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Q_p-SPACES ON BOUNDED SYMMETRIC DOMAINS

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Abstract. We generalize the theory of Q_p spaces, introduced on the unit disc in 1995 by Aulaskari, Xiao and Zhao, to bounded symmetric domains in $\mathbf{C}^d$, as well as to analogous Moebius-invariant function spaces and Bloch spaces defined using higher order derivatives; the latter generalization contains new results even in the original context of the unit disc.

1. Introduction

Let $\mathbf{D}$ be the unit disc in the complex plane $\mathbf{C}$. For $-\infty < p < \infty$, a holomorphic function f is said to belong to the space Q_p if

$$\sup_{x\in\mathbf{D}} \int_{\mathbf{D}} |f'(z)|^2 \Big(1 - \Big|\frac{x-z}{1-\overline{x}z}\Big|^2\Big)^p \,dz < \infty, \tag{1}$$

the square root of the last quantity being, by definition, the (semi)norm in Q_p. Here dz denotes the Lebesgue area measure. Since any Moebius map ϕ (i.e. biholomorphic self-map of $\mathbf{D}$) is of the form $\phi(z) = \epsilon\dfrac{x-z}{1-\overline{x}z}$, with $|\epsilon| = 1$ and $x \in \mathbf{D}$, the quantity (1) can be rewritten as

$$\begin{aligned}
\sup_{\phi\in\mathrm{Aut}(\mathbf{D})} &\int_{\mathbf{D}} |f'(z)|^2 \,(1-|\phi(z)|^2)^p \,dz \\
&= \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} \Delta|f|^2(z) \,(1-|\phi(z)|^2)^p \,dz \\
&= \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} (\widetilde{\Delta}|f|^2)(z) \,(1-|\phi(z)|^2)^p \,d\mu(z) \\
&= \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} \widetilde{\Delta}|f\circ\phi(z)|^2 \,(1-|z|^2)^p \,d\mu(z),
\end{aligned}$$

where $\widetilde{\Delta} = (1-|z|^2)^2 \dfrac{\partial^2}{\partial z \partial\overline{z}}$ and $d\mu(z) = \dfrac{dz}{(1-|z|^2)^2}$ are the $\mathrm{Aut}(\mathbf{D})$-invariant Laplacian and the $\mathrm{Aut}(\mathbf{D})$-invariant measure on $\mathbf{D}$, respectively, and $\mathrm{Aut}(\mathbf{D})$ stands for the group of all Moebius maps. (Note that we are using the normalization $\Delta = \partial\overline{\partial}$ for the Euclidean Laplacian, which differs from the usual one by a factor of 4.) From the last formula it is apparent that $f \in Q_p$ implies $f\circ\phi \in Q_p$ and f and $f\circ\phi$ have the same norm in Q_p, for all $\phi \in \mathrm{Aut}(\mathbf{D})$. That is, the space Q_p is *Moebius invariant.*

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The spaces Q_p were introduced in 1995 by Aulaskari, Xiao and Zhao [AXZ], who showed that

$$\begin{array}{rccl} & p>1 & \Longrightarrow & Q_p=\mathcal{B}, \quad \text{the Bloch space,} \\ & p=1 & \Longrightarrow & Q_p=BMOA, \\ (2) & 0\le p_1<p_2\le 1 & \Longrightarrow & Q_{p_1}\subsetneq Q_{p_2}, \\ & p=0 & \Longrightarrow & Q_p=\mathcal{D}, \quad \text{the Dirichlet space,} \\ & p<0 & \Longrightarrow & Q_p=\{\text{constants}\}. \end{array}$$

Thus the Q_p spaces provide a whole range of Möbius-invariant function spaces on $\mathbf{D}$ lying strictly between the Dirichlet space on the one hand, and $BMOA$ and the Bloch space

$$\mathcal{B}=\{f \text{ holomorphic on } \mathbf{D} : \sup_{z\in\mathbf{D}}(1-|z|^2)|f'(z)|<\infty\}$$

on the other hand.

The Q_p spaces subsequently attracted a lot of attention; see e.g. the book by Xiao [Xi] and the references therein. They were generalized to the unit ball $\mathbf{B}^d\subset\mathbf{C}^d$ in 1998 by Ouyang, Yang and Zhao [OYZ]:

$$(3)\qquad f\in Q_p(\mathbf{B}^d) \iff \sup_{\phi\in\mathrm{Aut}(\mathbf{B}^d)}\int_{\mathbf{B}^d}\widetilde{\Delta}|f\circ\phi|^2\,(1-\|z\|^2)^{pd}\,d\mu(z)<\infty,$$

where $\widetilde{\Delta}$ and $d\mu$ denote the $\mathrm{Aut}(\mathbf{B}^d)$-invariant Laplacian and the $\mathrm{Aut}(\mathbf{B}^d)$-invariant measure on $\mathbf{B}^d$, respectively, $\mathrm{Aut}(\mathbf{B}^d)$ being the group of all biholomorphic self-maps of $\mathbf{B}^d$. Again, these spaces are $\mathrm{Aut}(\mathbf{B}^d)$-invariant, and it was proved in [OYZ] that

$$\begin{array}{rccl} & p>1 & \Longrightarrow & Q_p=\mathcal{B}(\mathbf{B}^d), \quad \text{the Bloch space,} \\ & p=1 & \Longrightarrow & Q_p=BMOA(\mathbf{B}^d), \\ (4) & \dfrac{d-1}{d}<p_1<p_2\le 1 & \Longrightarrow & Q_{p_1}\subsetneq Q_{p_2}, \\ & p\le\dfrac{d-1}{d} & \Longrightarrow & Q_p=\{\text{constants}\}. \end{array}$$

Note that, in contrast to the disc, for $d>1$ the Dirichlet space does not turn up as one of the Q_p's, though in all other cases the situation is the same as for $\mathbf{D}$.

Other generalizations include Q_p spaces on smoothly bounded strictly pseudoconvex domains [AC] or the $F(p,q,s)$ spaces of Rättyä and Zhao [Ra], [Zh]. In this paper, we will consider a generalization in another direction. Note that the definitions (1) and (3) involve the invariant Laplacian $\widetilde{\Delta}$, the invariant measure $d\mu$, and the quantity $1-\|z\|^2$, whose power $(1-\|z\|^2)^{-d-1}$ is at the same time the density of $d\mu$ with respect to dz as well as — up to a constant factor — the Bergman kernel $K(z,z)$ of $\mathbf{B}^d$. Our generalization concerns the context where all of these ingredients still prevail — namely, the bounded symmetric domains.

Recall that a bounded domain $\Omega\subset\mathbf{C}^d$ is called *symmetric* if for any $x\in\Omega$ there exists $s_x\in\mathrm{Aut}(\Omega)$ such that $s_x\circ s_x=\mathrm{id}$ and x is an isolated fixed-point of s_x. One calls s_x the geodesic symmetry at x. A bounded symmetric domain is *irreducible* if it is not biholomorphic to a Cartesian product of another two nontrivial bounded symmetric domains. Any such domain can be realized as (i.e. is biholomorphic to) one which is circular with respect to the origin and convex. Its Bergman kernel

$K(x,y)$ is then of the form const $\cdot\, h(x,y)^{-p}$, where p is a positive integer, called the *genus* of Ω, and $h(x,y)$ is an irreducible polynomial, holomorphic in x and $\overline{y}$, such that $h(0,0)=1$. The measure $d\mu(z)=h(z,z)^{-p}\,dz=\text{const}\cdot K(z,z)\,dz$ on Ω is invariant under biholomorphic self-maps, i.e. $d\mu(\phi(z))=d\mu(z)$ for all $\phi\in\operatorname{Aut}(\Omega)$, the group of all biholomorphic self-maps of Ω (called *Moebius transformations*). Finally, a (linear) differential operator L on Ω is called *invariant* if

$$L(f\circ\phi)=(Lf)\circ\phi$$

for any $f\in C^\infty(\Omega)$ and any $\phi\in\operatorname{Aut}(\Omega)$.

Assume that L is an invariant differential operator such that

$$L|f|^2\geq 0 \qquad \text{for any } f \text{ holomorphic on } \Omega \tag{5}$$

and let $-\infty<\nu<\infty$. Then we define the (L-)Bloch space

$$\mathcal{B}_L := \{f \text{ holomorphic on } \Omega : \sup_\Omega L|f|^2<\infty\}, \tag{6}$$

and the $Q_{\nu,L}$-space

$$Q_{\nu,L} := \{f \text{ holomorphic on } \Omega : \sup_{\phi\in\operatorname{Aut}(\Omega)} \int_\Omega L|f\circ\phi|^2\, h^\nu\, d\mu<\infty\}, \tag{7}$$

the square roots of the indicated suprema being, by definition, the semi-norms in $\mathcal{B}_L$ and $Q_{\nu,L}$. Clearly, both $\mathcal{B}_L$ and $Q_{\nu,L}$ are Moebius invariant. Note that since $0<h(z,z)\leq 1\ \forall z\in\Omega$, we have

$$Q_{\nu_1}\subset Q_{\nu_2} \text{ continuously if } \nu_1<\nu_2.$$

For the unit disc and the unit ball, one has $h(z,z)=1-\|z\|^2$, and taking for L the invariant Laplacian, (6) and (7) reduce to the definitions of the ordinary Bloch space and Q_p spaces, respectively (the latter with $\nu=pd$).

Our goal in this paper is to provide counterparts, for general irreducible bounded symmetric domains, of the characterizations (2) and (4). In more detail, our results are the following.

First of all, we characterize the invariant differential operators L satisfying (5). It turns out that there exists a basis $\Delta_{\mathbf{m}}$ of the vector space of all invariant differential operators such that $L=\sum_{\mathbf{m}} l_{\mathbf{m}}\Delta_{\mathbf{m}}$ satisfies (5) if and only if $l_{\mathbf{m}}\geq 0\ \forall\mathbf{m}$. Here $\mathbf{m}$ runs through the set of all *signatures,* i.e. tuples $m_1,\dots,m_r$ of integers such that $m_1\geq m_2\geq\dots\geq m_r\geq 0$, r being the rank of Ω. It follows that

$$\mathcal{B}_L=\bigcap_{\mathbf{m}:\, l_{\mathbf{m}}>0}\mathcal{B}_{\mathbf{m}} \qquad\text{and}\qquad Q_{\nu,L}=\bigcap_{\mathbf{m}:\, l_{\mathbf{m}}>0} Q_{\nu,\mathbf{m}}, \tag{8}$$

with the norm in $\mathcal{B}_L$ equivalent to $\max_{\mathbf{m}:\, l_{\mathbf{m}}>0}\|\cdot\|_{\mathcal{B}_{\mathbf{m}}}$, and similarly for $Q_{\nu,L}$; here, for the sake of brevity, we have introduced the shorthand $\mathcal{B}_{\mathbf{m}}, Q_{\nu,\mathbf{m}}$ for $\mathcal{B}_{\Delta_{\mathbf{m}}}$ and $Q_{\nu,\Delta_{\mathbf{m}}}$. This reduces the study of $\mathcal{B}_L$ and $Q_{\nu,L}$ to $\mathcal{B}_{\mathbf{m}}$ and $Q_{\nu,\mathbf{m}}$, respectively.

For $\mathbf{m}=(1,0,\dots,0)$, the operator $\Delta_{\mathbf{m}}$ reduces to the ordinary invariant Laplacian $\widetilde{\Delta}$, and $\mathcal{B}_{\mathbf{m}}$ coincides with the Bloch space introduced by Timoney [Ti].

For Ω a domain of tube type with $s:=\frac{d}{r}$ an integer and $\mathbf{m}=(s,\dots,s)$, the Bloch space $\mathcal{B}_{\mathbf{m}}$ was studied by the first author [A2] in connection with Hankel operators on the top quotient of the composition series. (See Section 2 below the various definitions.)

Our first main result is then that for $\nu > p-1$, we have a full analogue of the first lines in (2) and (4), namely,

$$Q_{\nu,\mathbf{m}} = \mathcal{B}_{\mathbf{m}}, \qquad \text{with equivalent norms.} \tag{9}$$

Further, the Bloch space $\mathcal{B}_{\mathbf{m}}$ depends only on the *height* $q(\mathbf{m})$ of the signature $\mathbf{m}$, i.e.

$$\mathcal{B}_{\mathbf{m}} = \mathcal{B}_{\mathbf{n}} \qquad \text{if } q(\mathbf{m}) = q(\mathbf{n}) \text{ (with equivalent norms);}$$

here

$$q(\mathbf{m}) = \operatorname{card}\{j : \frac{j-1}{2}a \in \mathbf{Z} \text{ and } m_j > \frac{j-1}{2}a\} \tag{10}$$

where a is the characteristic multiplicity of Ω. (See again Section 2 below for the various definitions.)

Note that (9) and (10) give new results even in the original context of the unit disc $\mathbf{D}$: for instance, for any $k \geq 1$, the Bloch space seminorm is equivalent to the square root of

$$\sup_{x\in\mathbf{D}} \Delta^k |f\circ\phi_x|^2(0) = \sup_{x\in\mathbf{D}} |(f\circ\phi_x)^{(k)}(0)|^2,$$

where $\phi_x(z) = \frac{x-z}{1-\overline{x}z}$. Similarly for the unit ball.

For $\nu \leq p-1$, the situation turns out to be more subtle. The spaces $Q_{\nu,\mathbf{m}}$ always contain the set

$$\mathcal{N}_{\mathbf{m}} := \{f \text{ holomorphic on } \Omega : \Delta_{\mathbf{m}}|f|^2 \equiv 0\}.$$

(Again, as with $\mathcal{B}_{\mathbf{m}}$, this set in fact depends not on $\mathbf{m}$ but only on the height $q(\mathbf{m})$.) We say that $Q_{\nu,\mathbf{m}}$ is *trivial* if $Q_{\nu,\mathbf{m}} = \mathcal{N}_{\mathbf{m}}$. This is always the case if $\nu < 0$. It may happen that $Q_{\nu,\mathbf{m}}$ is trivial even for all $\nu \leq p-1$: for instance, this is the case for $\mathbf{m} = (1,0,\dots,0)$ — that is, when $\Delta_{\mathbf{m}}$ is just the invariant Laplacian — for any irreducible bounded symmetric domain Ω of rank > 1 (i.e. not biholomorphic to the ball $\mathbf{B}^d$); this is in sharp contrast with (2) and (4). On the other hand, it may happen that $Q_{\nu,\mathbf{m}}$ is nontrivial for all $\nu \geq 0$: this is the case, for instance, for tube type domains Ω and $\mathbf{m} = (s,\dots,s)$, if $s = d/r$ is an integer.

In general, there exists an integer or half-integer $\rho_{\mathbf{m}}$ such that

$$Q_{\nu,\mathbf{m}} \text{ is nontrivial} \iff \quad \nu \geq 0 \text{ and } \nu > p-1-\rho_{\mathbf{m}}.$$

We have $\rho_{(0,\dots,0)} = 0$ for any Ω, $\rho_{\mathbf{m}} = 2$ for $\Omega = \mathbf{D}$ and $\mathbf{m} = (1)$, $\rho_{\mathbf{m}} = 1$ for $\Omega = \mathbf{B}^d$, $d > 1$, and $\mathbf{m} = (1)$, $\rho_{(1,0,\dots,0)} = 0$ for Ω not biholomorphic to $\mathbf{B}^d$, and $\rho_{(s,\dots,s)} = p$ for Ω a tube type domain with $s = \frac{d}{r}$ an integer. For general Ω and $\mathbf{m}$, the exact value of $\rho_{\mathbf{m}}$ is, unfortunately, unknown.

The paper is organized as follows. In Section 2 we review various prerequisites on bounded symmetric domains. In Section 3 we establish several auxiliary results, including the characterization of invariant differential operators satisfying (5) and the proof of (8). The main results are established in Section 4. The last Section 5 contains some concluding remarks, open problems, and an additional material on certain Pieri-type coefficients.

A preliminary version of this paper, containing only a selection of the results and with the more difficult parts of their proofs omitted, appeared in the proceedings of the 13th ICFIDCAA conference [E]; the second author thanks the organizers for the invitation.

2. Bounded symmetric domains

Throughout the rest of this paper, Ω will be an irreducible bounded symmetric domain in $\mathbf{C}^d$ in its Harish-Chandra realization (i.e. a Cartan domain). As usual we denote by $G = \mathrm{Aut}(\Omega)$ the group of all biholomorphic self-maps of Ω, and by K the stabilizer in G of the origin $0 \in \Omega$. Then K consists precisely of the unitary maps on $\mathbf{C}^d$ that preserve Ω, and Ω is isomorphic to the coset space G/K. We further denote by r, a, b and p the rank, the characteristic multiplicities and the genus of Ω, respectively, so that

$$p = (r-1)a + b + 2, \qquad d = \frac{r(r-1)}{2}a + rb + r.$$

If $b = 0$, Ω is said to be of *tube type.*

Irreducible bounded symmetric domains were completely classified by E. Cartan. There are four infinite series of such domains plus two exceptional domains in $\mathbf{C}^{16}$ and $\mathbf{C}^{27}$. For future reference, we include a table with brief descriptions of these domains and with the corresponding values of r, a, b, p and d.

Domain	Description	
I_{mn}	$Z \in \mathbf{C}^{m\times n}$: $\|Z\|_{\mathbf{C}^n\to\mathbf{C}^m} < 1$ $r = m$, $a = 2$, $b = n-m$, $p = n+m$, $d = mn$	$n \geq m \geq 1$
II_n	$Z \in I_{nn}$, $Z = Z^t$ $r = n$, $a = 1$, $b = 0$, $p = n+1$, $d = \frac{1}{2}n(n+1)$	$n \geq 2$
III_m	$Z \in I_{mm}$, $Z = -Z^t$ $r = [\frac{m}{2}]$, $a = 4$, $b = 2(m-2r)$, $p = 2m-2$, $d = \frac{1}{2}m(m-1)$	$m \geq 5$
IV_n	$Z \in \mathbf{C}^{n\times 1}$, $\lvert Z^tZ\rvert < 1$, $1 + \lvert Z^tZ\rvert^2 - 2Z^*Z > 0$ $r = 2$, $a = n-2$, $b = 0$, $p = d = n$	$n \geq 5$
V	$Z \in \mathbf{O}^{1\times 2}$, $\|Z\| < 1$ $r = 2$, $a = 6$, $b = 4$, $p = 12$, $d = 16$	
VI	$Z \in \mathbf{O}^{3\times 3}$, $Z = Z^*$, $\|Z\| < 1$ $r = 3$, $a = 8$, $b = 0$, $p = 18$, $d = 27$	

The unit balls $\mathbf{B}^d = I_{1d}$ are the only bounded symmetric domains of rank 1, and the only bounded symmetric domains with smooth boundary.

For $x \in \Omega$, ϕ_x will denote the (unique) geodesic symmetry which interchanges x and the origin, i.e.

$$\phi_x \circ \phi_x = \mathrm{id},\ \phi_x(0) = x,\ \phi_x(x) = 0, \tag{11}$$

and ϕ_x has only an isolated fixed-point. (In fact, ϕ_x has only one fixed point, namely the geodesic mid-point between 0 and x.) We will also use the *transvections*

$$\gamma_x(z) := \phi_x(-z)$$

which map the origin into x. Note that from the definition of K it is immediate that any $\phi \in G$ is of the form $\phi = \phi_x k = \gamma_x k'$, where $k, k' \in K$ and $x \in \Omega$. (In fact $x = \phi(0)$.)

It is known that the ambient space $\mathbf{C}^d =: Z$ possesses a structure of *Jordan-Banach $*$-triple system* (or *JB*-triple* for short) for which Ω is the open unit ball.

That is, there exists a Jordan triple product

$$\{\cdot,\cdot,\cdot\} : Z \times Z \times Z \to Z, \qquad x, y, z \mapsto \{x, y, z\},$$

(linear and symmetric in x, z and anti-linear in y) such that

$$\Omega = \{z \in Z : \|\{z, z, \cdot\}\| < 1\}.$$

Moreover, if one uses the notation, for $x, y \in Z$,

$$\begin{aligned} D(x, y) &= \{x, y, \cdot\} : \ Z \to Z, \\ Q(x) &= \{x, \cdot, x\} : \ Z \to Z, \end{aligned}$$

then for every $x \in \Omega$, $D(x, x)$ is Hermitian and has nonnegative spectrum, and $iD(x, x)$ is a triple derivation. The linear operator

$$B(x, y) = I - 2D(x, y) + Q(x)Q(y) \tag{12}$$

on Z is called the *Bergman operator*.

Two vectors $x, y \in Z$ are said to be *orthogonal* (in the Jordan-theoretic sense) if $D(x, y) = 0$, and a vector $v \in Z$ is called a *tripotent* if $\{v, v, v\} = v$. Any maximal set $e_1, \dots, e_r$ of pairwise orthogonal nonzero tripotents is called a *Jordan frame*; its cardinality r is independent of the frame and equal to the rank of Ω. For any tripotent v, the ambient space admits the *Peirce decomposition*

$$Z = Z_0(v) \oplus Z_{1/2}(v) \oplus Z_1(v)$$

into the orthogonal components

$$Z_{j/2}(v) := \{z \in Z : D(v, v)z = \frac{j}{2} z\}.$$

(The orthogonality is only with respect to the inner product in $\mathbf{C}^d$, not in the triple-product (Jordan-theoretic) sense.) For any Jordan frame $e_1, \dots, e_r$, we similarly have the *joint Peirce decomposition*

$$Z = \bigoplus_{0 \le i \le j \le r} Z_{ij} \tag{13}$$

with

$$Z_{ij} = \{z \in Z : D(e_k, e_k)z = \frac{\delta_{ik} + \delta_{jk}}{2} \ \forall k = 1, \dots, r\}. \tag{14}$$

In terms of the Jordan triple data, the geodesic symmetries and transvections (11) are given by

$$\begin{aligned} \phi_x(z) &= x - B(x, x)^{1/2} B(z, x)^{-1} (z - Q(z)x) \\ &= x - B(x, x)^{1/2} (I - D(z, x))^{-1} z, \\ \gamma_x(z) &= x + B(x, x)^{1/2} B(z, -x)^{-1} (z + Q(z)x) \\ &= x + B(x, x)^{1/2} (I + D(z, x))^{-1} z. \end{aligned} \tag{15}$$

Given any Jordan frame $e_1, \dots, e_r$ — which we choose and fix once and for all from now on — any $z \in Z$ has a *polar decomposition*

$$z = k(t_1 e_1 + \dots + t_r e_r) \tag{16}$$

with $k \in K$ and $t_1 \ge t_2 \ge \dots \ge t_r \ge 0$; the numbers $t_1, \dots, t_r$, called the *singular numbers* of z, are determined uniquely, but k need not be (it is if all the t_j are distinct). Further, $z \in \Omega$ if and only if $t_1 < 1$, $z \in \partial\Omega$ if and only if $t_1 = 1$, and z

belongs to the Shilov boundary $\partial_e\Omega$ of Ω if and only if $t_1 = \cdots = t_r = 1$; that is, if and only if $z = ke$, where $e = e_1 + \cdots + e_r$ is a *maximal tripotent*.

Since the Jordan triple product is invariant under K (i.e. $\{kx, ky, kx\} = k\{x, y, z\}$ $\forall k \in K$), it is immediate from (14) that under the decomposition (13), the Bergman operator $B(z, z)$ with z as in (16) is given by

$$B(z,z)|_{Z_{ij}} = (1 - t_i^2)(1 - t_j^2)I|_{Z_{ij}} \tag{17}$$

(where $t_0 := 0$).

There exists a unique polynomial $h(x, y)$ on $\mathbf{C}^d \times \mathbf{C}^d$, holomorphic in x and anti-holomorphic in y, which is K-invariant, in the sense that

$$h(kx, ky) = h(x, y) \qquad \forall k \in K,$$

and satisfies

$$h(z, z) = \prod_{j=1}^{r} (1 - t_j^2) \quad \text{ for } z \text{ as in (16).}$$

It is known that $h(x, y)$ is irreducible, of degree r in x as well as in $\overline{y}$, and $h(x, 0) = h(0, x) = 1$ $\forall x \in \mathbf{C}^d$; also, $h(x, y)^p = \det B(x, y)$. Further, the measure

$$h(z, z)^{\nu-p}\, dz \tag{18}$$

is finite if and only if $\nu > p-1$, and the corresponding weighted Bergman kernel — i.e. the reproducing kernel of the space of all holomorphic functions on Ω square-integrable with respect to (18) — is equal to

$$K_\nu(x, y) = c_\nu h(x, y)^{-\nu} \tag{19}$$

for some constant c_ν. In particular, for $\nu = p$, the ordinary (i.e. unweighted) Bergman kernel of Ω is equal to

$$K(x, y) = \frac{1}{\mathrm{vol}(\Omega)}\, h(x, y)^{-p}.$$

Finally, the measure

$$d\mu(z) = \frac{dz}{h(z, z)^p} = \mathrm{vol}(\Omega)\, K(z, z)\, dz$$

is the unique (up to constant multiples) G-invariant measure on Ω.

In the polar coordinates (16), the measures (18) assume the form

$$\begin{aligned} &\int_\Omega f(z)\, h(z, z)^\nu\, d\mu(z) = \\ &c \int_{[0,1]^r} \int_K f(k(\sum_{j=1}^r t_j e_j)) \prod_{j=1}^r (1 - t_j^2)^{\nu-p} \prod_{j=1}^r t_j^{2b+1} \prod_{1\le i<j\le r} |t_i^2 - t_j^2|^a\, dk\, d\mathbf{t}, \end{aligned} \tag{20}$$

where $d\mathbf{t} = dt_1 \dots dt_r$, dk is the normalized Haar measure on the (compact) group K, and c is a constant whose exact value will not be needed and which we will therefore omit in the sequel. (Alternatively, choosing $c = 1$ amounts to a special choice of the invariant measure μ.)

For the sake of brevity, we will often abbreviate $h(z, z)$ just to $h(z)$ (or even to h) if there is no danger of confusion.

Let $\mathcal{P}$ denote the vector space of all (holomorphic) polynomials on $\mathbf{C}^d$. We endow $\mathcal{P}$ with the *Fock* (or *Fischer*) inner product

$$\begin{aligned}\langle f,g\rangle_F &:= \pi^{-d}\int_{\mathbf{C}^d} f(z)\,\overline{g(z)}\,e^{-|z|^2}\,dz \\ &= (f(\partial)g^*)(0) = (g^*(\partial)f)(0),\end{aligned} \tag{21}$$

where

$$g^*(z) := \overline{g(\overline{z})}.$$

This makes $\mathcal{P}$ into a pre-Hilbert space, and the action

$$f \mapsto f\circ k, \qquad k\in K,$$

is a unitary representation of K on $\mathcal{P}$. It is a deep result of W. Schmid [Sch] that this representation has a multiplicity-free decomposition into irreducibles

$$\mathcal{P} = \bigoplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}}$$

where $\mathbf{m}$ ranges over all <u>signatures</u>, i.e. r-tuples $\mathbf{m} = (m_1, m_2, \dots, m_r) \in \mathbf{Z}^r$ satisfying $m_1 \geq m_2 \geq \dots \geq m_r \geq 0$. Polynomials in $\mathcal{P}_{\mathbf{m}}$ are homogeneous of degree $|\mathbf{m}| := m_1 + m_2 + \dots + m_r$; in particular, $\mathcal{P}_{(0)}$ are the constants and $\mathcal{P}_{(1)}$ the linear polynomials. Any holomorphic function thus has a decomposition $f = \sum_{\mathbf{m}} f_{\mathbf{m}}$, $f_{\mathbf{m}} \in \mathcal{P}_{\mathbf{m}}$, which refines the usual homogeneous expansion.

Since the spaces $\mathcal{P}_{\mathbf{m}}$ are finite dimensional, they automatically possess a reproducing kernel: there exist polynomials $K_{\mathbf{m}}(x,y)$ on $\mathbf{C}^d\times\mathbf{C}^d$, holomorphic in x and $\overline{y}$, such that for each $f\in\mathcal{P}_{\mathbf{m}}$ and $y\in\mathbf{C}^d$, $f(y) = \langle f, K_{\mathbf{m}}(\cdot,y)\rangle_F$. In terms of any orthonormal basis $\{\psi_j\}_{j=1}^{d_{\mathbf{m}}}$ of $\mathcal{P}_{\mathbf{m}}$, where $d_{\mathbf{m}} := \dim\mathcal{P}_{\mathbf{m}}$, $K_{\mathbf{m}}$ is given by

$$K_{\mathbf{m}}(x,y) = \sum_{j=1}^{d_{\mathbf{m}}} \psi_j(x)\overline{\psi_j(y)}. \tag{22}$$

From the definition of the spaces $\mathcal{P}_{\mathbf{m}}$ it also follows that the kernels $K_{\mathbf{m}}(x,y)$ are K-invariant.

It is a consequence of Schur's lemma from representation theory that for any K-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{P}$, $\mathcal{P}_{\mathbf{m}}$ and $\mathcal{P}_{\mathbf{n}}$ are orthogonal if $\mathbf{m}\neq\mathbf{n}$, while on each $\mathcal{P}_{\mathbf{m}}$, $\langle\cdot,\cdot\rangle$ is proportional to $\langle\cdot,\cdot\rangle_F$. In particular, for the inner product

$$\langle f,g\rangle_\nu := c_\nu \int_\Omega f(z)\,\overline{g(z)}\,h(z,z)^\nu\,d\mu(z) \qquad (\nu > p-1),$$

(with c_ν as in (19)) we have, for any $f_{\mathbf{m}}\in\mathcal{P}_{\mathbf{m}}$ and $g_{\mathbf{n}}\in\mathcal{P}_{\mathbf{n}}$,

$$\langle f_{\mathbf{m}}, g_{\mathbf{n}}\rangle_\nu = \frac{\langle f_{\mathbf{m}}, g_{\mathbf{n}}\rangle_F}{(\nu)_{\mathbf{m}}} \tag{23}$$

(cf. [FK1]), where $(\nu)_{\mathbf{m}}$ is the *generalized Pochhammer symbol*

$$(\nu)_{\mathbf{m}} := (\nu)_{m_1}(\nu-\frac{a}{2})_{m_2}\dots(\nu-\frac{r-1}{2}a)_{m_r};$$

here

$$(\nu)_k := \nu(\nu+1)\dots(\nu+k-1) \qquad \Big(= \frac{\Gamma(\nu+k)}{\Gamma(\nu)} \text{ if } \nu\neq 0,-1,-2,\dots,\Big)$$

is the ordinary Pochhammer symbol.

A consequence of the relation (23) is the *Faraut-Koranyi formula*

$$h(x,y)^{-\nu} = \sum_{\mathbf{m}} (\nu)_{\mathbf{m}} K_{\mathbf{m}}(x,y) \tag{24}$$

relating the reproducing kernels K_ν from (19) and $K_{\mathbf{m}}$ from (22).

For a signature $\mathbf{m}$, consider the function

$$\nu \ \mapsto \ (\nu)_{\mathbf{m}}, \qquad \nu \in \mathbf{C}.$$

Let $q(\mathbf{m})$ (the *height* of the signature $\mathbf{m}$) be the multiplicity of zero of this function at $\nu = 0$:

$$q(\mathbf{m}) := \operatorname{card}\{j : \ m_j > \frac{j-1}{2} a \in \mathbf{Z}\}.$$

Denote by q the maximum possible value of $q(\mathbf{m})$; that is,

$$q = \begin{cases} r & a \text{ even}, \\ \left[\dfrac{r+1}{2}\right] & a \text{ odd}. \end{cases}$$

For $-1 \le \ell \le q$, let

$$\mathcal{M}_\ell = \{f = \sum_{\mathbf{m}} f_{\mathbf{m}} \text{ holomorphic} : \ f_{\mathbf{m}} = 0 \text{ if } q(\mathbf{m}) > \ell\}.$$

Thus, in particular,

$$\begin{gathered} \mathcal{M}_{-1} \subset \mathcal{M}_0 \subset \mathcal{M}_1 \subset \dots \subset \mathcal{M}_q, \\ \mathcal{M}_{-1} = \{0\}, \quad \mathcal{M}_0 = \{\text{constants}\}, \quad \mathcal{M}_q = \{\text{all holomorphic functions}\}. \end{gathered} \tag{25}$$

The sequence (25) is known as the *composition series* of Ω. It is a deep result of Ørsted (in the special case of $\Omega = I_{nn}$) and Faraut and Koranyi (in general) that

$$\text{each } \mathcal{M}_\ell \text{ is } G\text{-invariant} \tag{26}$$

and that for any G-invariant space E of holomorphic functions on Ω on which the action $f \mapsto f \circ k$ of K is strongly continuous,

$$E \setminus \mathcal{M}_{\ell-1} \neq \emptyset \implies \mathcal{P} \cap \mathcal{M}_\ell \subset E. \tag{27}$$

In other words, if E is not wholly contained in $\mathcal{M}_{\ell-1}$, then E contains every $\mathcal{P}_{\mathbf{m}}$ with $q(\mathbf{m}) = \ell$.

Standard references for the material in this section are [A1], [Lo], [FK1], [FK2], or [Up].

3. Invariant differential operators and some convolutions

Recall that we have called a (linear) differential operator L on Ω *invariant* if

$$L(f \circ \phi) = (Lf) \circ \phi \qquad \forall \phi \in G = \operatorname{Aut}(\Omega).$$

It is well known that on the unit disc, invariant differential operators are precisely the polynomials of the invariant Laplacian $\widetilde{\Delta} = (1-|z|^2)^2 \Delta$; the same is true for $\mathbf{B}^d$. For a general Cartan domain, the situation is more complicated: namely, there exist r commuting algebraically independent differential operators $\Delta_1, \dots, \Delta_r$, where r is the rank, which can be chosen to have orders $2, 4, \dots, 2r$, respectively, such that the algebra of all invariant differential operators consists precisely of all polynomials in $\Delta_1, \dots, \Delta_r$. In particular, the monomials $\Delta_1^{n_1} \dots \Delta_r^{n_r}$ form a linear basis of all invariant differential operators. However, often it is much more convenient to use another basis, the construction of which we now describe.

For any invariant differential operator L, let L_0 be the (non-invariant) constant-coefficient linear differential operator obtained upon freezing the coefficients of L at the origin; that is, $Lf(0) =: L_0 f(0)$. From the invariance of L it follows that

$$k \in G,\ k0 = 0 \implies L_0(f \circ k) = (L_0 f) \circ k$$

(i.e. L_0 is K-invariant) and

$$Lf(z) = L_0(f \circ \phi_z)(0). \tag{28}$$

Conversely, if L_0 is a K-invariant constant-coefficient differential operator, then the recipe (28) clearly defines an invariant differential operator L on Ω. Thus there is a one-to-one correspondence between invariant linear differential operators on Ω and K-invariant linear constant-coefficient differential operators on $\mathbf{C}^d$.

Further, any constant-coefficient linear differential operator L_0 can be written in the form $L_0 = p(\partial, \overline{\partial})$ for some polynomial p on $\mathbf{C}^d \times \mathbf{C}^d$. It is not difficult to see that such operator is K-invariant if and only if the polynomial p is K-invariant, in the sense that $p(x, \overline{y}) = p(kx, \overline{ky})$ for all $x, y \in \mathbf{C}^d$ and $k \in K$. Combining this with the observation in the preceding paragraph, we thus see that the recipe

$$p(x, \overline{y}) \mapsto L_p, \qquad L_p f(x) := p(\partial, \overline{\partial})(f \circ \phi_x)(0) = p(\partial, \overline{\partial})(f \circ \gamma_x)(0)$$

sets up a one-to-one correspondence between invariant differential operators on Ω and K-invariant sesqui-holomorphic polynomials on $\mathbf{C}^d \times \mathbf{C}^d$.

Example 1. Since K consists of unitary maps, the simplest K-invariant polynomial (apart from the constants) is $p(x, \overline{y}) = \langle x, y \rangle$. Then $p(\partial, \partial) = \sum_{j=1}^d \partial_j \overline{\partial}_j = \Delta$, and the corresponding invariant differential operator is

$$Lf(x) = \Delta(f \circ \phi_x)(0).$$

This operator is called the invariant Laplacian on Ω; it coincides with the Laplace-Beltrami operator with respect to the Bergman metric on Ω. Note that for f holomorphic,

$$L|f|^2(x) = \sum_{j=1}^d \Big| \frac{\partial (f \circ \phi_x)(0)}{\partial z_j} \Big|^2 = \|\partial (f \circ \phi_x)(0)\|^2 \tag{29}$$

is the norm-squared of what we might call the *invariant holomorphic gradient* of f.

We have seen in the preceding section that for each signature $\mathbf{m}$, the reproducing kernel $K_{\mathbf{m}}(x, y)$ of the Peter-Weyl space $\mathcal{P}_{\mathbf{m}}$ is a K-invariant polynomial on $\mathbf{C}^d \times \mathbf{C}^d$. By the discussion above, $K_{\mathbf{m}}$ therefore defines an invariant differential operator

$$\Delta_{\mathbf{m}} f(x) := K_{\mathbf{m}}(\partial, \partial)(f \circ \phi_x)(0). \tag{30}$$

Proposition 2. *The polynomials $K_{\mathbf{m}}(x, y)$ form a basis of the space of all K-invariant sesqui-holomorphic polynomials on $\mathbf{C}^d \times \mathbf{C}^d$. Consequently, the operators $\Delta_{\mathbf{m}}$ form a basis for the space of all invariant differential operators on Ω.*

Proof. Any polynomial $p(x, \overline{y})$ on $\mathbf{C}^d \times \mathbf{C}^d$ is uniquely determined by its restriction to $\Omega \times \Omega$ and, hence (by holomorphy), by its restriction to the Shilov boundary $\partial_e \Omega \times \partial_e \Omega$ of $\Omega \times \Omega$; that is, by its values $p(k_1 e, \overline{k_2 e})$ where e is a fixed maximal tripotent and $k_1, k_2 \in K$. By K-invariance, $p(k_1 e, \overline{k_2 e}) = p(k_2^{-1} k_1 e, \overline{e})$, so p is actually uniquely determined by its values $p(ke, \overline{e})$ for $k \in K$. Now $f(x) := p(x, \overline{e})$ is a holomorphic polynomial on $\mathbf{C}^d$, and $f(lx) = p(lx, \overline{e}) = p(lx, \overline{le}) = p(x, \overline{e}) = f(x)$ for any $l \in K$ which fixes e; that is, letting L stand for the stabilizer of e in K,

$f(x)$ is L-invariant. If $f = \sum_{\mathbf{m}} f_{\mathbf{m}}$ is the Peter-Weyl decomposition of f, it follows that each $f_{\mathbf{m}}$ is also L-invariant. However, it is known [FK1, Theorem 2.1] that the only L-invariant polynomial in $\mathcal{P}_{\mathbf{m}}$, up to constant multiples, is $K_{\mathbf{m}}(\cdot, e)$. Thus $f = \sum_{\mathbf{m}} c_{\mathbf{m}} K_{\mathbf{m}}(\cdot, e)$ for some constants $c_{\mathbf{m}} \in \mathbf{C}$, which implies (tracing back the arguments from the beginning of this proof) that $p(x, \overline{y}) = \sum_{\mathbf{m}} c_{\mathbf{m}} K_{\mathbf{m}}(x, y)$.

The uniqueness of the $c_{\mathbf{m}}$ is obvious. □

The following result makes it clear why the basis $\Delta_{\mathbf{m}}$ is very appropriate for our applications to the Q_ν-spaces.

Proposition 3. *An invariant differential operator*

$$L = \sum_{\mathbf{m}} l_{\mathbf{m}} \Delta_{\mathbf{m}}$$

satisfies $L|f|^2 \geq 0$ *for all holomorphic* f *if and only if*

$$l_{\mathbf{m}} \geq 0 \qquad \forall \mathbf{m}.$$

Proof. From (22) and (30) we see that for any f holomorphic,

$$\Delta_{\mathbf{m}} |f|^2(x) = \sum_j |\psi_j(\partial)(f \circ \phi_x)(0)|^2 \geq 0.$$

Thus $l_{\mathbf{m}} \geq 0\ \forall \mathbf{m}$ implies $L|f|^2 \geq 0$.

On the other hand, if $f = \sum_{\mathbf{n}} f_{\mathbf{n}}$ then

$$\begin{aligned} \Delta_{\mathbf{m}} |f|^2(0) &= \sum_j |\psi_j(\partial) f(0)|^2 \\ &= \sum_j |\langle \psi_j, f^* \rangle_F|^2 \\ &= \|f^*_{\mathbf{m}}\|_F^2 = \|f_{\mathbf{m}}\|_F^2. \end{aligned}$$

Thus if $l_{\mathbf{m}} < 0$ for some $\mathbf{m}$, then $L|f_{\mathbf{m}}|^2(0) < 0$ for any nonzero $f_{\mathbf{m}} \in \mathcal{P}_{\mathbf{m}}$. □

Recall that for any L as in the last proposition and $\nu \in \mathbf{R}$, we have defined the L-Bloch space and the $Q_{\nu,L}$-space, respectively, by

$$\mathcal{B}_L = \{f \text{ holomorphic on } \Omega : \sup_\Omega L|f|^2 < \infty\},$$

$$Q_{\nu,L} = \{f \text{ holomorphic on } \Omega : \sup_{\phi \in \mathrm{Aut}(\Omega)} \int_\Omega L|f \circ \phi|^2\, h^\nu \, d\mu < \infty\},$$

the square roots of the indicated suprema being, by definition, the semi-norms in these Moebius invariant spaces. We have also agreed to denote, for brevity, $\mathcal{B}_L$ and $Q_{\nu,L}$ simply by $\mathcal{B}_{\mathbf{m}}$ and $Q_{\nu,\mathbf{m}}$ if $L = \Delta_{\mathbf{m}}$.

Corollary 4. *For any* L *as in the preceding proposition and* $\nu \in \mathbf{R}$,

$$\mathcal{B}_L = \bigcap_{\mathbf{m}:\, l_{\mathbf{m}} > 0} \mathcal{B}_{\mathbf{m}}, \qquad Q_{\nu,L} = \bigcap_{\mathbf{m}:\, l_{\mathbf{m}} > 0} Q_{\nu,\mathbf{m}},$$

with the norm in $\mathcal{B}_L$ *equivalent to* $\max_{\mathbf{m}:\, l_{\mathbf{m}} > 0} \|\cdot\|_{\mathcal{B}_{\mathbf{m}}}$, *and similarly for* $Q_{\nu,L}$.

Proof. Immediate from the fact that there can be only finitely many $\mathbf{m}$ for which $l_{\mathbf{m}} \neq 0$, and the fact that

$$\|f\|^2_{\mathcal{B}_L} = \sup_\Omega \sum_{\mathbf{m}} l_{\mathbf{m}} \Delta_{\mathbf{m}} |f|^2$$

satisfies, on the one hand,

$$\|f\|^2_{\mathcal{B}_L} \geq l_{\mathbf{m}} \sup_{\Omega} \Delta_{\mathbf{m}} |f|^2 = l_{\mathbf{m}} \|f\|^2_{\mathcal{B}_{\mathbf{m}}}$$

for each $\mathbf{m}$, hence also

$$\|f\|^2_{\mathcal{B}_L} \geq \Big(\min_{\mathbf{m}\colon l_{\mathbf{m}}>0} l_{\mathbf{m}} \Big) \Big(\max_{\mathbf{m}\colon l_{\mathbf{m}}>0} \|f\|^2_{\mathcal{B}_{\mathbf{m}}} \Big);$$

and on the other hand

$$\begin{aligned} \|f\|^2_{\mathcal{B}_L} &\leq \Big(\sum_{\mathbf{m}} l_{\mathbf{m}} \Big) \sup_{\mathbf{m}\colon l_{\mathbf{m}}>0} \sup_{\Omega} \Delta_{\mathbf{m}} |f|^2 \\ &= \Big(\sum_{\mathbf{m}} l_{\mathbf{m}} \Big) \Big(\max_{\mathbf{m}\colon l_{\mathbf{m}}>0} \|f\|^2_{\mathcal{B}_{\mathbf{m}}} \Big). \end{aligned}$$

Similarly for $Q_{\nu,L}$. □

The next proposition will be useful on several occasions later on. Note that the integral there is nothing but the value at $\phi \in G$ of the convolution $h^\rho * h^\nu$ of the two functions h^ρ, h^ν on G (upon lifting them from $\Omega \cong G/K$ to G); thus the proposition gives a characterization of the pairs ρ, ν for which $h^\rho * h^\nu$ is bounded.

Proposition 5. *For $\rho, \nu \in \mathbf{R}$, the supremum*

$$\sup_{\phi \in G} \int_\Omega h(\phi(z))^\rho \, h(z)^\nu \, d\mu(z) \tag{31}$$

is finite if and only if

$$\nu \geq 0, \ \rho \geq 0, \ \text{and } \rho + \nu > p - 1.$$

Proof. For $\phi = \mathrm{id}$, the integral becomes

$$\int_\Omega h(z)^{\rho+\nu} \, d\mu(z),$$

which we know from Section 2 to be finite if and only if $\nu + \rho > p - 1$. Thus the supremum is certainly infinite if $\nu + \rho \leq p - 1$.

Next, assume that $\nu + \rho > p - 1$ and, say, $\rho < 0$. Then $\nu > p - 1$, so that $d\mu_\nu(z) := h(z)^\nu \, d\mu(z)$ is a finite measure. Let e be a maximal tripotent and $0 < t < 1$. It was shown in [AE, Section 4] that as $t \nearrow 1$, $\gamma_{te}(z) \to e$ for any $z \in \Omega$, and, hence, $h(\gamma_{te}(z))^\rho \to +\infty$. For each $N > 0$, denote temporarily $f_N(t,z) := \min\{N, h(\gamma_{te}(z))^\rho\}$. Then $f_N(t,z) \to N \ \forall z \in \Omega$ as $t \nearrow 1$, so by the Lebesgue Dominated Convergence Theorem

$$\int_\Omega f_N(t,z) \, d\mu_\nu(z) \to N \, \mu_\nu(\Omega).$$

Since $\int_\Omega h(\gamma_{te}(z))^\rho \, d\mu_\nu(z) \geq \int_\Omega f_N(t,z) \, d\mu_\nu(z)$ for any N, it follows that

$$\lim_{t \nearrow 1} \int_\Omega h(\gamma_{te}(z))^\rho \, d\mu_\nu(z) = +\infty.$$

Thus the supremum (31) is infinite in this case as well.

Owing to the invariance of the measure $d\mu$, the integral in (31) remains unchanged if ϕ is replaced by ϕ^{-1} and ρ and ν are interchanged. It follows that the supremum is infinite also if $\nu + \rho > p - 1$ and $\nu < 0$.

Thus it only remains to show that (31) is finite if $\nu \geq 0$, $\rho \geq 0$ and $\rho + \nu > p - 1$.

However, then by the Hölder inequality

$$\begin{aligned}
&\int_\Omega h(\phi(z))^\rho\, h(z)^\nu\, d\mu(z) \\
&\qquad \le \Big(\int_\Omega h(\phi(z))^{\rho+\nu}\, d\mu(z)\Big)^{\frac{\rho}{\rho+\nu}} \Big(\int_\Omega h(z)^{\rho+\nu}\, d\mu(z)\Big)^{\frac{\nu}{\rho+\nu}} \\
&\qquad = \Big(\int_\Omega h(z)^{\rho+\nu}\, d\mu(z)\Big)^{\frac{\rho}{\rho+\nu}} \Big(\int_\Omega h(z)^{\rho+\nu}\, d\mu(z)\Big)^{\frac{\nu}{\rho+\nu}} \\
&\qquad = \mu_{\rho+\nu}(\Omega) < \infty,
\end{aligned}$$

completing the proof. $\square$

Remark 6. An alternative way of proving that (31) is infinite if $\rho+\nu > p-1$ but, say, $\nu < 0$ is by noting that

$$\int_\Omega h(\phi(z))^\rho\, h(z)^\nu\, d\mu(z) = c_\nu^{-1} h(x)^\rho {}_2\mathcal{F}_1(\rho,\rho;\rho+\nu;x,x),$$

where $x = \phi^{-1}(0)$ and ${}_2\mathcal{F}_1$ is the Faraut-Koranyi-Yan hypergeometric function (cf. Section 4 of [FK1]). It is known that ${}_2\mathcal{F}_1(\alpha,\beta;\gamma;x,x) \approx h(x)^{\gamma-\alpha-\beta}$ if $\alpha+\beta-\gamma > \frac{r-1}{2}a$; since $\rho-\nu > p-1 > \frac{r-1}{2}a$, we thus see that the last integral is $\approx h(x)^\nu$ and, consequently, unbounded on Ω.

We conclude this section by describing the simplest Bloch and Q_ν-spaces.

Proposition 7. *Let $L = I$, the identity operator. Then*

$$\mathcal{B}_I = H^\infty(\Omega), \qquad \textit{the space of bounded analytic functions},$$

while

$$Q_{\nu,I} = \begin{cases} H^\infty, & \textit{if } \nu > p-1, \\ \{0\}, & \textit{if } \nu \le p-1. \end{cases}$$

Proof. Recall that

$$\mathcal{B}_I = \{f \text{ holomorphic on } \Omega : \sup_\Omega |f|^2 < \infty\},$$

$$Q_{\nu,I} = \{f \text{ holomorphic on } \Omega : \sup_{\phi\in G} \int_\Omega |f\circ\phi|^2\, h^\nu\, d\mu < \infty\}.$$

The assertion concerning $\mathcal{B}_I$ is thus trivial. For $Q_{\nu,I}$, we know by the Ørsted-Faraut-Koranyi theorem (27) that whenever $Q_{\nu,I}$ does not reduce to $\{0\}$, then it contains the function constant one. On the other hand, by the last proposition (with $\rho = 0$), $\mathbf{1} \in Q_{\nu,I}$ if and only if $\nu > p-1$. $\square$

Example 8. For $L = \Delta_{(1,0,\dots,0)} = \widetilde{\Delta}$, the invariant Laplacian on Ω, we have by (29)

$$\mathcal{B}_L = \{f \text{ holomorphic on } \Omega : \sup_{\phi\in G} \|\partial(f\circ\phi)(0)\| < \infty\},$$

and $Q_{\nu,L}$ consists of all holomorphic functions f on Ω for which

$$\sup_{\phi\in G} \int_\Omega \|\partial(f\circ\phi)(z)\|^2\, h(z)^\nu\, d\mu(z) < \infty.$$

The space $\mathcal{B}_L$ is the Bloch space studied by Timoney [Ti]. We will see in Theorem 18 below that unless Ω is (biholomorphic to) the unit disc $\mathbf{D}$ or the unit ball $\mathbf{B}^d$, $Q_{\nu,L}$ coincides with $\mathcal{B}_L$ for $\nu > p-1$, and reduces to the constant functions for $\nu \le p-1$.

Example 9. Let Ω be a tube type domain for which $s := \frac{d}{r}$ is an integer, and let $L = \Delta_{\mathbf{m}}$ where $\mathbf{m} = (s, \dots, s) =: (s^r)$. It is known that in this case the space $\mathcal{P}_{(s^r)}$ is one-dimensional and consists of multiples of $N(z)^s$, where N, the *Jordan determinant polynomial* (also called the *Koecher norm*), is a polynomial of degree r; the kernel $K_{\mathbf{m}}$ is given by $(s)_{\mathbf{m}} K_{\mathbf{m}}(x, y) = N(x)^s \overline{N(y)}^s$; and for any f holomorphic, $N(\partial)^s (f \circ \phi_x)(0) = (-1)^d h(x)^s N(\partial)^s f(x)$. Hence,

$$\Delta_{(s^r)} |f|^2 = h^p |N(\partial)^s f|^2$$

and

$$\mathcal{B}_{(s^r)} = \{f \text{ holomorphic on } \Omega : h^p |N(\partial)^s f|^2 \text{ is bounded}\}.$$

This is the so-called *top quotient Bloch space* studied by the first author in connection with generalized Hankel operators [A2]. (The terminology comes from the fact that the associated Bloch seminorm vanishes on $\mathcal{M}_{q-1}$, so that $\mathcal{B}_{(s^r)}$ really "lives" on the top quotient $\mathcal{M}_q / \mathcal{M}_{q-1}$ of the composition series.) It is the maximal $\mathrm{Aut}(\Omega)$-invariant space of holomorphic functions on Ω.

Further, for $\nu = 0$ we have, by the invariance of $d\mu$,

$$Q_{0,(s^r)} = \{f \text{ holomorphic on } \Omega : \int_\Omega |N(\partial)^s f(z)|^2 \, dz < \infty\},$$

which is, by definition, the *generalized Dirichlet space* of Ω. It is the unique $\mathrm{Aut}(\Omega)$-invariant Hilbert space of holomorphic functions on Ω (modulo $\mathcal{M}_{q-1}$).

We remark that the definitions (6) and (7) of $\mathcal{B}_L$ and $Q_{\nu,L}$ are special cases of a more general construction of Moebius invariant spaces, which goes as follows. Let X be any Banach space of holomorphic functions on Ω with the property that $f \in X$ and $\phi \in \mathrm{Aut}(\Omega)$ imply $f \circ \phi \in X$. We define $M(X)$ to be the space of all $f \in X$ for which $\|f\|_{M(X)} := \sup_{\phi \in \mathrm{Aut}(\Omega)} \|f \circ \phi\|_X < \infty$. Then $M(X)$ is $\mathrm{Aut}(\Omega)$-invariant. Of course, one can replace here "Banach space" with "semi-Banach space" (i.e. complete semi-normed space). This construction is very basic and generalizes the spaces defined by (6) and (7). Notice that if X is already $\mathrm{Aut}(\Omega)$-invariant then $M(X) = X$. Finally, the composition $f \mapsto f \circ \phi$ can be replaced by the weighted action $f \mapsto (\det \phi')^{\nu/p} (f \circ \phi)$, with some fixed real parameter ν, which leads to weighted analogues of all the above Moebius-invariant spaces (in particular, to "weighted" analogues of Bloch and Q_ν spaces). The authors hope to return to this topic in future.

4. Main results

Recall that we have defined, for a signature $\mathbf{m}$ and a real number ν,

$$\mathcal{B}_{\mathbf{m}} = \{f \text{ holomorphic on } \Omega : \sup_\Omega \Delta_{\mathbf{m}} |f|^2 < \infty\},$$

$$Q_{\nu,\mathbf{m}} = \{f \text{ holomorphic on } \Omega : \sup_{\phi \in G} \int_\Omega \Delta_{\mathbf{m}} |f \circ \phi|^2 \, h^\nu \, d\mu < \infty\},$$

the square roots of the indicated quantities being the seminorms in these spaces.

Lemma 10. *The involution*

$$f \mapsto f^*, \qquad f^*(z) := \overline{f(\overline{z})}, \tag{32}$$

maps each $\mathcal{P}_{\mathbf{m}}$ *into itself.*

Proof. Since $\mathcal{P}_{\mathbf{m}}$ is spanned by $K_{\mathbf{m}}(\cdot,y)$, $y\in\mathbf{C}^d$, the image $\mathcal{P}^*_{\mathbf{m}}$ of $\mathcal{P}_{\mathbf{m}}$ under (32) is spanned by $K_{\mathbf{m}}(\cdot,\overline{y})^*$, $y\in\mathbf{C}^d$. Thus it is enough to show that $K_{\mathbf{m}}(\cdot,\overline{y})^*=K_{\mathbf{m}}(\cdot,y)$ — that is, that $K_{\mathbf{m}}(\overline{y},\overline{x})=K_{\mathbf{m}}(x,y)$ for all x,y. As both sides are holomorphic in x and $\overline{y}$, and any such function is uniquely determined by its restriction to the diagonal $x=y$ [BM, Proposition II.4.7], it is in turn enough to show that $K_{\mathbf{m}}(\overline{z},\overline{z})=K_{\mathbf{m}}(z,z)$ $\forall z\in\mathbf{C}^d$. However, an examination of the list of Cartan's domains in the table in Section 2 reveals that they are all preserved by complex conjugation; hence, so are the stabilizer subgroup K and the Jordan triple product $\{\cdot,\cdot,\cdot\}$. It follows that $\overline{e}_1,\dots,\overline{e}_r$ is a Jordan frame whenever $e_1,\dots,e_r$ is, and that $\overline{z}=\overline{k}(t_1\overline{e}_1+\dots+t_r\overline{e}_r)$ if $z=k(t_1e_1+\dots+t_re_r)$. Since K acts transitively on the set of all Jordan frames, there must exist $k'\in K$ such that $k'\overline{k}\overline{e}_j=ke_j$ $\forall j$, i.e. $k'\overline{z}=z$. By K-invariance, this implies that $K_{\mathbf{m}}(\overline{z},\overline{z})=K_{\mathbf{m}}(k'\overline{z},k'\overline{z})=K_{\mathbf{m}}(z,z)$. □

Proposition 11. *If $\ell<q(\mathbf{m})$, then the $Q_{\nu,\mathbf{m}}$-seminorm vanishes on $\mathcal{M}_\ell$; thus $\mathcal{M}_\ell$ is contained in $Q_{\nu,\mathbf{m}}$ in a trivial way.*

The same is true also for the Bloch space $\mathcal{B}_{\mathbf{m}}$.

Proof. Choose an orthonormal basis $\{\psi_j\}_{j=1}^{d_{\mathbf{m}}}$ for $\mathcal{P}_{\mathbf{m}}$. Then, by (22),

$$\begin{aligned}\Delta_{\mathbf{m}}|f|^2(z)&=K_{\mathbf{m}}(\partial,\partial)|f\circ\phi_z|^2(0)\\&=\sum_j\psi_j(\partial)\overline{\psi_j(\partial)}|f\circ\phi_z|^2(0)\\&=\sum_j|\psi_j(\partial)(f\circ\phi_z)(0)|^2\\&=\sum_j|\langle f\circ\phi_z,\psi_j^*\rangle_F|^2.\end{aligned}\tag{33}$$

Since, by Lemma 10, $\{\psi_j^*\}$ is also a basis for $\mathcal{P}_{\mathbf{m}}$, this equals $\|P_{\mathbf{m}}(f\circ\phi_z)\|_F^2$, where $P_{\mathbf{m}}$ denotes the projection $g=\sum_{\mathbf{n}}g_{\mathbf{n}}\mapsto g_{\mathbf{m}}$ onto $\mathcal{P}_{\mathbf{m}}$.

Thus $f\in\mathcal{M}_\ell\implies f\circ\phi_z\in\mathcal{M}_\ell\implies P_{\mathbf{m}}(f\circ\phi_z)=0\implies\Delta_{\mathbf{m}}|f|^2=0\implies f\in\mathcal{B}_{\mathbf{m}}$ and $f\in Q_{\nu,\mathbf{m}}$. □

Remark 12. In Section 1 we used the notation

$$\mathcal{N}_{\mathbf{m}}:=\{f\text{ holomorphic on }\Omega:\Delta_{\mathbf{m}}|f|^2\equiv0\}$$

for the subspace of $\mathcal{B}_{\mathbf{m}}$ on which the $\mathbf{m}$-Bloch seminorm vanishes. It follows from the last proof that, in fact,

$$\mathcal{N}_{\mathbf{m}}=\mathcal{M}_{q(\mathbf{m})-1}.$$

We will see in a moment (cf. Corollary 15) that the Bloch spaces $\mathcal{B}_{\mathbf{m}}$ also depend only on the "height" $q(\mathbf{m})$ of $\mathbf{m}$.

Proposition 13. *If $\nu>p-1$, then $\mathcal{B}_{\mathbf{m}}\subset Q_{\nu,\mathbf{m}}$ continuously.*

Proof. Since the measure $d\mu_\nu:=h^\nu\,d\mu$ is finite for $\nu>p-1$, we have, for any $\phi\in G$,

$$\begin{aligned}\int_\Omega(\Delta_{\mathbf{m}}|f|^2)\circ\phi\,h^\nu\,d\mu&\le\mu_\nu(\Omega)\,\|\Delta_{\mathbf{m}}|f|^2\|_\infty\\&=\mu_\nu(\Omega)\,\|f\|^2_{\mathcal{B}_{\mathbf{m}}}.\end{aligned}$$

Taking supremum over all $\phi\in G$ yields the assertion. □

Theorem 14. *If $q(\mathbf{m}) \le q(\mathbf{n})$, then $Q_{\nu,\mathbf{m}} \subset \mathcal{B}_{\mathbf{n}}$ continuously.*

Proof. By the K-invariance of $\Delta_{\mathbf{m}}$ and h, the integral

$$\int_\Omega \Delta_{\mathbf{m}}(f\overline{g})\, h^\nu \, d\mu$$

is a positive-definite K-invariant bilinear form in $f, g \in \mathcal{P}$. As noted in Section 2, it is a consequence of Schur's lemma from representation theory that any such bilinear functional must be of the form

$$\sum_{\mathbf{k}} c_{\mathbf{mk}} \langle f_{\mathbf{k}}, g_{\mathbf{k}} \rangle_F ,$$

for some coefficients $c_{\mathbf{mk}} \ge 0$. Suppose we can show that

$$c_{\mathbf{mn}} > 0. \tag{34}$$

Since $\Delta_{\mathbf{n}}|f|^2(0) = \|P_{\mathbf{n}} f\|_F^2 = \|f_{\mathbf{n}}\|_F^2$, by (33), it will follow that

$$\Delta_{\mathbf{n}}|f|^2(0) \le \frac{1}{c_{\mathbf{mn}}} \int_\Omega \Delta_{\mathbf{m}}|f|^2 \; h^\nu \; d\mu.$$

Replacing f by $f \circ \phi_x$, this becomes

$$\Delta_{\mathbf{n}}|f|^2(x) \le \frac{1}{c_{\mathbf{mn}}} \int_\Omega \Delta_{\mathbf{m}}|f \circ \phi_x|^2 \; h^\nu \; d\mu.$$

Taking suprema over all $x \in \Omega$ gives the assertion.

It remains to prove (34). But by the properties of the composition series,

$$\begin{aligned} c_{\mathbf{mn}} = 0 \iff & \int_\Omega \Delta_{\mathbf{m}}|f_{\mathbf{n}}|^2 \; h^\nu \; d\mu = 0 \qquad \forall f_{\mathbf{n}} \in \mathcal{P}_{\mathbf{n}} \\ \iff & \Delta_{\mathbf{m}}|f_{\mathbf{n}}|^2(z) = 0 \quad \forall z \, \forall f_{\mathbf{n}} \\ \iff & \|P_{\mathbf{m}}(f_{\mathbf{n}} \circ \phi_z)\|_F^2 = 0 \quad \forall z \, \forall f_{\mathbf{n}} \quad \text{by (33)} \\ \iff & P_{\mathbf{m}}(f_{\mathbf{n}} \circ \phi_z) = 0 \quad \forall z \, \forall f_{\mathbf{n}} \\ \iff & P_{\mathbf{m}} \mathcal{M}_{q(\mathbf{n})} = 0 \qquad \text{by (27)} \\ \iff & q(\mathbf{m}) > q(\mathbf{n}). \end{aligned}$$

□

Corollary 15. *If $\nu > p-1$, then $Q_{\nu,\mathbf{m}} = \mathcal{B}_{\mathbf{m}}$, with equivalent norms.*
If $q(\mathbf{m}) \le q(\mathbf{n})$, then $\mathcal{B}_{\mathbf{m}} \subset \mathcal{B}_{\mathbf{n}}$ continuously.
If $q(\mathbf{m}) = q(\mathbf{n})$, then $\mathcal{B}_{\mathbf{m}} = \mathcal{B}_{\mathbf{n}}$, with equivalent norms.
If $q(\mathbf{m}) = q(\mathbf{n})$ and $\nu > p-1$, then $Q_{\nu,\mathbf{m}} = Q_{\nu,\mathbf{n}}$, with equivalent norms.

The last corollary exhausts the case $\nu > p-1$ completely. Let us now turn to $\nu \le p-1$.

In the sequel, similarly as we did with $h(z,z)$, we will often abbreviate $K_{\mathbf{m}}(z,z)$ just to $K_{\mathbf{m}}(z)$ (or even to $K_{\mathbf{m}}$).

Lemma 16. *For any signature $\mathbf{m}$, there exist constants $\alpha > 0$ and $c > 0$ such that $\Delta_{\mathbf{m}} K_{\mathbf{m}} \ge c\, h^\alpha$ on Ω.*

Proof. For $\mathbf{m} = (0, \dots, 0)$ this is trivial, so assume $|\mathbf{m}| > 0$. Then

$$\Delta_{\mathbf{m}} K_{\mathbf{m}}(x) = K_{\mathbf{m}}(\partial_z, \partial_z) K_{\mathbf{m}}(\phi_x(z), \phi_x(z))\,|_{z=0} \tag{35}$$

is an expression of the form

$$\sum_{\substack{\alpha,\beta,\\ \alpha_1,\dots,i_1,\dots,\\ \beta_1,\dots,j_1,\dots}} c_{\alpha\beta\alpha_1\dots i_1\dots\beta_1\dots j_1\dots}\,(\partial^\alpha\overline{\partial}^\beta K_{\mathbf{m}})(x,x)\prod_k \partial^{\alpha_k}(\phi_x)_{i_k}(0)\,\overline{\prod_l \partial^{\beta_l}(\phi_x)_{j_l}(0)},$$

with some constants $c_{\dots}$ (independent of x), and with the summation extending over multiindices $\alpha,\beta,\alpha_1,\dots,\beta_1,\dots$ satisfying $|\alpha_j|,|\beta_j|>0\ \forall j$. (Here $(\phi_x)_j(z)$ stands for the j-th coordinate of $\phi_x(z)$, $j=1,\dots,d$.) From (15) one can see that

$$\phi'_x(z) = -B(x,x)^{1/2}B(z,x)^{-1}, \tag{36}$$

B being the Bergman operator (12). Using the formula

$$(X^{-1})' = -X^{-1}X'X^{-1}$$

for the derivative of any invertible-operator-valued function $X(z)^{-1}$, it follows by iteration that for any multiindex γ, $|\gamma|>1$, and any $j=1,\dots,d$,

$$\partial^\gamma(\phi_x)_j(0) = B(x,x)^{1/2}p_{j\gamma}(x), \tag{37}$$

for some polynomial $p_{j\gamma}$ of (the coordinates of) x.

Since both $\Delta_{\mathbf{m}}$ and $K_{\mathbf{m}}$ are K-invariant, so is the function $\Delta_{\mathbf{m}}K_{\mathbf{m}}$; thus it is enough to evaluate it only for $x=t_1e_1+\dots+t_re_r$ for some Jordan frame $e_1,\dots,e_r$ and $t_1,\dots,t_r\in[0,1]$. By (17), the quantity (37) — and, hence, also (35) — will then be an expression of the form

$$\text{a polynomial in } t_1,\dots,t_r \text{ and } \sqrt{1-t_1^2},\dots,\sqrt{1-t_r^2}.$$

Making the substitution $t_j=1-\tau_j^2$, $\tau_j\in[0,1]$, $j=1,\dots,r$, this becomes

$$(\text{a polynomial in } \tau_1,\dots,\tau_r \text{ and } \sqrt{2-\tau_1^2},\dots,\sqrt{2-\tau_r^2}) =: G(\tau_1,\dots,\tau_r).$$

However, this is clearly a holomorphic function of $\tau_1,\dots,\tau_r$ on the polydisc $\{|\tau_j|<\sqrt{2}\ \forall j\}$. Let $\mathcal{V}_k$, $k=0,1,2,\dots$, be the set of all points in this polydisc where $G(\tau)$ has a zero of order at least k (i.e. vanishes together with all its partial derivatives of orders $<k$). Then there exists a k for which $\mathcal{V}_k\cap\overline{\mathbf{D}}^r=\emptyset$: otherwise the decreasing chain of compact subsets $\{\mathcal{V}_k\cap\overline{\mathbf{D}}^r\}_{k\ge0}$ would have a nonempty intersection, i.e. there would exist a point in $\overline{\mathbf{D}}^r$ where G vanishes together with its partial derivatives of all orders; as G is holomorphic this would mean that G vanishes identically, contradicting the fact that $G=\Delta_{\mathbf{m}}K_{\mathbf{m}}>0$ for $\tau\in(0,1)^r$. Now $\mathcal{V}_k\cap\overline{\mathbf{D}}^r=\emptyset$ means that $\lim_{\tau\to\sigma}\dfrac{|G(\tau)|}{\|\tau-\sigma\|^k}=+\infty\ \forall\sigma\in\overline{\mathbf{D}}^r$. Consider σ of the form $\sigma_1=\dots=\sigma_m=0$ and $\sigma_{m+1},\dots,\sigma_r\in(0,1)$. Then as $\tau\to\sigma$, we eventually have $|\tau_j|\le1$ for $j=m+1,\dots,r$ while $|\tau_j|\le\|\tau-\sigma\|$ for $j=1,\dots,m$; thus $\|\tau-\sigma\|^m\ge|\tau_1\dots\tau_r|$. Since $|\tau_1\dots\tau_r|\asymp h^{1/2}$ for $\tau_1,\dots,\tau_r\in[0,1]$, it follows that $h^{k/2m}\lesssim\|\tau-\sigma\|^k$ and

$$\frac{G}{h^{k/2m}}\gtrsim\frac{G}{\|\tau-\sigma\|^k}\to+\infty$$

as $\tau\to\sigma$. As $m\ge1$, this implies that

$$\frac{G}{h^{k/2}}\to+\infty \qquad \text{as } [0,1]^r\ni\tau\to\sigma.$$

It follows that the (continuous) function $G/h^{k/2} = \Delta_{\mathbf{m}} K_{\mathbf{m}}/h^{k/2}$ on Ω is positive on Ω and tends to $+\infty$ at $\partial\Omega$. Thus it must be bounded from below by some $c > 0$. Taking $\alpha = k/2$, the claim follows. $\square$

Theorem 17. *If $\nu < 0$, then $Q_{\nu,\mathbf{m}} = \mathcal{M}_{q(\mathbf{m})-1}$.*

Proof. From the Ørsted-Faraut-Koranyi theorem (27) we know that

$$\begin{aligned} \mathcal{M}_{q(\mathbf{m})-1} \subsetneq Q_{\nu,\mathbf{m}} &\implies \mathcal{P} \cap \mathcal{M}_{q(\mathbf{m})} \subset Q_{\nu,\mathbf{m}} \\ &\implies \mathcal{P}_{\mathbf{m}} \subset Q_{\nu,\mathbf{m}} \\ &\implies \sup_x \int_\Omega \Delta_{\mathbf{m}} |f|^2 \, (h \circ \phi_x)^\nu \, d\mu < \infty \qquad \forall f \in \mathcal{P}_{\mathbf{m}}. \end{aligned}$$

Since $K_{\mathbf{m}}(z,z) = \sum_j |\psi_j(z)|^2$ for any basis $\{\psi_j\}$ of $\mathcal{P}_{\mathbf{m}}$, we can continue by

$$\implies \sup_x \int_\Omega \Delta_{\mathbf{m}} K_{\mathbf{m}} \cdot (h \circ \phi_x)^\nu \, d\mu < \infty$$

(where we again write just $K_{\mathbf{m}}$ for $K_{\mathbf{m}}(z,z)$). By Lemma 16, we can in turn continue by

$$\implies \sup_x \int_\Omega h^\alpha \, (h \circ \phi_x)^\nu \, d\mu < \infty.$$

By Proposition 5, this is only possible if $\nu \geq 0$. $\square$

Recall that the only Cartan domain of rank 1 is the unit ball $\mathbf{B}^d$, $d \geq 1$. Thus the following theorem means that the situation for $r > 1$ differs radically from the one for $r = 1$, when Q_ν is nontrivial also for $p-2 < \nu \leq p-1$ (for the disc, even for $p-2 \leq \nu \leq p-1$) in view of (2) and (4).

Theorem 18. *For $r > 1$ and $\mathbf{m} = (1,0,\dots,0) =: (1)$, that is,*

$$f \in Q_{\nu,(1)} \iff \sup_{\phi \in G} \int_\Omega \widetilde{\Delta} |f \circ \phi|^2 \, h^\nu \, d\mu < \infty,$$

we have

$$Q_{\nu,(1)} = \begin{cases} \mathcal{B}_{(1)}, \ \textit{the Timoney Bloch space}, & \textit{if } \nu > p-1, \\ \{\textit{constants}\}, & \textit{if } \nu \leq p-1. \end{cases}$$

Proof. The constants are always contained in $Q_{\nu,(1)}$, by Theorem 11. As in the preceding proof, we have

$$\{\text{constants}\} \subsetneq Q_{\nu,(1)} \implies \sup_{\phi \in G} \int_\Omega \Delta_{(1)} K_{(1)} \, (h \circ \phi)^\nu \, d\mu < \infty,$$

that is,

$$\sup_{\phi \in G} \int_\Omega (\widetilde{\Delta} \| \cdot \|^2) \, (h \circ \phi)^\nu \, d\mu < \infty. \tag{38}$$

Since the coordinate functions $z_1, \dots, z_d$ are a basis of $\mathcal{P}_{(1)}$, we have by (29)

$$(\widetilde{\Delta} \| \cdot \|^2)(x) = \sum_{j=1}^d (\widetilde{\Delta} | \cdot_j |^2)(x) = \sum_{j,k=1}^d |\partial_k (\phi_x)_j(0)|^2.$$

However, by (36), $\partial_k(\phi_x)_j(0)$ is precisely the (j,k)-entry of the matrix $-B(x,x)^{1/2}$. Thus

$$(\widetilde{\Delta} \| \cdot \|^2)(x) = \| B(x,x)^{1/2} \|_{HS}^2$$

is the square of the Hilbert-Schmidt norm of the operator $B(x,x)^{1/2}$ on $\mathbf{C}^d$. If x has the polar decomposition (16), then we know from (17) that $B(x,x)^{1/2}$ is a diagonal operator with respect to the Peirce decomposition (13), with eigenvalues $(1-t_i^2)^{1/2}(1-t_j^2)^{1/2}$ on each Z_{ij}. Since

$$\dim Z_{ij} = \begin{cases} a & \text{for } 1 \le i < j \le r, \\ b & \text{for } i = 0 < j \le r, \\ 1 & \text{for } 1 \le i = j \le r, \\ 0 & \text{for } i = j = 0, \end{cases}$$

it follows that

$$\begin{aligned} \|B(x,x)^{1/2}\|_{HS}^2 &= a \sum_{1 \le i < j \le r} (1-t_i^2)(1-t_j^2) + b \sum_{1 \le j \le r} (1-t_j^2) + \sum_{j=1}^{r} (1-t_j^2)^2 \\ &:= F(t_1, \dots, t_r). \end{aligned} \tag{39}$$

Now taking $\phi = \mathrm{id}$ in (38), we get by (20)

$$\begin{aligned} &\int_\Omega (\widetilde{\Delta}\|\cdot\|^2)\, h^\nu \, d\mu = \int_\Omega \|B(z,z)^{1/2}\|_{HS}^2 \, h(z)^\nu \, d\mu(z) \\ &\quad = \int_{[0,1]^r} F(t_1, \dots, t_r) \prod_{j=1}^{r} (1-t_j^2)^{\nu-p} \prod_{j=1}^{r} t_j^{2b+1} \prod_{1 \le i<j \le r} |t_i^2 - t_j^2|^a \, dt_1 \, \dots \, dt_r \\ &\quad \ge \int_{t_1 = 1-1/2r}^{1} \int_{t_2=1/2r}^{2/2r} \int_{t_3=3/2r}^{4/2r} \dots \int_{t_r=(2r-3)/2r}^{(2r-2)/2r} F(t_1, \dots, t_r) \cdot \\ &\qquad (1-t_1)^{\nu-p} \Big(\frac{1}{r}\Big)^{(r-1)\nu p} \Big(\frac{1}{(2r)^{r-1}} \cdot \Big(1 - \frac{1}{2r}\Big)\Big)^{2b+1} \Big(\frac{1}{r} \cdot \frac{1}{2r} \Big)^{\frac{r(r-1)}{2} a} dt_1 \, \dots \, dt_r. \end{aligned}$$

Since $F(t) \ge 1 - t_2^2 \ge 1 - \frac{2}{2r}$ (here the hypothesis that $r > 1$ was used!) on the last domain of integration, we can continue the estimate with

$$\ge C_r \int_{1-1/2r}^{1} (1-t_1)^{\nu-p} \, dt_1.$$

But the last integral is finite only for $\nu > p-1$. Since we know that $Q_{\nu,(1)} = \mathcal{B}_{(1)}$ for such ν, by Corollary 15, this completes the proof. $\square$

Note that the proof shows that for $\nu \le p-1$, not only the supremum (38) is infinite, but in fact the integral occurring there is infinite for $\phi = \mathrm{id}$ and, hence, for any $\phi \in G$ (since $(h \circ \phi)/h$ is bounded and bounded away from zero on Ω for any fixed ϕ).

The methods of proofs of the last two theorems can be adapted a little to yield the following result.

Theorem 19. *Let*

$$\rho_{\mathbf{m}} = \sup \Big\{ \rho \ge 0 : \ \frac{\Delta_{\mathbf{m}} K_{\mathbf{m}}}{h^\rho} \ \textit{is bounded on } \Omega \Big\}. \tag{40}$$

Then this supremum is attained (i.e. is a maximum) and finite, $\rho_{\mathbf{m}}$ is always an integer or a half-integer, and $Q_{\nu,\mathbf{m}}$ is nontrivial (i.e. does not reduce to $\mathcal{M}_{q(\mathbf{m})-1}$) if and only if

$$\nu \ge 0 \quad \textit{and} \quad \nu > p - 1 - \rho_{\mathbf{m}}.$$

Proof. We have already seen in the proofs of Theorems 17 and 18 that

$$\mathcal{M}_{q(\mathbf{m})-1} \subsetneq Q_{\nu,\mathbf{m}} \implies \sup_{\phi\in G} \int_\Omega \Delta_{\mathbf{m}} K_{\mathbf{m}} \, (h\circ\phi)^\nu \, d\mu < \infty.$$

Conversely, if the last supremum is finite, then — since $\Delta_{\mathbf{m}} K_{\mathbf{m}} = \sum_j \Delta_{\mathbf{m}} |\psi_j|^2$ for any orthonormal basis ψ_j of $\mathcal{P}_{\mathbf{m}}$ — we have

$$\sup_{\phi\in G} \int_\Omega \Delta_{\mathbf{m}} |\psi_j|^2 \, (h\circ\phi)^\nu \, d\mu < \infty \qquad \forall j,$$

i.e. $\psi_j \in Q_{\nu,\mathbf{m}}$ $\forall j$, whence $\mathcal{P}_{\mathbf{m}} \subset Q_{\nu,\mathbf{m}}$, so $Q_{\nu,\mathbf{m}} \supsetneqq \mathcal{M}_{q(\mathbf{m})-1}$. We thus see that

$$Q_{\nu,\mathbf{m}} \text{ is nontrivial} \iff \sup_{\phi\in G} \int_\Omega \Delta_{\mathbf{m}} K_{\mathbf{m}} \, (h\circ\phi)^\nu \, d\mu < \infty. \tag{41}$$

Next, we have seen in the proof of Lemma 16 that

$$(\Delta_{\mathbf{m}} K_{\mathbf{m}})(k(t_1 e_1 + \dots + t_r e_r)) = F(t_1, \dots, t_r),$$

where F is a polynomial in $t_1, \dots, t_r$ and $\sqrt{1-t_1^2}, \dots, \sqrt{1-t_r^2}$, $0 \le t_j \le 1$; and that, hence,

$$F(t_1, \dots, t_r) = G(\tau_1, \dots, \tau_r), \qquad t_j = 1 - \tau_j^2,$$

where G is a polynomial in $\tau_1, \dots, \tau_r$ and $\sqrt{2-\tau_1^2}, \dots, \sqrt{2-\tau_r^2}$, $0 \le \tau_j \le 1$, and, consequently, extends to a holomorphic function in the polydisc $(\sqrt{2}\mathbf{D})^r = \{|\tau_j| < \sqrt{2} \ \forall j\}$. Let

$$G(\tau) = \sum_{\alpha \text{ multiindex}} g_\alpha \tau^\alpha$$

be the Taylor expansion of G around the origin. Let $k \ge 0$ be the greatest integer such that

$$g_\alpha = 0 \text{ whenever } \max\{\alpha_1, \dots, \alpha_r\} \le k.$$

Then

$$G(\tau) = (\tau_1 \dots \tau_r)^k H(\tau)$$

where

$$H(\tau) = \sum_\alpha c_\alpha \tau^\alpha, \qquad c_\alpha := g_{\alpha + (k,k,\dots,k)},$$

is still holomorphic in $(\sqrt{2}\mathbf{D})^r$, and there exists α such that $\alpha_j = 0$ for some j and $c_\alpha = 0$. Since F is symmetric in $t_1, \dots, t_r$, and thus G and H are symmetric in $\tau_1, \dots, \tau_r$, we may assume that $j = 1$. Thus $c_{0\alpha_2\dots\alpha_r} \ne 0$ for some $\alpha_2, \dots, \alpha_r$; consequently,

$$H(0, \tau_2, \dots, \tau_r) = \sum_{\alpha_2,\dots,\alpha_r=0}^\infty c_{0\alpha_2\dots\alpha_r} \tau_2^{\alpha_2} \dots \tau_r^{\alpha_r}$$

does not vanish identically, and therefore assumes nonzero values in any neighbourhood of the origin in $\mathbf{R}^{r-1}$. It follows that $H(\tau)/(\tau_1 \dots \tau_r)^\epsilon$ is unbounded in any neighbourhood of the origin for any $\epsilon > 0$. Consequently, $G(\tau)/(\tau_1 \dots \tau_r)^k$ is bounded on $(0,1)^r$, but $G(\tau)/(\tau_1 \dots \tau_r)^{k+\epsilon}$ is not bounded there for any $\epsilon > 0$. Since

$$h(k(t_1 e_1 + \dots + t_r e_r)) = \prod_{j=1}^r \tau_j^2 (2 - \tau_j^2) \asymp (\tau_1 \dots \tau_r)^2,$$

it follows that $(\Delta_{\mathbf{m}} K_{\mathbf{m}})/h^\rho$ is bounded for $\rho = k/2$, but not for any $\rho > k/2$. Thus the first part of the theorem follows, with $\rho_{\mathbf{m}} = k/2$; moreover, we see that the function

$$\frac{F(t_1, \dots, t_r)}{\prod_{j=1}^r (1-t_j^2)^{k/2}} =: E(t_1, \dots, t_r)$$

is continuous on $[0,1]^r$ and positive at some point t with $t_1 = 1$. By continuity, there must exist $t \in [0,1]^r$ such that

$$E(t) \ge 2\delta, \ t_1 = 1, \ t_2, \dots, t_r \in [2\delta, 1-2\delta], \text{ and } |t_j - t_k| \ge 2\delta \ \forall j \ne k,$$

for some $\delta > 0$. Let $\mathcal{U}$ be a cubical neighbourhood of this point in $[0,1]^r$ so small that

$$E(t) \ge \delta, \ t_1 \in [1-\delta, 1], \ t_2, \dots, t_r \in [\delta, 1-\delta], \text{ and } |t_j - t_k| \ge \delta \ \forall j \ne k,$$

for all $t \in \mathcal{U}$. We may assume that $\delta < \frac{1}{2}$, so that $\delta < 1 - \delta$. Proceeding as in the proof of Theorem 18, we then have

$$\begin{aligned}
\int_\Omega \Delta_{\mathbf{m}} K_{\mathbf{m}} \, h^\nu \, d\mu &= \int_{[0,1]^r} F(t) \prod_{j=1}^r (1-t_j^2)^{\nu-p} \prod_{j=1}^r t_j^{2b+1} \prod_{1 \le i < j \le r} |t_i^2 - t_j^2|^a \, dt_1 \, \dots \, dt_r \\
&= \int_{[0,1]^r} E(t) \prod_{j=1}^r (1-t_j^2)^{\nu-p+\rho_{\mathbf{m}}} \prod_{j=1}^r t_j^{2b+1} \prod_{1 \le i < j \le r} |t_i^2 - t_j^2|^a \, dt_1 \, \dots \, dt_r \\
&\ge \int_{\mathcal{U}} \dots \\
&\ge \int_{\mathcal{U}} \delta \cdot (1-t_1)^{\nu-p+\rho_{\mathbf{m}}} \delta^{(\nu-p+\rho_{\mathbf{m}})(r-1)} \cdot \delta^{(2b+1)r} \cdot (2\delta^2)^{\frac{r(r-1)}{2}a} \, dt_1 \, \dots \, dt_r \\
&\ge C_\delta \int_{1-\epsilon}^1 (1-t_1)^{\nu-p+\rho_{\mathbf{m}}} \, dt_1.
\end{aligned}$$

The last integral is finite only for $\nu - p + \rho_{\mathbf{m}} > -1$, i.e. $\nu > p - 1 - \rho_{\mathbf{m}}$. Thus, by (41), $Q_{\nu,\mathbf{m}}$ is trivial if $\nu \le p - 1 - \rho_{\mathbf{m}}$.

From Theorem 17, we also already know that $Q_{\nu,\mathbf{m}}$ is trivial for $\nu < 0$. Thus it remains to prove that the supremum in (41) is finite if $\nu \ge 0$ and $\nu > p - 1 - \rho_{\mathbf{m}}$.

However, from the boundedness of $(\Delta_{\mathbf{m}} K_{\mathbf{m}})/h^{\rho_{\mathbf{m}}}$ it follows that $\Delta_{\mathbf{m}} K_{\mathbf{m}} \le C h^{\rho_{\mathbf{m}}}$ for some $0 < C < \infty$, so for any $\phi \in G$

$$\int_\Omega (\Delta_{\mathbf{m}} K_{\mathbf{m}}) \, (h \circ \phi)^\nu \, d\mu \le C \int_\Omega h^{\rho_{\mathbf{m}}} \, (h \circ \phi)^\nu \, d\mu.$$

But by Proposition 5, the supremum of the right-hand side over all $\phi \in G$ is finite if $\nu \ge 0$ and $\nu + \rho_{\mathbf{m}} > p - 1$. The proof is complete. □

As in the preceding theorem, we even see that for $\nu \le p - 1 - \rho_{\mathbf{m}}$, not only the supremum in (41) is infinite, but in fact the integral there is infinite for any $\phi \in G$.

We finish this section by a result which characterizes the Bloch spaces $\mathcal{B}_{\mathbf{m}}$ as maximal spaces of holomorphic functions on each given quotient $\mathcal{M}_q/\mathcal{M}_\ell$ of the composition series. It generalizes the analogous characterizations for the ordinary Bloch space $\mathcal{B}_{(1)}$ of Timoney and for the top quotient Bloch space $B_{(s^r)}$ on tube-type domains.

Theorem 20. *Let X be any semi-Banach space of holomorphic functions on Ω such that*

1. *X is Moebius invariant, i.e. $f \in X$ and $\phi \in G$ imply $f \circ \phi \in X$ and $\|f \circ \phi\|_X = \|f\|_X$;*
2. *if σ is any finite Borel measure on the stabilizer subgroup K then the operator of convolution with σ,*

$$C_\sigma f(z) := \int_K f(k^{-1}z)\, d\sigma(k),$$

is bounded on X.

Let further $\mathbf{m}$ be any signature such that

$$f \in X \text{ and } \|f\|_X > 0 \qquad \text{for some } f \in \mathcal{P}_{\mathbf{m}}. \tag{42}$$

Then $X \subset \mathcal{B}_{\mathbf{m}}$ continuously.

Proof. From the hypothesis 2. and the representation

$$P_{\mathbf{m}} f(z) = \int_K f(k^{-1}z)\, \chi_{\mathbf{m}}(k)\, dk, \tag{43}$$

where $\chi_{\mathbf{m}}$ is the character of K associated with $\mathbf{m}$, it follows that the canonical projection $P_{\mathbf{m}}$ onto $\mathcal{P}_{\mathbf{m}}$ is bounded on X. From (42), we further have $\|P_{\mathbf{m}}\|_{X\to X} > 0$ and, by the property (27) of the composition series, $\mathcal{M}_{q(\mathbf{m})} \cap \mathcal{P} \subset X$; in particular, $\mathcal{P}_{\mathbf{m}} \subset X$. Finally, as $\mathcal{P}_{\mathbf{m}}$ is finite-dimensional,

$$\alpha_{\mathbf{m}} \|f\|_X \leq \|f\|_F \leq \beta_{\mathbf{m}} \|f\|_X \qquad \forall f \in \mathcal{P}_{\mathbf{m}}$$

for some constants $\alpha_{\mathbf{m}}$ and $\beta_{\mathbf{m}}$. Let $f \in X$ and $\phi \in G$. Then

$$\|f\|_X = \|f \circ \phi\|_X \geq \frac{\|P_{\mathbf{m}}(f \circ \phi)\|_X}{\|P_{\mathbf{m}}\|_{X\to X}} \geq \frac{\|P_{\mathbf{m}}(f \circ \phi)\|_F}{\beta_{\mathbf{m}} \|P_{\mathbf{m}}\|_{X\to X}}.$$

Taking supremum over all $\phi \in G$ and recalling that $\|P_{\mathbf{m}}(f \circ \phi)\|_F^2 = \Delta_{\mathbf{m}} |f|^2(\phi(0))$, we obtain

$$\|f\|_X \geq \frac{\|f\|_{\mathcal{B}_{\mathbf{m}}}}{\beta_{\mathbf{m}} \|P_{\mathbf{m}}\|_{X\to X}}.$$

This means that $X \subset \mathcal{B}_{\mathbf{m}}$ continuously. □

Note that $\mathcal{P}_{\mathbf{m}} \subset \mathcal{B}_{\mathbf{m}}$ for any $\mathbf{m}$. In fact, even

$$H^\infty(\Omega) \subset \mathcal{B}_{\mathbf{m}} \qquad \text{continuously} \tag{44}$$

for any $\mathbf{m}$; this can be seen as follows. From the representation (43), we obtain for any $z \in \Omega$ and f holomorphic on Ω

$$\begin{aligned} |P_{\mathbf{m}} f(z)| &= \Big| \int_K f(k^{-1}z) \chi_{\mathbf{m}}(k)\, dk \Big| \\ &\leq \|f\|_\infty \Big(\int_K |\chi_{\mathbf{m}}(k)|^2\, dk \Big)^{1/2} \\ &= \|f\|_\infty \end{aligned}$$

(where $\|\cdot\|_\infty$ stands for the supremum norm on Ω). Thus $\|P_{\mathbf{m}} f\|_\infty \leq \|f\|_\infty$. Also, since $\mathcal{P}_{\mathbf{m}}$ is finite-dimensional, the Fock norm is equivalent to the $L^2(\Omega)$-norm on $\mathcal{P}_{\mathbf{m}}$. Thus for any $\phi \in G$,

$$\begin{aligned} \|P_{\mathbf{m}}(f \circ \phi)\|_F^2 &\asymp \int_\Omega |P_{\mathbf{m}}(f \circ \phi)(z)|^2\, dz \leq \operatorname{vol}(\Omega)\, \|P_{\mathbf{m}}(f \circ \phi)\|_\infty^2 \\ &\leq \operatorname{vol}(\Omega)\, \|f \circ \phi\|_\infty^2 = \operatorname{vol}(\Omega)\, \|f\|_\infty^2. \end{aligned}$$

Taking supremum over all $\phi \in G$ and recalling that $\|P_{\mathbf{m}}(f\circ\phi)\|_F^2 = \Delta_{\mathbf{m}}|f|^2(\phi(0))$, it follows that

$$\|f\|_{\mathcal{B}_{\mathbf{m}}} \le \mathrm{vol}(\Omega)^{1/2}\,\|f\|_\infty,$$

proving (44).

Thus $\mathcal{B}_{\mathbf{m}}$ is maximal among the spaces of holomorphic functions that contain $\mathcal{P}_{\mathbf{m}}$ and whose seminorm does not vanish identically on $\mathcal{P}_{\mathbf{m}}$. Since, by the property (27) of the composition series, $\mathcal{P}_{\mathbf{m}} \subset \{f \in X : \|f\|_X = 0\}$ implies $\mathcal{P}_{\mathbf{n}} \subset \{f \in X : \|f\|_X = 0\}$ whenever $q(\mathbf{m}) = q(\mathbf{n})$, it follows that $\mathcal{B}_{\mathbf{m}} \subset \mathcal{B}_{\mathbf{n}}$ and, hence, by symmetry, $\mathcal{B}_{\mathbf{m}} = \mathcal{B}_{\mathbf{n}}$ with equivalent norms, whenever $q(\mathbf{m}) = q(\mathbf{n})$. This gives another proof of Theorem 14 as well as of the second and the third parts of Corollary 15.

5. Concluding remarks

Theorem 19 reduces the question of nontriviality of $Q_{\nu,\mathbf{m}}$ to the determination of the number $\rho_{\mathbf{m}}$, i.e. to a question concerning the boundary behaviour of the function $\Delta_{\mathbf{m}}K_{\mathbf{m}}$. Unfortunately, in general we do not have a complete answer to the latter question either.

For $\mathbf{m} = (0,\dots,0) =: (0)$, one has trivially $\Delta_{(0)}K_{(0)} = \mathbf{1}$ and $\rho_{(0)} = 0$, in agreement with Proposition 7; we will thus assume that $|\mathbf{m}| > 0$ from now on.

On the unit disc $\mathbf{D}$, the operator $\Delta_{\mathbf{m}}$, being a polynomial in the invariant Laplacian $\widetilde{\Delta} = (1-|z|^2)^2\Delta$, always contains the factor $(1-|z|^2)^2$, and thus $\rho_{\mathbf{m}} \ge 2\ \forall\mathbf{m}$. Since $p-1=1$ in this case, the spaces $Q_{\nu,\mathbf{m}}(\mathbf{D})$ are thus nontrivial if and only if $\nu \ge 0$. For $\nu > 1$, $Q_{\nu,\mathbf{m}} = \mathcal{B}(\mathbf{D})$, the Bloch space, by Corollary 15. For $0 \le \nu \le 1$, the spaces $Q_{\nu,\mathbf{m}}$ are the familiar spaces from (2) if $\mathbf{m} = (1)$, but we do not know anything about them for any other nonzero $\mathbf{m}$.

Conjecture 21. *For any $\mathbf{m} \ne (0)$ and any $0 \le \nu \le 1$, the spaces $Q_{\nu,\mathbf{m}}$ are independent of $\mathbf{m}$, i.e. $Q_{\nu,\mathbf{m}} = Q_{\nu,(1)}$ with equivalent norms.*

For the unit ball $\mathbf{B}^d$, $d>1$, $\Delta_{\mathbf{m}}$ are still polynomials in the invariant Laplacian $\widetilde{\Delta}$, but now $\widetilde{\Delta} = (1-\|z\|^2)(\Delta - \mathcal{R}\overline{\mathcal{R}})$, where $\mathcal{R}$ stands for the radial derivative, contains only the factor $(1-\|z\|^2)$ instead of $(1-|z|^2)^2$; thus $\rho_{\mathbf{m}} \ge 1$. Computations seem to indicate that $\rho_{\mathbf{m}} = 1$ for all $\mathbf{m}$, so that $Q_{\nu,\mathbf{m}}$ is nontrivial — i.e. does not reduce to the constants — if and only if $\nu > p-2 = d-1$. For $\mathbf{m} = (1)$, this recovers the familiar spaces from (4); for other nonzero $\mathbf{m}$, again nothing is known.

Conjecture 22. *For any $\mathbf{m} \ne (0)$, the space $Q_{\nu,\mathbf{m}}(\mathbf{B}^d)$, $d>1$, is nontrivial if and only if $\nu > d-1$, and then coincides with $Q_{\nu,(1)}(\mathbf{B}^d)$, with equivalent norms.*

For domains of higher rank, it is immediate from (39) that $\rho_{(1)} = 0$; and from Example 9 that

$$\Delta_{(s^r)}K_{(s^r)} = h^p|N(\partial)^s N^s|^2 = (s)^2_{(s^r)}h^p$$

so that $\rho_{(s^r)} = p$ for Ω a tube type domain with $\frac{d}{r} =: s$ an integer.

Using computer, we were also able to compute $\rho_{\mathbf{m}}$ for a few signatures $\mathbf{m}$ for the Cartan domains $\Omega = I_{22}$ and $\Omega = I_{23}$, that is, for the unit balls of all 2×2 and 2×3 complex matrices, respectively; the results are summarized in the table below, which gives the values of $\rho_{\mathbf{m}}$ and the corresponding ranges of ν for which $Q_{\nu,\mathbf{m}}$ is nontrivial. Note that in this case $r=2$, $a=2$, and $b=0$ and 1, respectively, so that $p=4$ for I_{22} and $p=5$ for I_{23}.

$\mathbf{m}$	$\Omega = I_{22}$	$\Omega = I_{23}$
$(0,0)$	$\rho_{\mathbf{m}} = 0, \quad \nu > p-1$	$\rho_{\mathbf{m}} = 0, \quad \nu > p-1$
$(1,0)$	$\rho_{\mathbf{m}} = 0, \quad \nu > p-1$	$\rho_{\mathbf{m}} = 0, \quad \nu > p-1$
$(1,1)$	$\rho_{\mathbf{m}} = 0, \quad \nu > p-1$	$\rho_{\mathbf{m}} = 0, \quad \nu > p-1$
$(2,0)$	$\rho_{\mathbf{m}} = 0, \quad \nu > p-1$	$\rho_{\mathbf{m}} = 0, \quad \nu > p-1$
$(2,1)$	$\rho_{\mathbf{m}} = 0, \quad \nu > p-1$	$\rho_{\mathbf{m}} = 0, \quad \nu > p-1$
$(2,2)$	$\rho_{\mathbf{m}} = 4, \quad \nu \geq 0$	$\rho_{\mathbf{m}} = 2, \quad \nu > p-3$
$(3,0)$	$\rho_{\mathbf{m}} = 0, \quad \nu > p-1$	
$(3,1)$	$\rho_{\mathbf{m}} = 0, \quad \nu > p-1$	
$(3,2)$	$\rho_{\mathbf{m}} = 4, \quad \nu \geq 0$	
$(3,3)$	$\rho_{\mathbf{m}} = 4, \quad \nu \geq 0$	

It is not completely clear from the table what $\rho_{\mathbf{m}}$ might be in general, except for the case of tube type domains.

Conjecture 23. *Let Ω be a tube type domain with $\frac{d}{r}$ an integer. Then $\rho_{\mathbf{m}} = 0$ if $q(\mathbf{m}) < q$, and $\rho_{\mathbf{m}} = p$ if $q(\mathbf{m}) = q$. Consequently, for $\nu \leq p-1$, $Q_{\nu,\mathbf{m}}$ is nontrivial if and only if $q(\mathbf{m}) = q$ and $\nu \geq 0$.*

Note that even for the non-tube type domain I_{23}, the table suggests that only the top quotient of the composition series, i.e. the signatures with $q(\mathbf{m}) = q$, are of interest.

Similarly to the disc and the ball, we also conjecture that

Conjecture 24. *For any real ν and any Ω, $Q_{\nu,\mathbf{m}} = Q_{\nu,\mathbf{n}}$ (with equivalent norms) whenever $q(\mathbf{m}) = q(\mathbf{n})$.*

By Corollary 15, the last conjecture is definitely valid for $\nu > p-1$.

Another conjecture which has emerged from the computations behind the last table is the following. Recall that we have shown in the proof of Lemma 16 that $\Delta_{\mathbf{m}} K_{\mathbf{m}}(k \sum_j t_j e_j)$ is always a polynomial in t_j and $\sqrt{1-t_j^2}$, $j = 1, \dots, r$.

Conjecture 25. *$\Delta_{\mathbf{m}} K_{\mathbf{m}}(k \sum_j t_j e_j)$ is actually a polynomial in $t_1, \dots, t_r$.*

Of course, all the above conjectures could probably be solved if we had some explicit formula for $\Delta_{\mathbf{m}} K_{\mathbf{m}}$. We conclude this paper by a result which, though short of giving such a formula, at least relates it to another well-known problem.

For any signatures $\mathbf{m}$ and $\mathbf{n}$, the product of the invariant differential operators $\Delta_{\mathbf{m}}$ and $\Delta_{\mathbf{n}}$ is again an invariant differential operator; by Proposition 2, there must therefore exist coefficients $q^{\mathbf{k}}_{\mathbf{mn}}$ (only finitely many of which are nonzero, for each fixed $\mathbf{m}$ and $\mathbf{n}$) such that

$$\Delta_{\mathbf{m}} \Delta_{\mathbf{n}} = \sum_{\mathbf{k}} q^{\mathbf{k}}_{\mathbf{mn}} \Delta_{\mathbf{k}}. \tag{45}$$

(We remark that, similarly, there exist $\gamma^{\mathbf{k}}_{\mathbf{mn}}$ such that

$$K_{\mathbf{m}} K_{\mathbf{n}} = \sum_{\mathbf{k}} \gamma^{\mathbf{k}}_{\mathbf{mn}} \Delta_{\mathbf{k}}.$$

The coefficients $\gamma^{\mathbf{k}}_{\mathbf{mn}}$ are known as the *Pieri* (or *branching*, or *Clebsch-Gordan*) coefficients; however, there seems to be no established name for $q^{\mathbf{k}}_{\mathbf{mn}}$. Obviously,

$\gamma^{\mathbf{k}}_{\mathbf{mn}} = q^{\mathbf{k}}_{\mathbf{mn}}$ if $|\mathbf{k}| = |\mathbf{m}| + |\mathbf{n}|$, by comparing the top order terms in (45). Similarly, $q^{\mathbf{k}}_{\mathbf{mn}}$ is nonzero only if $|\mathbf{k}| \le |\mathbf{m}| + |\mathbf{n}|$.)

Theorem 26. *For any* $\mathbf{m}$ *and* $\mathbf{n}$*,*

$$\Delta_{\mathbf{m}} K_{\mathbf{n}} = \sum_{\mathbf{k}} q^{\mathbf{n}}_{\mathbf{km}} \frac{d_{\mathbf{n}}}{d_{\mathbf{k}}} K_{\mathbf{k}}. \tag{46}$$

The series on the right-hand side converges absolutely and uniformly on $\overline{\Omega}$*.*

Here, as before, $d_{\mathbf{m}}$ stands for the dimension of the Peter-Weyl space $\mathcal{P}_{\mathbf{m}}$.

Proof. Arguing as we did (for $\mathbf{m} = \mathbf{n}$) in the proof of Lemma 16 shows that for any fixed Jordan frame $e_1, \dots, e_r$,

$$\Delta_{\mathbf{m}} K_{\mathbf{n}}(k(t_1 e_1 + \dots + t_r e_r)) = F(t_1, \dots, t_r)$$

where F is a polynomial in t_j and $\sqrt{1 - t_j^2}$, $j = 1, \dots, r$; in particular, F extends to a holomorphic function on the polydisc $\mathbf{D}^r \subset \mathbf{C}^r$ and is continuous on its closure, and the Taylor expansion

$$F(t) = \sum_{\alpha \text{ multiindex}} f_\alpha t^\alpha$$

of F converges absolutely and uniformly on $\overline{\mathbf{D}}^r$. It is known that the stabilizer subgroup K acts transitively on the set of all Jordan frames; since $\pm e_1, \dots, \pm e_r$ and $e_{\sigma(1)}, \dots, e_{\sigma(r)}$ are also Jordan frames, for any choice of the signs $\pm$ and for any permutation σ of $\{1, \dots, r\}$, respectively, F must be invariant under all signed permutations of the variables $t_1, \dots, t_r$. Consequently, we even have

$$F(t) = \sum_{\alpha} f_{2\alpha} t^{2\alpha} \tag{47}$$

and $f_{2\sigma(\alpha)} = f_{2\alpha}$ for any permutation σ of $\{1, \dots, r\}$. Let $F_{2m}(t) = \sum_{|\alpha|=m} f_{2\alpha} t^{2\alpha}$ be the $2m$-homogeneous part of (47), $m = 0, 1, 2, \dots$; then F_{2m} is a homogeneous symmetric polynomial of $t_1^2, \dots, t_r^2$ of degree m. On the other hand, it is known that

$$K_{\mathbf{k}}(k(t_1 e_1 + \dots + t_r e_r)) = j_{\mathbf{k}} J^{(2/a)}_{\mathbf{k}}(t_1^2, \dots, t_r^2)$$

where $j_{\mathbf{k}}$ is a positive constant and $J^{(2/a)}_{\mathbf{k}}$ is the Jack symmetric polynomial with parameter $\frac{2}{a}$ [MD, Section 10 of Chapter VI]; furthermore, the Jack polynomials $J^{(2/a)}_{\mathbf{k}}$, $|\mathbf{k}| = m$, form a basis of the space of homogeneous polynomials of degree m. This means that there must exist constants $c^{\mathbf{k}}_{\mathbf{mn}}$ such that

$$F_{2m} = \sum_{|\mathbf{k}|=m} c^{\mathbf{k}}_{\mathbf{mn}} K_{\mathbf{k}}.$$

Feeding this back into (47), we conclude that, indeed,

$$\Delta_{\mathbf{m}} K_{\mathbf{n}} = \sum_{\mathbf{k}} c^{\mathbf{k}}_{\mathbf{mn}} K_{\mathbf{k}} \tag{48}$$

with the series converging absolutely and uniformly on the closure of Ω. It remains to identify the constants $c^{\mathbf{k}}_{\mathbf{mn}}$.

Let $\{\psi_{\mathbf{k}j}\}_{j=1}^{d_{\mathbf{k}}}$ and $\{\psi_{\mathbf{l}i}\}_{i=1}^{d_{\mathbf{l}}}$ be any orthonormal bases of $\mathcal{P}_{\mathbf{k}}$ and $\mathcal{P}_{\mathbf{l}}$, respectively. Then by (22) and (21)

$$\begin{aligned}\Delta_{\mathbf{l}} K_{\mathbf{k}}(0) &= K_{\mathbf{l}}(\partial,\partial) K_{\mathbf{k}}(z,z)\,|_{z=0} \\ &= \sum_{j,i} |\psi_{\mathbf{l}i}(\partial)\psi_{\mathbf{k}j}(0)|^2 \\ &= \sum_{j,i} |\langle \psi_{\mathbf{l}i}, \psi_{\mathbf{k}j}^* \rangle_F|^2 \\ &= \sum_i \|P_{\mathbf{k}}\psi_{\mathbf{l}i}\|_F^2 \qquad \text{by Lemma 10} \\ &= \delta_{\mathbf{kl}}\, d_{\mathbf{l}}.\end{aligned}$$

Since, for any smooth function g, $\Delta_{\mathbf{k}} g(0)$ depends only on the $(|\mathbf{k}|,|\mathbf{k}|)$-homogeneous part of the Taylor expansion of g, it is legitimate to apply $\Delta_{\mathbf{l}}$ to the series in (48) term-by-term. This yields

$$\Delta_{\mathbf{l}}(\Delta_{\mathbf{m}} K_{\mathbf{n}})(0) = \sum_{\mathbf{k}} c_{\mathbf{mn}}^{\mathbf{k}}\, \Delta_{\mathbf{l}} K_{\mathbf{k}}(0) = c_{\mathbf{mn}}^{\mathbf{l}}\, d_{\mathbf{l}}.$$

On the other hand, by (45),

$$\Delta_{\mathbf{l}}\Delta_{\mathbf{m}} K_{\mathbf{n}}(0) = \sum_{\mathbf{k}} q_{\mathbf{lm}}^{\mathbf{k}}\, \Delta_{\mathbf{k}} K_{\mathbf{n}}(0) = q_{\mathbf{lm}}^{\mathbf{n}}\, d_{\mathbf{n}}.$$

Thus $c_{\mathbf{mn}}^{\mathbf{l}} = \frac{d_{\mathbf{n}}}{d_{\mathbf{l}}}\, q_{\mathbf{lm}}^{\mathbf{n}}$, completing the proof. □

Note that Conjecture 25 is thus tantamount to the fact that the series in (46) terminates.

The following assertion is an immediate consequence of the symmetry $q_{\mathbf{km}}^{\mathbf{n}} = q_{\mathbf{mk}}^{\mathbf{n}}$, combined with the property (26) of the composition series, which implies that $\Delta_{\mathbf{m}} K_{\mathbf{n}} \equiv 0$ if $q(\mathbf{n}) < q(\mathbf{m})$ (cf. the proof of Proposition 11).

Proposition 27. $q_{\mathbf{km}}^{\mathbf{n}} = 0$ *if* $q(\mathbf{n}) < q(\mathbf{m})$ *or* $q(\mathbf{n}) < q(\mathbf{k})$.

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