Affine Moment Invariants

polynomial of moments

$$I = (\sum_{j=1}^{n_t} c_j \prod_{\ell=1}^{r} \mu_{p_{j\ell},q_{j\ell}})/\mu_{00}^{r+w} \ .$$

Cayley – Aronhold differential equation

$$\sum_p \sum_q p\mu_{p-1,q+1} \frac{\partial I}{\partial \mu_{pq}} = 0 \ .$$

Graph Method

$$I(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1y_2 - x_2y_1)(x_1y_3 - x_3y_1)(x_2y_3 - x_3y_2)^2$$
$$f(x_1, y_1)f(x_2, y_2)f(x_3, y_3)dx_1dy_1dx_2dy_2dx_3dy_3$$
$$= \mu_{20}\mu_{21}\mu_{03} - \mu_{20}\mu_{12}^2 - \mu_{11}\mu_{30}\mu_{03} + \mu_{11}\mu_{21}\mu_{12} + \mu_{02}\mu_{30}\mu_{12} - \mu_{02}\mu_{21}^2$$

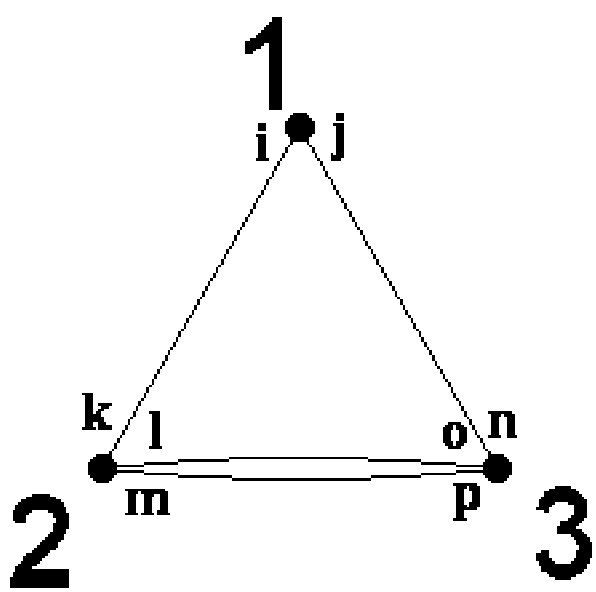

Figure 1: The graph corresponding to the invariant

Theorem All affine moment invariants in the polynomial form can be expressed as linear combinations of some invariants generated by the graph method

$$I^{(e)} = \sum_{P=1}^{n} c_P I_P^{(g)} \ .$$

$I^{(e)}$ – general affine moment invariant
$I_P^{(g)}, \ P = 1, \dots n$ set of invariants generated by the graph method with the same structure as $I^{(e)}$.

Tensor method

Moment tensor

$$M^{i_1 i_2 \cdots i_r} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^{i_1} x^{i_2} \cdots x^{i_r} f(x^1, x^2) dx^1 dx^2 \ , x^1 = x, x^2 = y$$

$M^{i_1 i_2 \cdots i_r} = m_{pq}$ if p indices equal 1 and q indices equal 2.
In affine transform

$$M^{i_1 i_2 \cdots i_r} = |J| p_{\alpha_1}^{i_1} p_{\alpha_2}^{i_2} \cdots p_{\alpha_r}^{i_r} \hat{M}^{\alpha_1 \alpha_2 \cdots \alpha_r}$$

– relative oriented contravariant tensor with the weight -1.

Covariant unit polyvector

$\epsilon_{i_1 i_2 \dots i_n}$ – skew-symmetric tensor over all indices and $\epsilon_{12 \dots n} = 1$.
2D: $\epsilon_{12} = 1$, $\epsilon_{21} = -1$, $\epsilon_{11} = 0$ and $\epsilon_{22} = 0$.
In affine transform

$$\hat{\epsilon}_{12 \dots n} = J \epsilon_{12 \dots n}$$

– relative affine invariant with weight 1.

$$I(f) = M^{ij} M^{klm} M^{nop} \epsilon_{ik} \epsilon_{jn} \epsilon_{lo} \epsilon_{mp}$$

Proof:

$I^{(e)}$ can be decomposed

$$I^{(e)} = K_{i_1 i_2 \ldots i_{2w}} B^{i_1 i_2 \ldots i_{2w}} \ ,$$

$$B^{i_1 i_2 \ldots i_{2w}} = M^{i_1 i_2 \ldots i_{d_1}} M^{i_{d_1+1} i_{d_1+2} \ldots i_{d_1+d_2}} \cdots M^{i_{2w-d_r+1} i_{2w-d_r+2} \ldots i_{2w}} / m_{00}^{w+r} \ ,$$

If some product of moments occurs m times in B, then the corresponding components of K must be multiplied by $1/m$.

$$K_{i_1 i_2 \ldots i_{2w}} B^{i_1 i_2 \ldots i_{2w}} = \sum_{P=1}^{(2w)!} c_P \epsilon_{\{i_1 i_2} \epsilon_{i_3 i_4} \cdots \epsilon_{i_{2w-1} i_{2w}\}_P} B^{i_1 i_2 \ldots i_{2w}} \ ,$$

$\{i_1 i_2 \cdots i_{2w}\}_P$ – P-th permutation

$$K_{i_1 i_2 \ldots i_{2w}} = \sum_{P=1}^{(2w)!} c_P \epsilon_{\{i_1 i_2} \epsilon_{i_3 i_4} \cdots \epsilon_{i_{2w-1} i_{2w}\}_P} \ .$$

Contravariant unit polyvector – skew-symmetric tensor over all indices and $\epsilon^{12\ldots n} = 1$

It is multiplied as contravariant tensor, e.g.

$$\epsilon_{i_1 i_2} \epsilon^{i_1 i_2} = 2 \ .$$

$$K_{i_1 i_2 \ldots i_{2w}} \epsilon^{x_1 x_2} \epsilon^{x_3 x_4} \cdots \epsilon^{x_{2w-1} x_{2w}} = \sum_{P=1}^{(2w)!} c_P^* \delta_{\{i_1}^{x_1} \delta_{i_2}^{x_2} \cdots \delta_{i_{2w}\}_P}^{x_{2w}} \ ,$$

$c_P^* = 2^w c_P$

2^{4w} equations for $(2w)!$ unknowns, but many of the equations are linearly dependent.

Rank $((2w)! - s)$, where $s > 0$

$$\sum_{P=1}^{(2w)!} \delta_{\{i_1}^{x_1} \delta_{i_2}^{x_2} \cdots \delta_{i_{2w}\}_P}^{x_{2w}} \lambda_P = 0 \ .$$

s linearly independent solutions

$$\lambda_P = \lambda_P^\sigma,\ \sigma = 1, 2, \dots, s$$

Add s equations

$$\sum_{P=1}^{(2w)!} \lambda_P^\sigma \lambda_P = 0$$

$2^{4w} + s$ equations.

Solution $\lambda_P = \lambda_P^0$, $P = 1, 2, \dots, (2w)!$

Linear combination of the solutions λ_P^σ

$$\lambda_P^0 = \sum_{\sigma=1}^{s} \alpha_\sigma \lambda_P^\sigma,\ P = 1, 2, \dots, (2w)!$$

$$\sum_{\sigma=1}^{s} \alpha_\sigma \sum_{P=1}^{(2w)!} \lambda_P^\sigma \lambda_P = 0$$

$$\sum_{P=1}^{(2w)!} \lambda_P^0 \lambda_P = 0$$

$$\sum_{P=1}^{(2w)!} (\lambda_P^0)^2 = 0$$

i.e. $\lambda_1^0 = \lambda_2^0 = \cdots = \lambda_{(2w)!}^0 = 0$

Rank $(2w)!$

$$\sum_{P=1}^{(2w)!} \lambda_P^\sigma \delta_{\{i_1}^{x_1} \delta_{i_2}^{x_2} \cdots \delta_{i_{2w}\}_P}^{x_{2w}} = 0\ .$$

$p_{x_1 x_2 \dots x_{2w}}$ – arbitrary tensor of covariance $2w$

$$\delta_{i_1}^{x_1} \delta_{i_2}^{x_2} \dots \delta_{i_{2w}}^{x_{2w}} p_{x_1 x_2 \dots x_r} = p_{i_1 i_2 \cdots i_{2w}}$$

$$\sum_{P=1}^{(2w)!} \lambda_P^\sigma p_{\{i_1 i_2} \cdots i_{2w}\}_P = 0$$

$$\sum_{P=1}^{(2w)!} \lambda_P^\sigma c_P^* \delta_{\{i_1}^{x_1} \delta_{i_2}^{x_2} \cdots \delta_{i_{2w}\}_P}^{x_{2w}} = 0,\ \sigma = 1, 2, \dots, s\ \Box$$

References

[1] G. B. Gurevich, *Foundations of the Theory of Algebraic Invariants*. Groningen, The Netherlands: Nordhoff, 1964.