ and any $\mathbf{r} \in \wedge^1 \mathrm{R}^m$.

Proof One finds that if $\boldsymbol{\pi} \in \mathcal{D}^r(\mathrm{R}^{m+n})$ is given by $\boldsymbol{\pi} = \mathbf{P}^{\#} \boldsymbol{\sigma} \wedge \mathbf{Q}^{\#} \boldsymbol{\rho}$ where $0 \le k \le r$, $\boldsymbol{\pi} \in \mathcal{D}^{r-k}(\Omega)$ and $\boldsymbol{\rho} \in \wedge^k \mathrm{R}^m$ then

$$\langle \mathfrak{graph}(T, \mathbf{g}), \boldsymbol{\pi} \rangle = (-1)^{(r-k)k} \int_M \langle \mathbf{M}_k \boldsymbol{\sigma}, \boldsymbol{\rho} \rangle \, d\mathcal{H}^r. \tag{3.6.4}$$

Let now k and $\boldsymbol{\tau}$ be as in the statement and let $\boldsymbol{\omega} \in \mathcal{D}^{k-1}(\mathrm{R}^{m+n})$ be defined by $\boldsymbol{\omega} = \mathbf{P}^{\#}\boldsymbol{\tau} \wedge \mathbf{Q}^{\#}\boldsymbol{\rho}$. One easily finds that $\mathrm{D}\,\boldsymbol{\omega} = \mathbf{P}^{\#}\,\mathrm{D}\,\boldsymbol{\tau} \wedge \mathbf{Q}^{\#}\boldsymbol{\rho}$; applying (3.6.4) twice one finds that (3.4.10) reads

$$(-1)^{(r-k)k} \int_M \langle \mathbf{M}_k\,\mathrm{D}\,\boldsymbol{\tau}, \boldsymbol{\rho}\rangle\, d\mathcal{H}^r = (-1)^{k(r-k-1)} \int_N \langle \mathbf{N}_k \boldsymbol{\tau}, \boldsymbol{\rho}\rangle\, d\mathcal{H}^{r-1}$$

and the arbitrariness of $\boldsymbol{\rho}$ gives (3.6.2).

To prove (3.6.3), let $\varphi : \mathrm{R}^{m+n} \to \mathrm{R}$ be defined by $\varphi(\mathbf{x}, \mathbf{y}) = \langle \mathbf{r}, \mathbf{y}\rangle$ for each $(\mathbf{x}, \mathbf{y}) \in \mathrm{R}^{m+n}$ and let $\boldsymbol{\omega} \in \mathcal{D}^{k-1}(\mathrm{R}^{m+n})$ be defined by $\boldsymbol{\omega} = \varphi\mathbf{P}^{\#}\boldsymbol{\tau} \wedge \mathbf{Q}^{\#}\boldsymbol{\rho}$. One finds that

$$\mathrm{D}\,\boldsymbol{\omega} = (-1)^{r-k-1}\mathbf{P}^{\#}\boldsymbol{\tau} \wedge \mathbf{Q}^{\#}\mathbf{r} \wedge \boldsymbol{\rho} + \varphi\mathbf{P}^{\#}\,\mathrm{D}\,\boldsymbol{\tau} \wedge \mathbf{Q}^{\#}\boldsymbol{\rho}.$$

Evaluating $\langle \mathfrak{graph}(T, \mathbf{g}), \mathrm{D}\,\boldsymbol{\omega}\rangle$ and $\langle \mathfrak{graph}(\partial T, \boldsymbol{\theta}), \boldsymbol{\omega}\rangle$ via (3.6.4) and equating the results we obtain (3.6.3) in the same way as in the preceding part of the proof. □

***Proof of Remark* 2.3.3** We apply (3.6.2) with $r = n$, $T = \mathbf{A}_n\mathcal{L}^n \llcorner E$, $\mathbf{A}_n := \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n$, $\boldsymbol{\theta} = \mathbf{y}$, $\boldsymbol{\tau} = \boldsymbol{\Omega}^n \llcorner \boldsymbol{\xi}$ where $\boldsymbol{\xi} \in \mathcal{D}_{k+1}(\Omega)$ and $\boldsymbol{\Omega}^n = \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n$. Then $M = E$, $N = \mathrm{bd}_* E \cap \Omega$, $\boldsymbol{\alpha} = \mathbf{A}_n$, $\boldsymbol{\beta} = \boldsymbol{\Omega}^n \llcorner \mathfrak{m}$. Using $\mathrm{D}\,\boldsymbol{\tau} = (-1)^k\boldsymbol{\Omega}^n \llcorner \partial\boldsymbol{\xi}$ and algebrain manipulations, one finds that (3.6.2) reduces to

$$\int_E \wedge_k \mathrm{D}\,\mathbf{y}\,\partial\boldsymbol{\xi}\, d\mathcal{L}^n = (-1)^{k+1} \int_{\mathrm{bd}_*(E,\Omega)} (\wedge_k \nabla \mathfrak{y} \wedge \mathfrak{m})\boldsymbol{\xi}\, d\mathcal{H}^{n-1},$$

which in comparison with (2.3.1) gives (2.3.4). □

Proposition 3.6.2. *Let T^i, T, $i = 1, \ldots$, be integral r dimensional currents in Ω and let $\boldsymbol{\theta}^i, \boldsymbol{\theta} : \Omega \to \mathrm{R}^m$, $i = 1, \ldots$, be lipschitzian functions such that*

$$\left.\begin{aligned} &T^i \to T \quad \text{in } \mathcal{M}(\Omega, \wedge_r \mathrm{R}^n)), \\ &\sup\{\mathbf{M}(T^i) + \mathbf{M}(\partial T^i) : i = 1, \ldots\} < \infty, \\ &\boldsymbol{\theta}^i \to \boldsymbol{\theta} \quad \text{uniformly on } \Omega, \\ &\sup\{\mathrm{Lip}(\boldsymbol{\theta}^i) : i = 1, \ldots\} < \infty. \end{aligned}\right\} \tag{3.6.5}$$

Then

(i)

$$\mathfrak{graph}(T^i, \boldsymbol{\theta}^i) \to \mathfrak{graph}(T, \boldsymbol{\theta}) \quad \text{in } \mathcal{M}(\mathrm{R}^{m+n}, \wedge_r \mathrm{R}^{m+n}); \tag{3.6.6}$$

(ii) *if Ω is bounded and $\hat{\mathfrak{f}} : \mathrm{H}(m, n, r) \to \mathrm{R}$ is a continuous nonnegative polyconvex function then we have*

$$\liminf_{i\to\infty} I(\hat{\mathfrak{f}}; T^i, \boldsymbol{\theta}^i) \ge I(\hat{\mathfrak{f}}; T, \boldsymbol{\theta}). \tag{3.6.7}$$

Proof Write

$$T_i = \boldsymbol{\alpha}_i\,\mathcal{H}^r \llcorner M_i, \quad T = \boldsymbol{\alpha}\,\mathcal{H}^r \llcorner M,$$
$$\partial T_i = \boldsymbol{\beta}_i\,\mathcal{H}^{r-1} \llcorner N_i, \quad \partial T = \boldsymbol{\beta}\,\mathcal{H}^{r-1} \llcorner N,$$

with M, $\boldsymbol{\alpha}$, N, $\boldsymbol{\beta}$ satisfying (3.6.1) and with M_i, $\boldsymbol{\alpha}_i$, N_i, $\boldsymbol{\beta}_i$, satisfying the obvious analogs of (3.6.1). Let $\mathbf{g}_i$, $\mathbf{g}$, $\mathbf{h}_i$, $\mathbf{h}$ be the restrictions of $\boldsymbol{\theta}$ to M_i, M, N_i, and N, respectively. Let

$$\mathbf{M}_k^i = \wedge_k \nabla\,\mathbf{g}_i \,\lrcorner\, \boldsymbol{\alpha}_i, \quad \mathbf{N}_k^i = \wedge_k \nabla\,\mathbf{h}_i \,\lrcorner\, \boldsymbol{\beta}_i$$

and let $\mathbf{M}_k$, $\mathbf{N}_k$ be given by analogous expressions without the index i. Prove first that for each k satisfying $0 \le k \le r$ we have

$$\mathbf{M}_k^i\,\mathcal{H}^r \llcorner M_i \rightharpoonup^* \mathbf{M}_k\,\mathcal{H}^r \llcorner M, \tag{3.6.8}$$

$$\mathbf{N}_k^i\,\mathcal{H}^{r-1} \llcorner N_i \rightharpoonup^* \mathbf{N}_k\,\mathcal{H}^{r-1} \llcorner N \tag{3.6.9}$$

in $\mathcal{M}(\Omega, \mathrm{Lin}(\wedge_{r-k}\mathrm{R}^n, \wedge_k\mathrm{R}^m))$ and $\mathcal{M}(\Omega, \mathrm{Lin}(\wedge_{r-k-1}\mathrm{R}^n, \wedge_k\mathrm{R}^m))$, respectively. Proceeding by induction on k, we note that for $k = 0$ this follows from $(3.6.5)_{1,2}$. Assume now that the assertions hold for some $k \ge 0$ and prove it for $k+1$. By (3.6.3) we have

$$\int_{M_i} \mathbf{M}_{k+1}^i \boldsymbol{\tau}\,d\mathcal{H}^r \llcorner \mathbf{r} + \int_{M_i} \langle \mathbf{r}, \mathbf{g}_i\rangle \mathbf{M}_k^i\,\mathrm{D}\,\boldsymbol{\tau}\,d\mathcal{H}^r = (-1)^{r-k}\int_{N_i} \langle \mathbf{r}, \mathbf{h}_i\rangle \mathbf{N}_k^i \boldsymbol{\tau}\,d\mathcal{H}^{r-1} \tag{3.6.10}$$

for any $\mathbf{r} \in \wedge^1\mathrm{R}^m$ and any $\boldsymbol{\tau} \in \mathcal{D}^{r-k-1}(\Omega)$. The induction hypothesis and $(3.6.5)_3$ imply that the second and third integrals in (3.6.10) converge to

$$\int_{M_i} \langle \mathbf{r}, \mathbf{g}\rangle \mathbf{M}_k\,\mathrm{D}\,\boldsymbol{\tau}\,d\mathcal{H}^r, \quad \int_N \langle \mathbf{r}, \mathbf{h}\rangle \mathbf{N}_k \boldsymbol{\tau}\,d\mathcal{H}^{r-1},$$

respectively, which in comparison with (3.6.3) implies

$$\int_{M_i} \mathbf{M}_{k+1}^i \boldsymbol{\tau}\,d\mathcal{H}^r \llcorner \mathbf{r} \to \int_M \mathbf{M}_{k+1} \boldsymbol{\tau}\,d\mathcal{H}^r \llcorner \mathbf{r}.$$

As $\mathbf{r} \in \wedge^1\mathrm{R}^m$ is arbitrary and the involved measures have bounded masses by $(3.6.5)_2$, we have (3.6.8) with k replaced by $k+1$. The application of the just proved assertion to $\partial T^i, \partial T$ gives (3.6.9) with k replaced by $k+1$. This completes the proof of (3.6.8) and (3.6.9).

(i): To prove (3.6.6), we note that the total mass of the measure $\mathfrak{graph}(T^i, \boldsymbol{\theta}^i)$ is given by

$$\mathbf{M}\big(\mathfrak{graph}(T^i, \boldsymbol{\theta}^i)\big) = \int_{M_i} |\wedge_r (\mathbf{C} + \mathbf{D}\nabla \mathbf{g}_i)||\boldsymbol{\alpha}|\,d\mathcal{H}^r$$

and thus $\mathbf{M}\big(\mathfrak{graph}(T^i, \boldsymbol{\theta}^i)\big)$ is bounded independently of i by $(3.6.5)_{2,4}$. In view of this it suffices to verify

$$\langle \mathfrak{graph}(T^i, \boldsymbol{\theta}^i), \boldsymbol{\omega}\rangle \to \langle \mathfrak{graph}(T, \boldsymbol{\theta}), \boldsymbol{\omega}\rangle \tag{3.6.11}$$

on a dense set of $\boldsymbol{\omega}$, which in turn implies that it suffices to verify (3.6.11) for each $\boldsymbol{\omega}$ of the form (3.4.11). However, from $(3.6.5)_3$ follows that $\varphi(\mathbf{x}, \boldsymbol{\theta}^i(\mathbf{x})) \to \varphi(\mathbf{x}, \boldsymbol{\theta}(\mathbf{x}))$ uniformly in $\mathbf{x} \in \Omega$ which in combination with (3.6.8) and (3.4.12) gives (3.6.11).

(ii): To prove (3.6.7), we consider separately the case $r \le n-1$ and $r = n$.

If $r \le n-1$, we let Ψ be as in (3.5.8) and note that since $\hat{\mathfrak{f}}$ is nonnegative, Φ can be chosen nonnegative as well. Then (3.6.8) and the Reshetnyak lowesemicontinuity theorem give (3.6.7).

If $r = n$, we have $T^i = \varphi_i \mathbf{A}_n \mathcal{L}^n$, $T = \varphi \mathbf{A}_n \mathcal{L}^n$ for some $\varphi_i, \varphi \in L^1(\mathcal{L}^n, \mathrm{Z})$ and (3.6.8) means that $\varphi_i \to \varphi$ in $L^1(\mathcal{L}^n, \mathrm{R})$. We deduce that for any k with $0 \le k \le s$ we have

$$\wedge_k \nabla \boldsymbol{\theta}^i \rightharpoonup^* \wedge_k \nabla \boldsymbol{\theta} \quad \text{in } L^\infty(\mathrm{R}^n, \mathrm{Lin}(\wedge_k \mathrm{R}^n, \wedge_k \mathrm{R}^m)).$$

Let now $\Psi^\pm$ be as in (3.5.9), which we can choose nonnegative. Let $\Xi : \mathrm{R} \times Z_n$ be defined by

$$\Xi(t, \mathbf{\Theta}_0, \dots, \mathbf{\Theta}_s) = [t]_+ \Psi^+(\mathbf{\Theta}_0, \dots, \mathbf{\Theta}_s) + [t]_- \Psi^-(\mathbf{\Theta}_0, \dots, \mathbf{\Theta}_s)$$

for each $t \in \mathrm{R}$ and $(\mathbf{\Theta}_0, \dots, \mathbf{\Theta}_s) \in Z_n$ where $[t]_\pm$ denote the positive and negative parts. Then

$$I(\mathfrak{f}; T^i, \boldsymbol{\theta}^i) = \int_{\mathrm{R}^n} \Xi(\varphi_i, \wedge_0 \nabla \boldsymbol{\theta}^i, \dots, \wedge_s \nabla \boldsymbol{\theta}^i)\, d\mathcal{L}^n. \tag{3.6.12}$$

The function Ξ is nonnegative and for each t, the function $\Xi(t, \cdot)$ is convex. In the integrand in (3.6.12), we have a L^1 convergence in the first argument and the weak convergence in the remaining arguments. It then follows from [2; Theorem 5.8, Chapter 5] that

$$\liminf_{i \to \infty} \int_{\mathrm{R}^n} \Xi(\varphi_i, \wedge_0 \nabla \boldsymbol{\theta}^i, \dots, \wedge_s \nabla \boldsymbol{\theta}^i)\, d\mathcal{L}^n \geq \int_{\mathrm{R}^n} \Xi(\varphi, \wedge_0 \nabla \boldsymbol{\theta}, \dots, \wedge_s \nabla \boldsymbol{\theta})\, d\mathcal{L}^n.$$

This completes the proof. □

Appendices

A.1 Differentiation on manifolds and on rectifiable sets

The main text uses the derivatives of maps defined on sets $M \subset \mathrm{R}^n$ of dimension $< n$ in two different ways: (a) for the response function $\hat{\mathbb{f}}$, which is a function defined on the manifold G; (b) for fields defined on the phase interface $\mathcal{S} \subset \mathrm{R}^n$. The phase interface is interpreted either as a smooth $n-1$ surface or as a $\mathcal{H}^{n-1}$ rectifiable set in Chapter 2 and even as a $\mathcal{H}^r$ rectifiable set with $0 \le r \le n$ in Chapter 3.

We are thus lead to consider both the classical derivatives of maps on manifolds and approximate derivatives of maps on rectifiable sets. In the first situation we deal with manifolds of (at least) class 1 embedded in finite dimensional vectorspaces [9; Subsections 3.1.19–3.1.20], which we call simply manifolds or synonymously surfaces and consider classically differentiable maps on these surfacers. We define derivatives (gradients) of maps on manifolds, which we call surface derivatives or surface gradients. In the second situation we deal with $\mathcal{H}^r$ rectifiable sets, consider lipschitzian maps and review the approximate surface derivatives [9; Subsections 3.1.1–3.1.10 and 3.1.22].

Throughout the section, let V, W be finite dimensional inner product spaces.

Let f be a map with the domain $\mathrm{dom}\, f$ which is a relatively open subset of a manifold $\mathcal{M}$ in V with the range $\mathrm{ran}\, f$ in W. If $x \in \mathcal{M}$, we denote by $\mathrm{Tan}(\mathcal{M}, x)$ the tangent space to $\mathcal{M}$ at x, a k dimensional subspace of V where k is the dimension of $\mathcal{M}$. We say that f is differentiable at $x \in \mathrm{dom}\, f$ if there exists a $\mathrm{D} f(x) \in \mathrm{Lin}(V, W)$, called the derivative of f at x, such that

$$\mathrm{D} f(x) P = \mathrm{D} f(x) \tag{A.1.1}$$

where P is the orthogonal projection onto $\mathrm{Tan}(\mathrm{dom}\, f, x)$ and

$$\lim_{\substack{y \to x \\ y \in \mathrm{dom}\, f,\, y \neq x}} |f(y) - f(x) - \mathrm{D} f(x)(y-x)|/|y-x| = 0. \tag{A.1.2}$$

The map $\mathrm{D} f(x)$ is uniquely determined. We note that $\mathrm{D} f(x)$ is a linear transformation defined on the entire space V and not just on the tangent space; however, it vanishes on the orthogonal complement of the tangent space by (A.1.1). This convection renders the derivatives of f at different points of $\mathcal{M}$ belong to the same linear space $\mathrm{Lin}(V, W)$. Other authors (e.g., [9; Subsection 3.1.22]) mean by the derivative the restriction of the derivative in the present sense to the tangent space at the given point. If the range W of f is R, we identify $\mathrm{D} f(x) \in \mathrm{Lin}(V, \mathrm{R})$ with an equally denoted vector in V, such that

$$\mathrm{D} f(x)a = a \cdot \mathrm{D} f(x)$$

for each $a \in V$; then $\mathrm{D} f(x) \in \mathrm{Tan}(\mathcal{M}, x)$.

Let r be an integer, $0 \le r \le \dim V$, and let f be a map such that $\mathrm{dom}\, f$ is a $\mathcal{H}^r$ rectifiable subset of V and $\mathrm{ran}\, f \subset W$. If $x \in \mathrm{dom}\, f$, we denote by $\mathrm{Tan}^r(\mathrm{dom}\, f, x)$ the approximate r dimensional tangent cone to $\mathrm{dom}\, f$ at x. We say that f is approximately r differentiable at $x \in \mathrm{dom}\, f$ if $\mathrm{Tan}^r(\mathrm{dom}\, f, x)$ is an r dimensional subspace of V and there exists a $\mathrm{D} f(x) \in \mathrm{Lin}(V, W)$, called the approximate derivative of f at x, such that (A.1.1) holds with P the projection onto $\mathrm{Tan}^r(\mathrm{dom}\, f, x)$ and we have the limit (A.1.2) in the approximate sense, i.e., for each $\varepsilon > 0$ the r dimensional density of the set

$$\{y \in \mathrm{dom}\, f : |f(y) - f(x) - \mathrm{D} f(x)(y - x)|/|y - x| \ge \varepsilon\}$$

at the point x vanishes. A lipschitzian map on a $\mathcal{H}^r$ rectifiable set has the r approximate derivative at $\mathcal{H}^r$ a.e. point of $\mathrm{dom}\, f$. When the integer r is clear from the context, we abbreviate and speak of approximate differentiability and approximate derivative in place of r approximate differentiability and r approximate derivative. We also use the term approximate surface gradient whenever appropriate.

If $T \in \mathrm{Lin}(V, W)$ is an injective map, we define the pseudoinverse $T^{-1} \in \mathrm{Lin}(W, V)$ as the unique linear map such that

$$T^{-1}T = P, \quad T T^{-1} = Q$$

where P and Q are the orthogonal projections onto $(\ker T)^{\perp}$ and $\mathrm{ran}\, T$. One has

$$(T^{-1})^{-1} = T.$$

If T maps V bijectively onto W then the pseudoinverse coincides with the usual inverse.

We retain the symbol D for the derivative of the response functions $\hat{f}_\alpha$, $\alpha = 1, 2$, and $\hat{\mathbb{f}}$. However, if the variable x in the definition above has the meaning of the referential position $\mathbf{x}$ of a material point of Ω, we write ∇ for D in case of a map f defined on an open subset of Ω and $\mathbb{\nabla}$ for D in case f is a map defined on the phase interface $\mathcal{S}$ in Ω. If $\boldsymbol{\phi}$ is a local parametrization of $\mathcal{S}$ then f is differentiable at $\mathbf{x} \in \mathcal{S}$ if and only if $f \circ \boldsymbol{\phi}$ is differentiable at $\mathbf{p} := \boldsymbol{\phi}^{-1}(\mathbf{x})$ and then

$$\mathbb{\nabla} f(\mathbf{x}) = \nabla(f \circ \boldsymbol{\phi})(\mathbf{p})\nabla\boldsymbol{\phi}(\mathbf{p})^{-1}.$$

If g is a extension of f to a neighborhood of $\mathbf{x}$ in R^n that is differentiable at $\mathbf{x}$ then

$$\mathbb{\nabla} f(\mathbf{x}) = \nabla g(\mathbf{x})\mathbb{P}(\mathbf{x})$$

where $\mathbb{P}(\mathbf{x})$ is the orthogonal projection onto the tangent space of $\mathcal{M}$ at $\mathbf{x}$. If f is a map defined in a neighborhood of $\mathcal{S}$ we use the notation

$$\mathbb{\nabla} f := \mathbb{\nabla}(f|\mathcal{S}).$$

A.2 Multilinear algebra

The exposition in the main text relies on notations and concepts of multilinear algebra. We use the conventions from [9; Chapter One] with some extensions; see also [18; Section 1.7].

If r is an integer with $0 \le r \le n$, we denote by $\wedge_r \mathrm{R}^n$ the inner product space of all r vectors in R^n, i.e., the set of all r linear completely antisymmetric maps $\boldsymbol{\alpha}$ from the dual space of R^n into R. Likewise, we denote by $\wedge^r \mathrm{R}^n$ the inner product space of all r covectors in R^n, i.e., the set of all r linear completely antisymmetric maps $\boldsymbol{\omega}$ from R^n into R. We put $\wedge_0 \mathrm{R}^n = \mathrm{R}$, note that $\wedge_1 \mathrm{R}^n$ is canonically isomorphic with R^n and recall that $\wedge_n \mathrm{R}^n$ is unidimensional. We also put $\wedge_r \mathrm{R}^n = \{\mathbf{0}\}$ if r is an integer with $r < 0$ or $r > n$. We use the same conventions for $\wedge^r \mathrm{R}^n$. We denote by $\langle \boldsymbol{\alpha}, \boldsymbol{\omega} \rangle$ the duality pairing of an r vector $\boldsymbol{\alpha}$ with an r covector and $\boldsymbol{\omega}$.

We use the symbol $\wedge$ to denote the wedge products of a family of r vectors with varying r and the wedge product of a family of r convectors with varying r. If $\mathbf{a}_1, \ldots, \mathbf{a}_r$ are vectors in R^n we abbreviate $\prod_{i=1}^r \mathbf{a}_i := \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_r$ the wedge product of $\mathbf{a}_1, \ldots, \mathbf{a}_r$, an element of $\wedge_r \mathrm{R}^n$. In addition to the wedge product, we need the contraction $\llcorner$ of vectors by convectors and vice versa. If $\boldsymbol{\alpha} \in \wedge_r$, $\boldsymbol{\omega} \in \wedge_s$ with $s \le r$ then the contraction $\boldsymbol{\alpha} \llcorner \boldsymbol{\omega}$ of $\boldsymbol{\alpha}$ by $\boldsymbol{\omega}$ is an $r - s$ vector satisfying $\langle \boldsymbol{\alpha} \llcorner \boldsymbol{\omega}, \boldsymbol{\sigma} \rangle = \langle \boldsymbol{\alpha}, \boldsymbol{\sigma} \wedge \boldsymbol{\omega} \rangle$ for all $\boldsymbol{\sigma} \in \wedge^{r-s} \mathrm{R}^n$. Similarly, if $\boldsymbol{\alpha}$ is an r vector in R^n and $\boldsymbol{\beta}$ and s vector with $s \le r$ we define a contraction $\boldsymbol{\alpha} \llcorner \boldsymbol{\beta}$ of $\boldsymbol{\alpha}$ by $\boldsymbol{\beta}$ to be an $r - s$ vector in R^n such that

$$(\boldsymbol{\alpha} \llcorner \boldsymbol{\beta}) \cdot \boldsymbol{\gamma} = \boldsymbol{\alpha} \cdot (\boldsymbol{\gamma} \wedge \boldsymbol{\beta})$$

for each $r - s$ vector $\boldsymbol{\gamma}$.

Furthermore, we need exterior products and powers of linear transformations. If r, s are positive integers, a permutation π of $1, \ldots, r + s$ is said to be a shuffle of type r, s if π is increasing on $\{1, \ldots, r\}$ and on $\{r + 1, \ldots, r + s\}$. Denote by $\mathrm{Sh}(r, s)$ the set of all shuffles of type r, s. One has $\operatorname{card} \mathrm{Sh}(r, s) = (r + s)!/r!s!$. If

$$\boldsymbol{\Phi} \in \mathrm{Lin}(\wedge_r \mathrm{R}^n, \wedge_r \mathrm{R}^m), \quad \boldsymbol{\Psi} \in \mathrm{Lin}(\wedge_s \mathrm{R}^n, \wedge_s \mathrm{R}^m) \tag{A.2.1}$$

where r, s are nonegative integers, then there exists a unique

$$\boldsymbol{\Phi} \wedge \boldsymbol{\Psi} \in \mathrm{Lin}(\wedge_{r+s} \mathrm{R}^n, \wedge_{r+s} \mathrm{R}^m)$$

such that

$$(\boldsymbol{\Phi} \wedge \boldsymbol{\Psi})\Big(\prod_{i=1}^{r+s} \mathbf{a}_i\Big) = C(r, s) \sum_{\pi \in \mathrm{Sh}(r,s)} \operatorname{sgn} \pi \ \boldsymbol{\Phi}\Big(\prod_{i=1}^{r} \mathbf{a}_{\pi(i)}\Big) \wedge \boldsymbol{\Psi}\Big(\prod_{i=r+1}^{r+s} \mathbf{a}_{\pi(i)}\Big)$$

for any $\mathbf{a}_1, \ldots, \mathbf{a}_{r+s} \in \mathrm{R}^n$ where $C(r, s) = r!s!/(r + s)!$. We call $\boldsymbol{\Phi} \wedge \boldsymbol{\Psi}$ the exterior product of $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$. The exterior product is commutative, i.e.,

$$\boldsymbol{\Phi} \wedge \boldsymbol{\Psi} = \boldsymbol{\Psi} \wedge \boldsymbol{\Phi},$$

and associative, i.e., assuming (A.2.1) and $\boldsymbol{\Omega} \in \mathrm{Lin}(\wedge_t \mathrm{R}^n, \wedge_t \mathrm{R}^m)$ then

$$(\boldsymbol{\Phi} \wedge \boldsymbol{\Psi}) \wedge \boldsymbol{\Omega} = \boldsymbol{\Phi} \wedge (\boldsymbol{\Psi} \wedge \boldsymbol{\Omega})$$

so that we can use unambiguously the notation $\boldsymbol{\Phi} \wedge \boldsymbol{\Psi} \wedge \boldsymbol{\Omega}$. If k is a positive integer and $\boldsymbol{\Phi} \in \mathrm{Lin}(\wedge_r \mathrm{R}^n, \wedge_r \mathrm{R}^m)$, we define $\wedge_k \boldsymbol{\Phi} \in \mathrm{Lin}(\wedge_{rk} \mathrm{R}^n, \wedge_{rk} \mathrm{R}^m)$ by $\wedge_k \boldsymbol{\Phi} = \boldsymbol{\Phi} \wedge \cdots \wedge \boldsymbol{\Phi}$ with k terms of the product. In particular, if $\mathbf{A} \in \mathrm{Lin}(\mathrm{R}^n, \mathrm{R}^m)$

then $\wedge_r \mathbf{A} \in \mathrm{Lin}(\wedge_r \mathrm{R}^n, \wedge_r \mathrm{R}^m)$. If $m = n$ then $\wedge_n \mathbf{A}\,\boldsymbol{\beta} = \det \mathbf{A}\,\boldsymbol{\beta}$ for each n vector $\boldsymbol{\beta}$; in the same situation, $\wedge_{n-1}\mathbf{A} = *\operatorname{cof} \mathbf{A} *$ where $*$ is the Hodge operator mapping $\wedge_r \mathrm{R}^n$ isometrically onto $\wedge_{n-r}\mathrm{R}^n$. We put $\wedge_0 \mathbf{A} = 1$ in all situations. Clearly, $\wedge_r \mathbf{A} = \mathbf{0}$ if $r > \min\{m, n\}$. Gnerally, we have the binomial theorem

$$\wedge_k(\boldsymbol{\Phi} + \boldsymbol{\Psi}) = \sum_{i=0}^{k} \binom{k}{i} \wedge_i \boldsymbol{\Phi} \wedge \wedge_{k-i} \boldsymbol{\Psi} \tag{A.2.2}$$

for each $\boldsymbol{\Phi}, \boldsymbol{\Psi} \in \mathrm{Lin}(\wedge_r \mathrm{R}^n, \wedge_r \mathrm{R}^m)$.

If $\boldsymbol{\Theta} \in \mathrm{Lin}(\wedge_r \mathrm{R}^n, \wedge_r \mathrm{R}^m)$ and $\mathbf{a} \in \mathrm{R}^n$ we define $\boldsymbol{\Theta} \wedge \mathbf{a} \in \mathrm{Lin}(\wedge_{r+1}\mathrm{R}^n, \wedge_r \mathrm{R}^m)$ by

$$(\boldsymbol{\Theta} \wedge \mathbf{a})\boldsymbol{\beta} = \boldsymbol{\Theta}(\boldsymbol{\beta} \llcorner \mathbf{a})$$

for each $r+1$ vector $\boldsymbol{\beta}$ in R^n. If $\boldsymbol{\Psi} \in \mathrm{Lin}(\wedge_s \mathrm{R}^n, \wedge_s \mathrm{R}^m)$ and $\boldsymbol{\alpha} \in \wedge_r \mathrm{R}^n$ with $r \ge s$, let

$$\boldsymbol{\Psi} \lrcorner\, \boldsymbol{\alpha} \in \mathrm{Lin}(\wedge^{r-s}\mathrm{R}^n, \wedge_s \mathrm{R}^m)$$

be defined by

$$(\boldsymbol{\Psi} \lrcorner\, \boldsymbol{\alpha})\boldsymbol{\omega} = \boldsymbol{\Psi}(\boldsymbol{\alpha} \llcorner \boldsymbol{\omega})$$

for each $\boldsymbol{\omega} \in \wedge^{r-s}\mathrm{R}^n$.

If $\Omega \subset \mathrm{R}^n$ is open, we denote by $\mathscr{D}_r(\Omega)$ the set of all infinitely differentiable r vectorfields $\boldsymbol{\xi} : \mathrm{R}^n \to \wedge_r \mathrm{R}^n$ whose support is compact and contained in Ω. We define the ***interior derivative*** $\partial\boldsymbol{\xi}$ of $\boldsymbol{\xi}$ as an element of $\mathscr{D}_{r-1}(\Omega)$ given by

$$\partial\boldsymbol{\xi} = (-1)^r \sum_{i=1}^{n} \mathrm{D}_i \,\boldsymbol{\xi} \llcorner \mathbf{e}^i,$$

where D_i denote the partial derivatives and $\mathbf{e}_1, \dots, \mathbf{e}_n$ is the standard basis in R^n. If $\mathbf{a}$ is a 1 vectorfield then $\partial\mathbf{a} = -\operatorname{div}\mathbf{a}$. The factor $(-1)^r$ is chosen so as to render valid the integration by parts formula

$$\int_{\mathrm{R}^n} \partial\boldsymbol{\xi} \cdot \boldsymbol{\omega}\, d\mathscr{L}^n = \int_{\mathrm{R}^n} \boldsymbol{\xi} \cdot \mathrm{D}\,\boldsymbol{\omega}\, d\mathscr{L}^n$$

for every smooth $r-1$ form $\boldsymbol{\omega}$ on R^n where $\mathrm{D}\,\boldsymbol{\omega}$ denotes the exterior derivative.

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