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# Nodal $\mathcal{O}(h^4)$-superconvergence of piecewise trilinear FE approximations

Antti Hannukainen[1,*], Sergey Korotov[1,†], Michal Křížek[2,‡]

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[1] Institute of Mathematics, Helsinki University of Technology
P.O. Box 1100, FIN–02015 Espoo, Finland
e-mail: antti.hannukainen, sergey.korotov@hut.fi

[2] Institute of Mathematics, Academy of Sciences
Žitná 25, CZ–115 67 Prague 1, Czech Republic
e-mail: krizek@math.cas.cz

**Abstract:** We construct and analyse a nodal $\mathcal{O}(h^4)$-superconvergent FE scheme for approximating the Poisson equation with homogeneous boundary conditions in three-dimensional domains by means of piecewise trilinear functions. The scheme is based on averaging the equations that arise from FE approximations on uniform cubic, tetrahedral, and prismatic partitions. This approach presents a three-dimensional generalization of a two-dimensional averaging of linear and bilinear elements which also exhibits nodal $\mathcal{O}(h^4)$-superconvergence. The obtained superconvergence result is illustrated by two numerical examples.

**Keywords:** higher order error estimates, tetrahedral and prismatic elements, superconvergence, averaging operators

**MSC:** 65N30

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# 1 Introduction

We consider the Poisson equation with homogeneous Dirichlet boundary condition

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1}$$

Assume that $\Omega \subset \mathbb{R}^3$ is a bounded rectangular domain and that the right-hand side function $f \in C^4(\overline{\Omega})$.

The weak form of problem (1) reads: Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega), \tag{2}$$

where $(\cdot,\cdot)$ denotes the scalar products in both $L_2(\Omega)$ and $(L_2(\Omega))^3$.

In [15], Schatz discovered the nodal $\mathcal{O}(h^4)$-superconvergence of quadratic elements on uniform tetrahedral partitions (i.e., for each internal edge $e$ the patch of tetrahedra sharing $e$ is a point symmetric set with respect to the midpoint of $e$). This result was later extended by Schatz, Sloan, and Wahlbin [16] to locally symmetric meshes. Since each uniform tetrahedralization is locally point-symmetric with respect to the midpoints of edges, the $\mathcal{O}(h^4)$-superconvergence of quadratic tetrahedral elements holds at these midpoints as well.

Linear triangular elements also exhibit nodal $\mathcal{O}(h^4)$-superconvergence (ultraconvergence) on uniform triangulations consisting solely of equilateral triangles. This result was obtained by Lin and Wang in [14] (see also [2]). It is based on the fact that the corresponding stiffness FE matrix is the same as the matrix associated to the standard 7-point finite difference scheme, which is $\mathcal{O}(h^4)$-accurate. However, this result cannot be extended to three-dimensional space, since the regular tetrahedron is not a space-filler (see [4, 11]).

The study of superconvergence by a computer-based approach developed by Babuška et al. [1] requires to examine harmonic polynomials in the plane. Note that the dimension of the space of harmonic polynomials of degree $k \in \{1, 2, \dots\}$ in two variables is only 2, whereas the dimension of such a space in three variables is $2k+1$. This makes superconvergence analysis for $d = 3$ much more difficult (see [17]) than for $d = 2$. The likelihood of $2k+1$ polynomial graphs passing through a common point is much smaller than the probability of two intersecting polynomial graphs.

A suitable averaging of gradients of FE solutions leads to superconvergence, see [5, 6]. In this paper we show that an averaging of stiffness matrices of several kind of elements exhibits also a superconvergence. In particular, here we will present an averaging of linear algebraic equations arising from

FE approximations of problem (2) on uniform partitions of $\overline{\Omega}$ into cubes, tetrahedra, and triangular prisms, respectively. The method is an extension of the nodal $\mathcal{O}(h^4)$-superconvergence result for the Poisson equation in two-dimensional domains, where the stiffness matrices corresponding to linear and bilinear elements are appropriately averaged [12, 13] to obtain the matrix associated to the standard 9-point finite difference scheme. To the authors' knowledge, extension of this result to the three-dimensional case has not yet been studied. Note that the size (and also the band-width) of the resulting matrix will be the same as for the stiffness matrix corresponding to trilinear finite elements, which produces only $\mathcal{O}(h^2)$-accuracy in the maximum norm at nodes.

# 2 Construction of the averaged FE scheme

## 2.1 Preliminaries

Assume that $\mathcal{T}_h$ is a face-to-face partition of the domain $\overline{\Omega}$ into cubes. We denote the set of interior nodes of $\mathcal{T}_h$ by $\mathcal{N}_h = \{z_i\}_{i=1}^N$, where $N = N(h)$ and $h$ is the length of any edge.

In order to introduce the relevant FD and FE schemes, we shall use the compact notation from [9]. To this end, the nodes in the FD stencil (see Figure 1) are divided into three separate groups (midpoints of faces, vertices, and midpoints of edges) and the following conventional summations

$$\begin{gathered}\Diamond U_0 = U_1 + U_2 + U_3 + U_4 + U_{13} + U_{14},\\ \bigcirc U_0 = U_{19} + U_{20} + U_{21} + U_{22} + U_{23} + U_{24} + U_{25} + U_{26},\\ \Box U_0 = U_5 + U_6 + U_7 + U_8 + U_9 + U_{10} + U_{11} + U_{12} + U_{15} + U_{16} + U_{17} + U_{18}\end{gathered}$$

are used, where the value $U_0$ corresponds to the (central) vertex $z_i$ and $U_1, \ldots, U_{26}$ stand for the neighbouring vertices as sketched in Figure 1.

Using this notation, the fourth-order accurate 19-point FD scheme can be written as follows (see [9, p. 600])

$$24U_0 - 2\Diamond U_0 - \Box U_0 = 6h^2 f(z_i) + h^4 \Delta f(z_i). \tag{3}$$

From now on, we will use the notation $u_h$ for the finite element solution and $\vec{u}_h$ for the vector of its nodal values at $z_i$, i.e.,

$$(\vec{u}_h)_i = u_h(z_i). \tag{4}$$

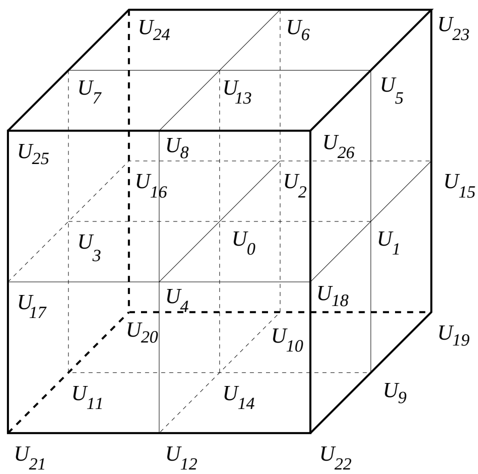


Figure 1: Numbering of nodes with respect to the central node $U_0$.

The nodal superconvergence will be measured in the discrete $\ell^2$-norm

$$\|u - u_h\|_h = \left( h^3 \sum_{i=1}^{N} \left(u(z_i) - u_h(z_i)\right)^2 \right)^{1/2}.$$

We will use the same notation also for vectors $\vec{x} = (x_1, \ldots, x_N)^\top$:

$$\|\vec{x}\|_h = \left( h^3 \sum_{i=1}^{N} x_i^2 \right)^{1/2}, \tag{5}$$

and for the induced matrix norm. Notice that $\|\vec{x}\|_h = h^{3/2} \|\vec{x}\|_2$, where $\|\cdot\|_2$ is the standard Euclidean norm.

The principal idea in the derivation of the superconvergent FE scheme in our work is to ensure a certain "closeness" between two systems of linear algebraic equations

$$\Delta_h \vec{U}_h = \vec{f}_h \quad \text{and} \quad A_h \vec{u}_h = \vec{F}_h, \tag{6}$$

arising from FD scheme (3) and from the averaged FE scheme corresponding to (2), respectively. In more detail, we will construct the matrix $A_h$ so that

$$A_h = h\Delta_h, \tag{7}$$

and we prove that the right-hand side vectors in (6) satisfy the estimate

$$\|h\vec{f}_h - \vec{F}_h\|_\infty \le Ch^7, \tag{8}$$

where $\|\cdot\|_\infty$ stands for the maximum norm and $(\vec{f}_h)_i = 6h^2 f(z_i) + h^4 \Delta f(z_i)$.

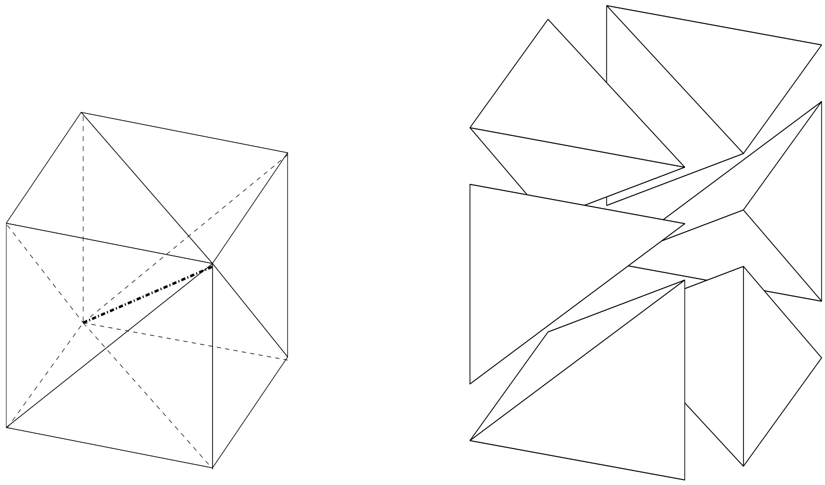

Figure 2: The Kuhn partition of a cube into six non-obtuse tetrahedra.

## 2.2 Averaging of FE approximations

In this subsection, we construct the FE scheme for which requirements (7) and (8) are satisfied. In order to meet requirement (7), we shall use cubic, tetrahedral, and prismatic FE partitions.

Applying the FE discretization on a uniform cubic partition gives the following equation for entries that appear in one row of the stiffness matrix

$$A^c = \frac{8h}{3}\, u_0 - \frac{h}{12} \bigcirc u_0 - \frac{h}{6} \Box u_0 \tag{9}$$

at each node located at the interior of the domain, where $u_0 = u(z_i)$ for simplicity. The coefficient at the diamond term $\Diamond u_0$ is zero since the scalar product of the gradients of two basis functions related to adjacent nodes is zero.

A cube can be decomposed into six tetrahedra that share a spatial diagonal (see Figure 2). Since each cube has four spatial diagonals, there exists four different such uniform tetrahedral partitions (see Figure 3). Although all these partitions yield the same local equation, the averaging requires usage of all associated partitions. This is due to the symmetry requirement, which will be stated later in the proof of Lemma 1. Applying the FE method on any of the four tetrahedral partitions gives the following contributions to the local stiffness matrix

$$A^t = 6h\, u_0 - h \Diamond u_0. \tag{10}$$

An easy calculation shows that the coefficients standing at the terms $\bigcirc u_0$ and $\square u_0$ are zero.

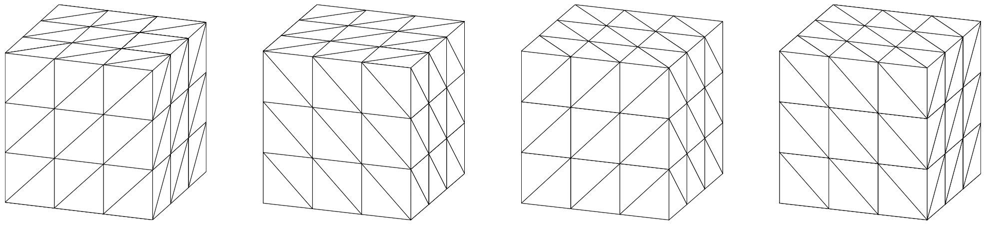

Figure 3: Four different types of tetrahedral partitions.

We will employ prismatic elements in a similar manner as tetrahedral elements. Again, due to the symmetry requirement we need to use all six possible uniform prismatic partitions (see Figure 4). Individual local equations do not fit into our notational framework, but summing all local equations gives

$$A^p = 22hu_0 - \frac{5h}{3}\diamond u_0 - \frac{h}{4}\bigcirc u_0 - \frac{5h}{6}\square u_0. \tag{11}$$

This local equation will be applied in the averaging.

The global stiffness matrix arising from cubic elements is denoted by $A_c$, from tetrahedral elements by $A_t^j$ $(j = 1, \dots, 4)$, and from prismatic elements by $A_p^k$ $(k = 1, \dots, 6)$. Notice that

$$A_t^1 = A_t^2 = A_t^3 = A_t^4. \tag{12}$$

By summing all these matrices with appropriate weights as follows

$$A = -9A_c - \frac{3}{4}\sum_{j=1}^{4} A_t^j + 3\sum_{k=1}^{6} A_p^k, \tag{13}$$

we obtain the matrix $A$ for the averaged FE scheme. The corresponding combination of local equations (9), (10), and (11) gives the equation

$$A = 24hu_0 - 2h\diamond u_0 - h\square u_0 \tag{14}$$

for the averaged FE scheme. The above matrix is the same (up to the factor $h$) as the matrix of the 19-point FD scheme (3), i.e., we have $A_h = h\Delta_h$. The matrices $A_c, A_t^j, A_p^k$ are symmetric and positive definite. In Lemma 2, we prove that the averaged matrix $A_h$ is also symmetric and positive definite, even though some weights in (13) are negative.

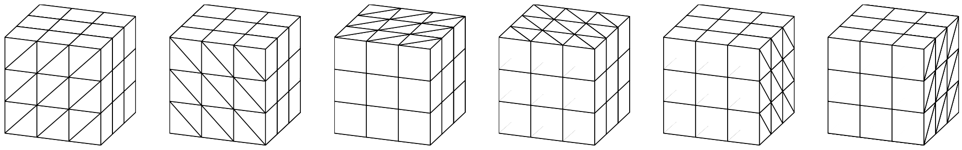

Figure 4: Six different types of prismatic partitions.

# 3 Superconvergence properties

In the previous section, we presented an averaged FE scheme with the same system matrix (up to the factor $h$) as the 19-point finite difference formula. In order to prove the superconvergence property for approximation obtained by the proposed method, we show that condition (8) is satisfied.

In what follows, we set

$$v_i = -9c_i - \frac{3}{4}\sum_{j=1}^{4} t_i^j + 3\sum_{k=1}^{6} p_i^k, \tag{15}$$

where $c_i, t_i^j$, and $p_i^k$ are cubic, tetrahedral, and prismatic basis functions related to the node $z_i$ ($i = 1, \dots, N$), respectively. Notice that $v_i$ is piecewise trilinear on each cube that is partitioned into 24 subtetrahedra (see Figure 5).

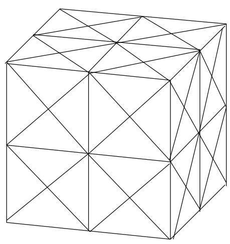

Figure 5: Support of the averaged basis function $v_i$ consists of 8 cubes each of which is partitioned into $24 = 6 \times 4$ tetrahedra.

Now, we prove estimate (8).

**Lemma 1.** *Let $f \in C^4(\overline{\Omega})$. Then*

$$\|h\vec{f}_h - \vec{F}_h\|_\infty = \max_{i=1,\dots,N} |\,(f, v_i) - 6h^3 f(z_i) - h^5 \Delta f(z_i)| \le Ch^7 \|f\|_{C^4(\overline{\Omega})} \text{ as } h \to 0, \tag{16}$$

*where $v_i$ is the basis function defined in* (15).

**P r o o f :** Without loss of generality, we can assume $z_i = 0$. Let $S = \langle -h, h\rangle^3$ be the support of $v_i$. Due to the symmetry in the partitions, the averaged basis function $v_i$ is even with respect to all coordinate axes, i.e.,

$$v_i\left(x_1, x_2, x_3\right) = v_i\left(-x_1, x_2, x_3\right) = v_i\left(x_1, -x_2, x_3\right) = v_i\left(x_1, x_2, -x_3\right). \tag{17}$$

Consequently, the integral over the support of the averaged basis function $v_i$ multiplied with any odd function vanishes.

We can expand $f$ as follows

$$\begin{aligned} f(x_1, x_2, x_3) = f(0) + \sum_p f_{,p}(0)x_p + \sum_{p,q} \frac{1}{2!} f_{,pq}(0)x_p x_q \\ + \sum_{p,q,r} \frac{1}{3!} f_{,pqr}(0)x_p x_q x_r + \sum_{p,q,r,s} R_{pqrs}(x)x_p x_q x_r x_s, \end{aligned} \tag{18}$$

where the remainder $R_{pqrs}(x)$ satisfies

$$|R_{pqrs}(x)| \le \|f\|_{C^4(\overline{\Omega})}, \quad p, q, r, s \in \{1, 2, 3\}. \tag{19}$$

The higher order terms can be bounded with the triangle inequality and (19) as

$$\left| \int_S \sum_{p,q,r,s} R_{pqrs}(x)x_p x_q x_r x_s \; dx \right| \le \int_S \sum_{p,q,r,s} |R_{pqrs}(x)| \, |x_p x_q x_r x_s| \; dx \tag{20}$$

$$\le C\|f\|_{C^4(\overline{\Omega})} \, h^4 \sum_{p,q,r,s} \int_S \; dx \le C' h^7 \|f\|_{C^4(\overline{\Omega})}.$$

Using the above expansion (18) to compute $(f, v_i)$, all integrals over odd terms $x_p, x_p x_q (p \ne q), x_p x_q x_r, \ldots$ in equation (18) vanish. We are left only with the following even terms

$$\left(f\left(0\right), v_i\right), \; \left(f_{11}\left(0\right) x_1^2, v_i\right), \; \left(f_{22}\left(0\right) x_2^2, \; v_i\right), \left(f_{33}\left(0\right) x_3^2, v_i\right). \tag{21}$$

The values of these four terms above can be explicitly computed using

$$(1, c_i) = h^3, \qquad (1, t_i) = 4h^3, \qquad (1, p_i) = 6h^3, \tag{22}$$

$$\left(x_i^2, c_i\right) = \frac{1}{6}h^5, \qquad \left(x_i^2, t_i\right) = \frac{2}{3}h^5, \qquad \left(x_i^2, p_i\right) = h^5, \tag{23}$$

where $t_i$ is the sum of four tetrahedral basis functions and $p_i$ is the sum of six prismatic basis functions. By combining the above results with weights from (13), we immediately obtain

$$(f, v_i) = 6h^3 f + h^5 \Delta f + H.O.T., \tag{24}$$

where higher order terms were estimated in (20). This gives

$$|\,(f, v_i) - 6h^3 f - h^5 \Delta f| \leq Ch^7 \|f\|_{C^4(\overline{\Omega})}\,. \tag{25}$$

■

**Lemma 2.** *The averaged matrix $A_h$ is symmetric and positive definite.*

**P r o o f :** We prove that

$$\eta^T A_h \eta \geq 2\eta^T A_t \eta, \tag{26}$$

where the matrix $A_t$ is the global FE matrix arising from the tetrahedral partition (see (12)).

Using the local equation (14), we can compute

$$\eta^T A_h \eta = 24h \sum_{i=1}^{N} \eta_i^2 - 2h \sum_{i=1}^{N} \sum_{z_j \in \Diamond z_i} \eta_i \eta_j - h \sum_{i=1}^{N} \sum_{z_j \in \Box z_i} \eta_i \eta_j, \tag{27}$$

where notation $z_j \in \Diamond z_i$ denotes that we perform summation over those $j$ for which the node $z_j$ is a midpoint of some face of the cube centered at $z_i$. The symbol $z_j \in \Box z_i$ has a similar meaning. Using the estimate $2\eta_i\eta_j \leq \eta_i^2 + \eta_j^2$, we have

$$h \sum_{i=1}^{N} \sum_{z_j \in \Box z_i} \eta_i \eta_j \leq \frac{h}{2} \sum_{i=1}^{N} \sum_{z_j \in \Box z_i} \left(\eta_i^2 + \eta_j^2\right) \tag{28}$$

As $\eta_i^2$ is present in the sum at most 24-times, we have

$$h \sum_{i=1}^{N} \sum_{z_j \in \Box z_i} \eta_i \eta_j \leq 12h \sum_{i=1}^{N} \eta_i^2. \tag{29}$$

Combining equations (27), (29), and (10) completes the proof. ■

**Theorem 1.** *If $f \in C^4(\overline{\Omega})$ then*

$$\|u(z_i) - u_h(z_i)\|_h = \mathcal{O}(h^4) \quad as \quad h \to 0. \tag{30}$$

**P r o o f :** Based on Lemma 2, the matrix $A_h$ is symmetric and positive definite. Thus

$$\|A_h^{-1}\|_2 = \lambda_{min}^{-1},$$

where $\lambda_{min}$ is the smallest eigenvalue of $A_h$ and $\|\cdot\|_2$ is the standard spectral matrix norm. A lower bound for the smallest eigenvalue follows from Lemma 2 and the standard estimate of the smallest eigenvalue of $A_t$,

$$\lambda_{min} = \min_{0 \neq x \in \mathbb{R}^N} \frac{x^T A_h x}{x^T x} \geq \min_{0 \neq x \in \mathbb{R}^N} \frac{2x^T A_t x}{x^T x}. \tag{31}$$

We have $\lambda_{min} \geq Ch^3$, and thus,

$$\|A_h^{-1}\|_2 \leq Ch^{-3}. \tag{32}$$

Now, it is easy to see that (7) and (8) imply the nodal $\mathcal{O}(h^4)$-superconvergence. Indeed, we have by the triangle inequality, formulae (4), (6), and the embedding $\ell_\infty \hookrightarrow \ell_2$ that

$$\begin{aligned} \|u - u_h\|_h = \|\vec{u} - \vec{u}_h\|_h &\leq \|\vec{u} - \vec{U}_h\|_h + \|\vec{U}_h - \vec{u}_h\|_h \\ &\leq \|\vec{u} - \vec{U}_h\|_h + \|A_h^{-1}\|_2 \|h\vec{f}_h - \vec{F}_h\|_h \\ &\leq \|\vec{u} - \vec{U}_h\|_h + C\|A_h^{-1}\|_2 \|h\vec{f}_h - \vec{F}_h\|_\infty \leq C' h^4, \end{aligned}$$

where $C = \sqrt{\operatorname{meas}\Omega}$ is independent of $h$. The last inequality is based on Lemma 1, (32), and the result by Bramble [3, p. 219-220]. ∎

# 4 Numerical experiments

In this section, we present two numerical tests which are performed on the cubic domain $\Omega = (0,1)^3$.

**Test 1:** The load function $f$ is chosen so that the exact solution is

$$u(x,y,z) = x\,(1-x)\,y\,(1-y)\,z\,(1-z)\,. \tag{33}$$

Clearly, the resulting load function $f$ has the regularity required in Theorem 1, and the nodal superconvergence property should be present. The convergence of the discretization is measured in the stronger norm

$$\|u - u_h\|_\infty = \max_{i=1,\ldots,N} |u(z_i) - u_h(z_i)|$$

and visualized in Figure 6. In the end, the $\mathcal{O}(h^4)$-superconvergence in the $\|\cdot\|_\infty$-norm is observed in Table 1. This phenomenon is probably due to the

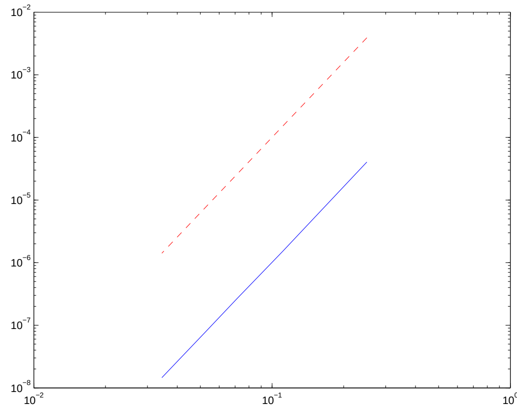


Figure 6: Error measured in the maximum norm for Test 1. The dashed line demonstrates $\mathcal{O}(h^4)$ convergence rate.

| $h$ | Cubic | Tetrahedral | Prismatic | Averaging |
|---|---|---|---|---|
| 0.250000 | 0.00162990 | 0.00140550 | 0.00050729 | 4.0509e-05 |
| 0.111110 | 0.00029282 | 0.00028510 | 9.70820e-05 | 1.5284e-06 |
| 0.071429 | 0.00012392 | 0.00012245 | 4.11000e-05 | 2.6828e-07 |
| 0.052632 | 6.66230e-05 | 6.62270e-05 | 2.21820e-05 | 7.8594e-08 |

Table 1: Nodal convergence of different approximations

existence of an improved bound for the matrix norm of the inverse of the averaged scheme.

**Test 2:** In the second test, we set the problem whose exact solution is

$$u(x, y, z) = \sin \pi x \quad \sin \pi y \sin \pi z. \tag{34}$$

The error is measured in the same norm as in Test 1. In order to calculate the entries of the stiffness matrix and the load vector, we employed higher order numerical quadrature formulae on tetrahedra from references [7, 8, 10]. Numerical results are presented in Table 2.

Tables 1 and 2 illustrate theoretical results of Theorem 1.

| $h$ | Cubic | Tetrahedral | Prismatic | Averaging |
|---|---|---|---|---|
| 0.250000 | 0.1075200 | 0.0967160 | 0.0315570 | 0.00020997 |
| 0.111110 | 0.0195730 | 0.0193270 | 0.0064288 | 9.47060e-06 |
| 0.071429 | 0.0084242 | 0.0083504 | 0.0027805 | 1.73330e-06 |
| 0.052632 | 0.0045193 | 0.0045066 | 0.0015014 | 5.09400e-07 |

Table 2: Nodal convergence of different approximations

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