Invariants for Recognition of Degraded 1-D Digital Signals

Jan Flusser and Tomáš Suk

Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
Pod vodárenskou věží 4, 182 08 Praha 8, Czech Republic
E-mail: flusser@utia.cas.cz

Abstract

The paper is devoted to the feature-based recognition of degraded signals acquired by a linear time-invariant system. The proposed approach consists of describing signals by features which are invariant with respect to the degradation and recognizing signals in the feature space. Neither impulse response identification nor signal restoration are required. Thanks to this, the new approach is in many applications much more effective than the classical one based on blind deconvolution. Two sets of appropriate blur-invariant features (the first one in time domain, the other one in spectral domain) are introduced in this paper. The proof of their invariance is given and their performance is illustrated by numerical experiments.

1. Introduction

One of very frequent tasks in digital signal processing is the classification of 1-D finite (i.e. time-limited) signals (experimental curves) with respect to the template curves stored in a database. This task appears in EEG and ECG processing, speech recognition as well as in many other application areas.

The crucial problem is that the acquired curve represents often only a degraded version of the original signal. Standard noise-free model of a time-invariant linear system [1] describes the acquisition process by a convolution of an unknown original signal $f(t)$ with the system impulse response $h(t)$:

$$g(t) = (f * h)(t) \tag{1}$$

where $g(t)$ represents the observed (i.e. blurred) signal. However, the impulse response $h(t)$ is in our case unknown.

Classical "blind-restoration" approach to this problem consists of estimation of the impulse response via blind deconvolution [2] or via modeling of the blurred signal as an ARMA stochastic process [3]-[5]. After the impulse response has been identified, the blurred signal is restored by inverse filter, Wiener filter or by any other similar technique (see [1] for a survey). Finally, the restored signal is compared with each template from the database.

In this paper, a different approach is presented. All signals (the blurred one as well as those in the database) are represented by a set of features, which are invariant to the impulse response $h(t)$ and which are able to distinguish among different signals. The blurred signal is then classified in the Euclidean feature space (by minimum distance rule for instance) without any impulse response identification and signal restoration.

Blur-invariant features introduced in this paper are based on central moments of a signal. The relation between moments of the original and blurred signals is shown and a recursive formula for computing invariants is derived. The performance of the invariants is demonstrated experimentally. Finally, invariants defined in the spectral domain are introduced and the relationship between both types of features is shown.

2. Signal Characterization by its Central Moments

Let $f(t)$ be a continuous function with bounded support. Its p-th order central moment $\mu_p^{(f)}$ is defined by the integral

$$\mu_p^{(f)} = \int_{-\infty}^{\infty} (t - c^{(f)})^p f(t)dt, \tag{2}$$

where $c^{(f)}$ is a mean value (centroid) of $f(t)$:

$$c^{(f)} = \frac{\int_{-\infty}^{\infty} t \cdot f(t)dt}{\int_{-\infty}^{\infty} f(t)dt}.$$

The moments of a blurred signal $g(t)$ given by (1) can be expressed in terms of moments of the original signal and the impulse response:

$$\mu_p^{(g)} = \sum_{k=0}^{p} \binom{p}{k} \mu_{p-k}^{(f)} \mu_k^{(h)}. \tag{3}$$

To prove this, we only apply the definitions (1) and (2) and the fact that $c^{(g)} = c^{(f)} + c^{(h)}$.

3. Invariants in the Time Domain

In this Section, we derive moment-based signal features which are independent of the degradation, i.e. independent of the type and parameters of $h(t)$. Feature S is called *blur invariant* if and only if $S^{(g)} \equiv S^{(f*h)} = S^{(f)}$ for any $h(t)$.

Let's deal with *symmetric* and *energy-preserving* impulse responses only, i.e. let $h(t) = h(-t)$ and $\mu_0^{(h)} = 1$. Then, due to the symmetry, $\mu_p^{(h)} = 0$ if p is odd.

Theorem 1: Let p be an odd integer and let $K = (p-1)/2$. Then the feature S_p defined recursively as

$$S_p = \mu_p - \frac{1}{\mu_0} \sum_{n=1}^{K} \binom{p}{2n} S_{p-2n} \cdot \mu_{2n} \tag{4}$$

is a blur invariant for any odd p.

Proof:

- $p = 1$: $S_1^{(g)} = S_1^{(f)} = 0$ regardless of $f(t)$ and $h(t)$.
- $p = 3$:

$$S_3^{(g)} = \mu_3^{(g)} = \sum_{k=0}^{3} \binom{3}{k} \mu_{3-k}^{(f)} \mu_k^{(h)} = \mu_3^{(f)} = S_3^{(f)}.$$

- Provided the Theorem is valid for $S_1, S_3, \cdots, S_{p-2}$. Then

$$\begin{aligned}
S_p^{(g)} &= \mu_p^{(g)} - \frac{1}{\mu_0^{(g)}} \sum_{n=1}^{K} \binom{p}{2n} S_{p-2n}^{(g)} \cdot \mu_{2n}^{(g)} = \\
&= \sum_{k=0}^{p} \binom{p}{k} \mu_{p-k}^{(f)} \mu_k^{(h)} - \\
&\quad - \frac{1}{\mu_0^{(f)}} \sum_{n=1}^{K} \binom{p}{2n} S_{p-2n}^{(f)} \sum_{k=0}^{2n} \binom{2n}{k} \mu_{2n-k}^{(f)} \mu_k^{(h)} = \\
&= S_p^{(f)} + \sum_{k=1}^{p} \binom{p}{k} \mu_{p-k}^{(f)} \mu_k^{(h)} - \\
&\quad - \frac{1}{\mu_0^{(f)}} \sum_{n=1}^{K} \binom{p}{2n} S_{p-2n}^{(f)} \sum_{k=1}^{2n} \binom{2n}{k} \mu_{2n-k}^{(f)} \mu_k^{(h)} = \\
&= S_p^{(f)} + \sum_{k=1}^{p} \binom{p}{k} \mu_{p-k}^{(f)} \mu_k^{(h)} - \\
&\quad - \frac{1}{\mu_0^{(f)}} \sum_{n=1}^{K} \sum_{k=1}^{2n} \binom{p}{k} \binom{p-k}{2n-k} S_{p-2n}^{(f)} \cdot \mu_{2n-k}^{(f)} \mu_k^{(h)}.
\end{aligned}$$

Introducing $m = \left[\frac{k+1}{2}\right]$ and changing the order of a summation, we get

$$\begin{aligned}
S_p^{(g)} &= S_p^{(f)} + \sum_{k=1}^{p-1} \binom{p}{k} \mu_{p-k}^{(f)} \mu_k^{(h)} - \\
&- \frac{1}{\mu_0^{(f)}} \sum_{k=1}^{2K} \sum_{n=m}^{K} \binom{p}{k} \binom{p-k}{2n-k} S_{p-2n}^{(f)} \cdot \mu_{2n-k}^{(f)} \mu_k^{(h)} = \\
&= S_p^{(f)} + \sum_{k=1}^{p-1} \binom{p}{k} \mu_k^{(h)} (\mu_{p-k}^{(f)} - \\
&\quad - \frac{1}{\mu_0^{(f)}} \sum_{n=m}^{K} \binom{p-k}{2n-k} S_{p-2n}^{(f)} \cdot \mu_{2n-k}^{(f)}).
\end{aligned}$$

Using a shorter notation, we get

$$S_p^{(g)} = S_p^{(f)} + \sum_{k=1}^{p-1} \binom{p}{k} \mu_k^{(h)} \cdot D_{p,k}.$$

If k is odd then $\mu_k^{(h)} = 0$. If k is even then $m = k/2$ and from the definition (4) follows that

$$S_{p-k}^{(f)} = \mu_{p-k}^{(f)} - \frac{1}{\mu_0^{(f)}} \sum_{n=1}^{\frac{p-k-1}{2}} \binom{p-k}{2n} S_{p-k-2n}^{(f)} \cdot \mu_{2n}^{(f)}$$

and, consequently,

$$\begin{aligned}
D_{p,k} &= \mu_{p-k}^{(f)} - \frac{1}{\mu_0^{(f)}} \sum_{n=m}^{\frac{p-1}{2}} \binom{p-k}{2n-k} S_{p-2n}^{(f)} \cdot \mu_{2n-k}^{(f)} = \\
&= \mu_{p-k}^{(f)} - \frac{1}{\mu_0^{(f)}} \sum_{n=0}^{\frac{p-k-1}{2}} \binom{p-k}{2n} S_{p-k-2n}^{(f)} \cdot \mu_{2n}^{(f)} = \\
&= S_{p-k}^{(f)} - \frac{1}{\mu_0^{(f)}} S_{p-k}^{(f)} \cdot \mu_0^{(f)} = 0.
\end{aligned}$$

Thus $S_p^{(g)} = S_p^{(f)}$ for any odd p. □

If we use regular moments

$$m_p^{(f)} = \int_{-\infty}^{\infty} t^p \cdot f(t) dt \tag{5}$$

instead of central moments, eq. (4) as well as Theorem 1 are still valid but the invariants S_p are no longer invariant to a time-shift.

Note that there were not defined any invariants if p is even.

4. Numerical Experiment

To demonstrate the performance of the above derived invariants, the following experiment was carried out.

Fig. 1a shows the original signal (the number of samples $N = 256$) and Fig. 1b shows its degraded version. The degradation was introduced by an averaging over 9-point neighborhood. Another signal (different from the previous ones) is shown in Fig. 1c to demonstrate the discriminability of the features.

The invariants $S_3, S_5, \cdots, S_{15}$ were calculated for each signal and normalized by the factor $\mu_0 \cdot N^p$. We denote the normalized invariants as S'_p. Their values are summarized in Table I. It can be seen that the feature values of the first two signals are almost the same whereas the corresponding values of the third signal are significantly different. These facts illustrate both the invariance and discrimination power of the features.

Tab. I : The values of the normalized invariants S'_p (multiplied by 10^3) in the case of the original signal, the blurred signal and another signal (see Fig. 1)

	Original signal	Blurred signal	Another signal
S'_3	3.9774	3.9774	10.6681
S'_5	1.4234	1.4246	3.3798
S'_7	0.4131	0.4140	0.9236
S'_9	0.1119	0.1123	0.2441
S'_{11}	0.0295	0.0297	0.0642
S'_{13}	0.0077	0.0078	0.0170
S'_{15}	0.0020	0.0020	0.0045

5. Invariants in the Spectral Domain

In this Section, blur invariants in the Fourier spectral domain are investigated.

Due to the properties of the Fourier transform, the corresponding relation to eq. (1) in spectral domain has a form

$$G(u) = F(u) \cdot H(u), \tag{6}$$

where $G(u)$, $F(u)$ and $H(u)$ are the Fourier transforms of the functions $g(t)$, $f(t)$ and $h(t)$, respectively. Considering the amplitude and phase separately, we get

$$|G(u)| = |F(u)| \cdot |H(u)| \tag{7}$$

and

$$\text{ph}G(u) = \text{ph}F(u) + \text{ph}H(u). \tag{8}$$

Due to the symmetry of $h(t)$, its Fourier transform $H(u)$ is also symmetric and real (this means the phase of $H(u)$ is only a two-valued function):

$$H(-u) = H(u),$$

$$\text{ph}H(u) \in \{0; \pi\}.$$

It is well-known that the phase is carrying significant information about the original signal [1]. Thus a lot of signal classification methods are based on phase matching. Unfortunately, ph$G(u)$ itself is not invariant with respect to blur and therefore it cannot be directly used for signal recognition. To overcome this, let us consider the ratio

$$r_G(u) = \frac{\text{Im}G(u)}{\text{Re}G(u)},$$

which is in a fact equal to tan(ph$G(u)$). It is evident that

$$r_G(u) = \frac{H(u) \cdot \text{Im}F(u)}{H(u) \cdot \text{Re}F(u)} = r_F(u).$$

Thus $r_G(u)$ is invariant with respect to convolution of the original signal with any symmetric impulse response $h(t)$.

The following Theorem shows the close relationship between $r_G(u)$ and moment invariants S_p.

Theorem 2: Using the notation given above, the following relation holds for any signal $g(t)$:

$$r_G(u) = \sum_{n=0}^{\infty} c_n u^n, \tag{9}$$

where

$$c_n = \frac{(-1)^{(n-1)/2} \cdot (-2\pi)^n}{n! \cdot m_0} S_n \tag{10}$$

if n is odd and $c_n = 0$ if n is even.

Proof:

The definition of Fourier transform implies that the spectrum of any function g(t) can be expressed by means of moments as a power series:

$$G(u) = \int_{-\infty}^{\infty} g(t) \cdot e^{-2\pi i u t} dt =$$

$$= \int_{-\infty}^{\infty} g(t) \cdot \sum_{n=0}^{\infty} \frac{(-2\pi i u)^n}{n!} t^n dt = \sum_{n=0}^{\infty} \frac{(-2\pi i)^n}{n!} m_n u^n.$$

Thus

$$r_G(u) = \frac{\text{Im}G(u)}{\text{Re}G(u)} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n (-2\pi)^{2n+1}}{(2n+1)!} m_{2n+1} u^{2n+1}}{\sum_{n=0}^{\infty} \frac{(-1)^n (-2\pi)^{2n}}{(2n)!} m_{2n} u^{2n}}.$$

Since the series in the numerator and denominator are absolutely convergent, their ratio can be also expressed as a power series

$$r_G(u) = \sum_{n=0}^{\infty} c_n u^n$$

that must accomplish the relation

$$\begin{aligned}&\textstyle\sum_{n=0}^{\infty}\frac{(-1)^n(-2\pi)^{2n+1}}{(2n+1)!}m_{2n+1}u^{2n+1} = \\ &= \textstyle\sum_{n=0}^{\infty}c_n u^n \cdot \sum_{n=0}^{\infty}\frac{(-1)^n(-2\pi)^{2n}}{(2n)!}m_{2n}u^{2n}.\end{aligned} \qquad (11)$$

It follows immediately from (11) that $c_n = 0$ for any even n.

Let's prove that

$$c_n = \frac{(-1)^{(n-1)/2}\cdot(-2\pi)^n}{n!\cdot m_0}S_n$$

for any odd n.

- $n = 1$
  It follows from (12) that $c_1 = -2\pi m_1/m_0$. On the other hand,

  $$\frac{(-1)^0\cdot(-2\pi)^1}{1!\cdot m_0}S_1 = \frac{-2\pi m_1}{m_0} = c_1.$$

- Let's suppose the assertion has been proven for $c_1, c_3, \cdots, c_{p-2}$. It follows from (11) that

  $$\begin{aligned}&\frac{(-1)^{(p-1)/2}\cdot(-2\pi)^p}{p!}m_p = \\ &= c_p m_0 + \sum_{n=1}^{(p-1)/2}\frac{(-1)^n(-2\pi)^{2n}}{(2n)!}c_{p-2n}m_{2n}.\end{aligned}$$

  Using (11) we get

$$\begin{aligned}c_p m_0 &= \frac{(-1)^{(p-1)/2}\cdot(-2\pi)^p}{p!}m_p - \\ &- \sum_{n=1}^{(p-1)/2}\frac{(-1)^{(p-1)/2}(-2\pi)^p}{(2n)!\cdot(p-2n)!\cdot m_0}S_{p-2n}\cdot m_{2n}, \\ c_p &= \frac{(-1)^{(p-1)/2}\cdot(-2\pi)^p}{p!\cdot m_0}\cdot \\ &(m_p - \frac{1}{m_0}\sum_{n=1}^{(p-1)/2}\binom{p}{2n}S_{p-2n}\cdot m_{2n}), \\ c_p &= \frac{(-1)^{(p-1)/2}\cdot(-2\pi)^p}{p!\cdot m_0}S_p. \qquad \square\end{aligned}$$

6. Discussion and Conclusion

In this paper, a new technique for recognition of signals degraded by a linear time-invariant system is presented. The proposed approach does not require impulse response identification and signal restoration.

Two groups of invariant features for signal representation were introduced. The features belonging to the first group are composed from central moments of the signal in time domain.

One of the advantageous properties of this kind of features is their high robustness with respect to additive random noise. Since the moments are computed by an integral over the whole signal, zero-mean random noise is averaged out and does not affect significantly the values of the invariants. On the other hand, the discriminative power of these features is limited – if, for instance, $g(t)$ is symmetric then all invariants S_p are equal to zero.

The second group of invariants is defined in spectral domain as tangent of the signal phase. It has been shown that both groups of features are equivalent in the following sense: a set $\{S_n\}_{n=1}^{\infty}$ of moment invariants in the time domain is unambiguously determined by phase tangent and vice versa. However, the numerical properties of both features are slightly different in the case of discrete signals. From computational point of view, the spectral features seem to be superior. If the signal has N samples, the computation of the spectral features by FFT takes only $O(N\cdot\log N)$ operations, whereas the computational cost of $S_1, S_3, \cdots, S_N$ through (4) is $O(N^2)$. On the other hand, only first p ($p << N$) features $S_1, S_3, \cdots, S_p$ are often sufficient for signal classification. Moreover, lower-order invariants $S_1, S_3, \cdots, S_p$ are more robust with respect to additive noise than the spectral features.

The proposed invariant-based approach to the signal recognition could be very effective also in image recognition. Large amount of effort has been spent to find blur invariants in 2-D. Although the situation is much more complicated than in 1-D, the first results have been presented recently in [6] and [7].

7. Acknowledgement

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8. References

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