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Extension of Moment Features' Invariance to Blur

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Abstract Moment invariants are features calculated on an image, which do not change their values after a transformation of the image. This paper focuses on the so called combined invariants, which obey additional requirement of invariance to image blurring. Our first contribution is a review of achievements most relevant to the derivation of algebraic, moment and combined invariants. The review explains and develops parallels between the moment and the blur invariants. Gradually, it reveals new properties, simplifying construction of the combined invariants, but having more general extent. Resulting substitution rules for easy construction of the combined invariants from other invariants are thus the main results of this paper. All the conclusions can be understood without knowledge of the tensor calculus. This paper addresses construction of the combined invariants in arbitrary dimension.

Keywords Blur invariants · Moment invariants · Algebraic invariants · Image registration · N-D imaging

1 Introduction

Analysis and interpretation of an image which was acquired by a real (i.e. non-ideal) imaging system is the key problem in many application areas such as remote sensing, astronomy and medicine, among others. Since real imaging systems as well as imaging conditions are usually imperfect, the observed image represents only a degraded version of the original scene. Various kinds of degradations (geometric as well as radiometric) are introduced into the image during the acquisition by such factors as imaging geometry, lens aberration, wrong focus, motion of the scene, systematic and random sensor errors, etc.

Generally, relation between the ideal image f and the observed image g is described as $g = \mathcal{D}(f)$, where $\mathcal{D}$ is a degradation operator. In the case of a linear shift-invariant imaging system, $\mathcal{D}$ usually has a form of

$$g = \tau(f) * h + n, \tag{1}$$

where h is the point-spread function (PSF) of the system, n is an additive random noise, τ is a transform of spatial coordinates due to projective imaging geometry and $*$ denotes an N-D convolution. Knowing the image g, our objective is to analyze the unknown scene f. Descriptors invariant to the degradation are thus needed in pattern recognition, fulfilling $I(g) = \Lambda I(f)$. Λ can be a function of only the degradation parameters, or $\Lambda = 1$.

Moment invariants belong to the most popular invariant features. Moment forms invariant to geometric deformations were used by Dudani [1] and Belkasim [2] to recognize aircraft silhouette. Wong and Hall [3], Goshtasby [4] and Flusser and Suk [5] employed moment invariants in template matching and registration of satellite images, Mukundan [6, 7] applied them to estimate the position and the attitude of the object in 3-D space, Sluzek [8] proposed to use local moment invariants in industrial quality inspection and many authors used moment invariants for character recognition [2, 9–12]. Maitra [13] and Hupkens [14] made them

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invariant also to contrast changes, Wang [15] proposed illumination invariants particularly suitable for texture classification. Li [16] and Wong [17] presented the systems of invariants up to the orders of nine and five, respectively.

There is also an alternative approach to deriving invariants called *image normalization* (see [18] and [19] for instance). First, the object is brought into certain "normalized" or "canonical" position, which is independent of the actual position, rotation, skewing, etc. of the original object. In this way, the influence of the deformation is eliminated. Since the normalized position is the same for all objects differing from each other just by the assumed transform, the moments of the normalized object are in fact invariants of the original object. Although this approach seems to be different, it finally ends up with the same moment functions as the "explicit" invariants.

All the above mentioned invariants deal with geometric distortion of the objects. Much less attention has been paid to invariants with respect to changes of the image intensity function (we call them radiometric invariants) and to combined radiometric-geometric invariants. In fact, just the invariants both to radiometric and geometric image degradations are necessary to resolve practical object recognition tasks because usually both types of degradations are present in input images.

Van Gool et al. introduced so-called affine-photometric invariants of gray-level [20] and color [21] images. These features are invariant to the affine transform and to the change of contrast and brightness of the image simultaneously. A pioneer work on this field was done by Flusser and Suk [22] who derived invariants to convolution with an arbitrary centrosymmetric PSF. From the geometric point of view, their descriptors were invariant to translation only. Despite of this, the invariants have found successful applications in face recognition on out-of-focused photographs [23], in normalizing blurred images into the canonical forms [24, 25], in template-to-scene matching of satellite images [22], in blurred digit and character recognition [15, 26], in registration of images obtained by digital subtraction angiography [27] and in focus/defocus quantitative measurement [28]. Other sets of blur invariants (but still only shift-invariant) were proposed for some particular kinds of PSF—axisymmetric blur invariants [29] and motion blur invariants [30, 31]. A significant improvement motivated by a problem of registration of blurred images was made by Flusser et al. They introduced so-called combined blur-rotation invariants [32] and combined blur-affine invariants [33] and reported their successful usage in satellite image registration [34] and in camera motion estimation [35]. These invariants were generalized to arbitrary dimensions in [36] and [37]. Later, method how to add blur invariance to a general N-D rotation [38] was presented.

From the practitioners' point of view, it is very useful to find as many simplifications of the moment invariant construction, as possible. For construction of the plain moment invariants, the Fundamental Theorem of Moment Invariants (FTMI) was introduced in [39]. Provided, the so called algebraic invariants to certain geometric transformations are known, it allows to construct the analogous moment invariants.

The FTMI uses results of the theory of algebraic invariants, which was introduced in the 19-th century [40]. For a survey about its main results and its connections to tensor calculus, see [41]. Although the center of gravity of the theory of algebraic invariants is the so-called algebraic forms, it is fully equivalent to the tensor view of it. The algebraic forms approach is more acceptable to people without knowledge of tensor calculus. On the other hand, tensor calculus shows more clearly, how to construct invariants from diverse objects used in image processing. This contribution aims to take advantage of both approaches.

A similar practical tool for construction of the combined invariants has been missing. Substitution rules for easy construction of the combined invariants from other invariants are the main results of this paper. This paper is intended to help practitioners in the combined invariant construction by providing links to existing works deriving other invariants and by substantial simplification of subsequent construction of the combined invariants. This contribution should promote usage of the blur invariance in new circumstances, due to good understanding of the underlying principles. Validity of the derived results, as well as invariance and discriminability of the derived combined invariants, are documented on examples from the literature. All the published specialized applications are thus for the first time unified here. Finally, this contribution generally addresses construction of the combined invariants in arbitrary dimension.

The main results of this paper are illustrated by several examples. In the next section, notation and some preliminary results have to be presented for later convenience. It includes brief derivation of the plain blur invariants and it recalls basic concepts of the theory of algebraic forms.

2 Preliminaries

In this section we recall some basic definitions and lemma which are important to introduce the following theory. We will use the notation proposed in [36]:

Notation: For $N \geq 1$, given the $x_i \in \mathbb{R}$, $p_i \in \mathbb{N}_0$, $k_i \in \mathbb{N}_0$ ($\mathbb{R}$ and $\mathbb{N}_0$ denote the sets of real numbers and non-negative integers, respectively). Then

$$\mathbf{x} \equiv (x_1, \ldots, x_N)^T$$

denotes N-dimensional vector of coordinates,

$$\mathbf{p} \equiv (p_1, \ldots, p_N)^T, \qquad \mathbf{k} \equiv (k_1, \ldots, k_N)^T$$

denote N-dimensional vectors of parameters. Logical relations are defined analogically to

$$\mathbf{p} < \mathbf{k} \quad \Longrightarrow \quad p_i < k_i \quad \text{for } \forall i.$$

The following notation is further introduced:

$$\mathbf{dx} \equiv dx_1 \cdots dx_N, \qquad |\mathbf{p}| \equiv \sum_{i=1}^{N} p_i,$$

$$\mathbf{x}^{\mathbf{p}} \equiv \prod_{i=1}^{N} x_i^{p_i}, \quad \mathbf{p}! \equiv \prod_{i=1}^{N} (p_i!),$$

$$\binom{\mathbf{p}}{\mathbf{k}} \equiv \prod_{i=1}^{N} \binom{p_i}{k_i}.$$

In our notation, $|\mathbf{p}|$ thus refers to the sum of the vector components, not to the usual euclidean length.

In the following text other N-dimensional vectors are used (they are denoted by bold symbols). We believe, it will be clear from the context, whether it is a coordinate or a parameter vector.

Definition 1 By N-dimensional *image function* (or *image*) we understand any real function $f(\mathbf{x}) \in L_1(\mathbb{R}^N)$ which has bounded support and non-zero integral.

Definition 2 *Ordinary geometric moment* $m_{\mathbf{p}}$ of order $|\mathbf{p}|$ of the image $f(\mathbf{x})$ is defined by the integral

$$m_{\mathbf{p}}[f] = \int_{R^N} \mathbf{x}^{\mathbf{p}} f(\mathbf{x}) \mathbf{dx}. \tag{2}$$

In the following text, provided there is no risk of confusion, we will not always denote, which function the moments are calculated on (later also Fourier transform and other moment forms).

Definition 3 *Central geometric moment* $\mu_{\mathbf{p}}$ of order $|\mathbf{p}|$ of the image $f(\mathbf{x})$ is defined as the ordinary geometric moment of the image obtained from $f(\mathbf{x})$ by shifting its center of gravity to the origin of the coordinate system.

Definition 4 Fourier transform (or spectrum) $F(\mathbf{u})$ of the image $f(\mathbf{x})$ is defined as

$$F[f](\mathbf{u}) = \int_{R^N} f(\mathbf{x}) e^{-2\pi i \mathbf{u}\cdot\mathbf{x}} \mathbf{dx}, \tag{3}$$

where i is the imaginary unit.

By means of expansion of the exponential in the Fourier transform (3) we get the following theorem.

Theorem 5 *Fourier transform of an image $f(\mathbf{x})$ can be expanded into power series*

$$F[f](\mathbf{u}) = \sum_{\mathbf{0}\le\mathbf{k}} \frac{(-2\pi i)^{|\mathbf{k}|}}{\mathbf{k}!} m_{\mathbf{k}}[f] \mathbf{u}^{\mathbf{k}} \tag{4}$$

where the coefficients $m_{\mathbf{k}}[f]$ are ordinary geometric moments.

3 Blur Invariants

The combined invariants are moment forms invariant both to geometric degradations and blur. In this section, blur invariants, which have been published, are reviewed. Blur can be often modeled by convolution. In this paper, we will consider only convolution with kernels which are constant throughout the image and which are centrally symmetrical, i.e.

$$h(\mathbf{x}) = h(-\mathbf{x}). \tag{5}$$

The following two theorems are of fundamental importance for derivation of the corresponding invariants.

Theorem 6 *Tangent of the Fourier transform phase is blur invariant.*

Proof To prove this theorem it is sufficient to realize that the phase of the Fourier transform of $h(\mathbf{x})$, as of centrally symmetrical function, can equal only 0 or π. □

Theorem 7 *Tangent of the Fourier transform phase of an image $f(\mathbf{x})$ can be expanded into power series* (*except of the points in which $F(\mathbf{u}) = 0$ or* $\mathrm{ph}F(\mathbf{u}) = \pm\pi/2$)

$$\tan(\mathrm{ph}F[f](\mathbf{u})) = \frac{\mathrm{Im}F[f](\mathbf{u})}{\mathrm{Re}F[f](\mathbf{u})} = \sum_{\mathbf{0}\le\mathbf{k}} c_{\mathbf{k}}[f]\mathbf{u}^{\mathbf{k}}, \tag{6}$$

where the $c_{\mathbf{k}}$'s are also blur invariants.

Proof The $\tan(\mathrm{ph}F(\mathbf{u}))$ is a ratio of two absolutely convergent power series, thus it can be also expressed as the power series. Invariance of $c_{\mathbf{k}}$'s follows from comparison of coefficients of the same monomials $\mathbf{u}^{\mathbf{k}}$. □

After expansion, (6) can be simplified using substitution

$$Q_{\mathbf{p}} = \frac{c_{\mathbf{p}}\mathbf{p}!}{(-1)^{(|\mathbf{p}|-1)/2}(-2\pi)^{|\mathbf{p}|}}. \tag{7}$$

$Q_{\mathbf{p}}$'s are an equivalent set of blur invariants. Since $m_{\mathbf{0}} \neq 0$ (see Definition 1), we can straightforwardly derive the following theorem from (6).

Theorem 8 *The moment forms $Q_{\mathbf{p}}$ are invariant to an N-dimensional convolution with a centrally symmetrical kernel.*

$$Q_{\mathbf{p}} = \frac{m_{\mathbf{p}}}{m_{\mathbf{0}}} - \frac{1}{m_{\mathbf{0}}} \sum_{\substack{\mathbf{0}\le\mathbf{n}\le\mathbf{p} \\ 0<|\mathbf{n}|<|\mathbf{p}|}} \binom{\mathbf{p}}{\mathbf{n}} Q_{\mathbf{p}-\mathbf{n}} m_{\mathbf{n}} \quad \text{for odd } |\mathbf{p}|, \qquad (8)$$

$$Q_{\mathbf{p}} = 0 \qquad \text{for even } |\mathbf{p}|.$$

We would like to remark that (8) implies immediately independence of $Q_{\mathbf{p}}$'s for odd $|\mathbf{p}|$.

4 Algebraic Invariants

Normalized moments and many other objects used in pattern recognition can be described as tensor coordinates in certain coordinate system. Although tensor calculus is mentioned several times here, its knowledge is not necessary for understanding the results of this paper. However, you might wish to recall the basics from [41].

Given spatial coordinates $\mathbf{x}$ are transformed into $\mathbf{x}'$ as follows:

$$\mathbf{x}' = A\mathbf{x}. \qquad (9)$$

Intuitively speaking, tensor is an object, whose coordinates (denoted schematically a) are under (9) linearly transformed into new coordinates a'.

An important task of pattern recognition is to look for functions of the tensor coordinates, which stay invariant to the transformation (9), or which change in an easy to handle way. This is formulated in the next definition.

Definition 9 Given $I(a, b, \ldots)$ is a function of coordinates of tensors a, b, ... (not necessarily of all of them). If $I(a, b, \ldots)$, under the transformation of spatial coordinates (9), fulfills the condition

$$I(a', b', \ldots) = \Lambda I(a, b, \ldots) \qquad (10)$$

where Λ is only a function of the transformation parameters from (9), then it is called *relative algebraic invariant*. If $\Lambda = 1$ then $I(a, b, \ldots)$ is an *absolute* invariant. For detailed discussion about Λ, see [41].

There are standard methods how to calculate invariants from the tensor coordinates. If (9) is a general linear transformation, the methods are well described in [41]. It is clear, it helps to identify the moments and other interesting objects as tensor coordinates (or to adapt them so), if possible. Then either the mentioned methods can be applied on them or the objects can be substituted into invariants found in the literature.

However, in the literature concerning moment invariants, a different approach has been traditionally used. Instead of working with invariants on tensor coordinates, invariants on coefficients of algebraic forms have been preferred.

Definition 10 *n-ary algebraic form of order p* is defined as a homogeneous polynomial

$$f = \sum_{|\mathbf{p}|=p} \frac{p!}{\mathbf{p}!} a_{\mathbf{p}} \mathbf{u}^{\mathbf{p}} \qquad (11)$$

of elements of n-dimensional vector of variables $\mathbf{u}$, where $p = |\mathbf{p}|$.

The notion of invariant here is analogous. Given $I(a)$ is a function of coefficients $a_{\mathbf{p}}$ (not necessarily of all of them), which originate from the n-ary algebraic form (11) of order p. After substitution of a general linear transformation

$$\mathbf{u} = A^T \mathbf{u}' \qquad (12)$$

into the algebraic form (11), new coefficients $a'_{\mathbf{p}}$ corresponding to monomials $\mathbf{u}'^{\mathbf{p}}$ appear.

$$f = \sum_{|\mathbf{p}|=p} \frac{p!}{\mathbf{p}!} a_{\mathbf{p}} \mathbf{u}^{\mathbf{p}} = \sum_{|\mathbf{p}|=p} \frac{p!}{\mathbf{p}!} a'_{\mathbf{p}} \mathbf{u}'^{\mathbf{p}}. \qquad (13)$$

If $I(a)$ fulfills the condition (10), then it is an invariant of the form.

Relation of moments to tensor coordinates is more visible, if we introduce the tensor notation of moments and other forms (denoted by tilde). For example, a 2-D moment of order 3 ($p + q = 3$; indices i_1, i_2, i_3 can be either 1 or 2; number of 1's is p, number of 2's is q; $x_i = x^i$ here) can be written as

$$\begin{aligned} m_{p,q} &\equiv \int_{R^2} x_1^p x_2^q f(x_1, x_2) dx_1 dx_2 \\ &= \int_{R^2} x^{i_1} x^{i_2} x^{i_3} f(x^1, x^2) dx^1 dx^2 \\ &\equiv \widetilde{m}^{i_1, i_2, i_3}. \end{aligned} \qquad (14)$$

It is possible to look at the value of the form f (13) as at a simultaneous (absolute) invariant of the vector $\mathbf{u}$ and of the set of coefficients $a_{\mathbf{p}}$ under the transformation A. The vector $\mathbf{u}$ is transformed as a covariant vector (i.e. the transformation is determined by (12)). We can use a basic fact from the tensor calculus, see e.g. [41]: If the form f stays invariant and $\mathbf{u}$ is transformed like a covariant vector under the transformation A, and if the coefficients of the form are symmetric in all indices (in tensor notation), then the coefficients $a_{\mathbf{p}}$, $|\mathbf{p}| = p$ are coordinates of a tensor. This tensor is contravariant (of order p) and symmetric.

This observation turns our attention back to investigation of invariant properties of tensors (contravariant and symmetric tensors are relevant here). If we find an invariant on a tensor, we can use it for an arbitrary object, which was identified as a tensor (simultaneous invariants on several tensors are analogous). Of course, the substituted objects must have the same properties, i.e. to be contravariant and symmetric and to have the same order.

It is really possible to make a tensor from moments. Another tensor is formed by blur invariant moment forms (8). The following section demonstrates it.

5 Fundamental Theorems

The Fundamental Theorem of Moment Invariants (FTMI) was formulated, because moments do not generally behave like tensors. The reason why it could be formulated is, that moments can be normalized to behave like tensors. Although several authors, e. g. [42–44], had realized this fact, it did not become widely used.

The main goal of this section is to prepare the way to the main results of this paper. Connections with previous research in the field of moment invariants are mentioned here. Although the contents of this section may seem redundant, we believe the following summary is more natural and understandable for people with little or no knowledge of tensor calculus.

A closer look at expansions from Theorems 5 and 7 reveals that they can be rewritten as sums of n-ary algebraic forms of orders $p \geq 0$. We can transform the spatial coordinates (9) in these expansions. We can find the new coefficients corresponding to the new monomials $\mathbf{u}'^{\mathbf{p}}$—these are transformed by (12). After these steps, we are ready to use already known invariants on the form (11) to construct analogous invariants containing either moments $m_{\mathbf{k}}$ or blur invariants $c_{\mathbf{k}}$ (resp. $Q_{\mathbf{k}}$).

Both moments $m_{\mathbf{k}}$ and blur invariants $c_{\mathbf{k}}$ are closely related to Fourier transform, see (4) and (6). Thus, recalling transformational properties of the Fourier transform seems a natural starting point.

It is desirable to rewrite a Fourier transform after the transformation (9) again as a Fourier transform of any function. Using (12), we get

$$\int_{R^N} f(\mathbf{x}) e^{-2\pi i \mathbf{u}\cdot\mathbf{x}} \mathbf{dx} = \frac{1}{|J|} \int_{R^N} f'(\mathbf{x}') e^{-2\pi i \mathbf{u}'\cdot\mathbf{x}'} \mathbf{dx}' \tag{15}$$

where

$$f(\mathbf{x}) = f'(\mathbf{x}')$$

and J is the Jacobian of the transformation (9). The relation (15) can be rewritten as

$$F[f](\mathbf{u}) = \frac{1}{|J|} F[f'](\mathbf{u}'). \tag{16}$$

In other words, if an image is transformed according to the transformation (9), then the Fourier transform remains the same (except the factor $\frac{1}{|J|}$) if the spectral coordinates are transformed according to (12). Using example of 2-D rotations, if an image is rotated, then its Fourier transform is rotated as well, by the same angle.

Derivation of the fundamental theorem now follows from (16) immediately. The FTMI was published e.g. in [45]. For completeness and relation to the rest of this paper, we rederive it here again.

Fundamental Theorem of Moment Invariants

Applying Theorem 5 to both sides of (16), we get

$$\sum_{\mathbf{0}\leq\mathbf{k}} \frac{(-2\pi i)^{|\mathbf{k}|}}{\mathbf{k}!} m_{\mathbf{k}}[f] \mathbf{u}^{\mathbf{k}} = \frac{1}{|J|} \sum_{\mathbf{0}\leq\mathbf{k}} \frac{(-2\pi i)^{|\mathbf{k}|}}{\mathbf{k}!} m_{\mathbf{k}}[f'] \mathbf{u}'^{\mathbf{k}}. \tag{17}$$

The monomials $\mathbf{u}^{\mathbf{k}}$ of the same order are under linear transformation (12) transformed among themselves (e.g. from a quadratic monomial we get a linear combination of quadratic monomials again). Therefore, as we mentioned before, it is possible to separate the equality (17) according to the order $|\mathbf{k}|$ of the monomials. For one particular order $k = |\mathbf{k}|$ we thus get

$$\sum_{|\mathbf{k}|=k} \frac{k!}{\mathbf{k}!} m_{\mathbf{k}}[f] \mathbf{u}^{\mathbf{k}} = \frac{1}{|J|} \sum_{|\mathbf{k}|=k} \frac{k!}{\mathbf{k}!} m_{\mathbf{k}}[f'] \mathbf{u}'^{\mathbf{k}}, \tag{18}$$

what is an expression for the transformation of n-ary algebraic form of order $k = |\mathbf{k}|$, see Definition 10.

In agreement with Definitions 10 and 9, the coefficients corresponding to the monomials $\mathbf{u}^{\mathbf{p}}$

$$a_{\mathbf{p}} = m_{\mathbf{p}}[f] \tag{19}$$

are after the transformation (12) changed to

$$a'_{\mathbf{p}} = \frac{1}{|J|} m_{\mathbf{p}}[f'] \tag{20}$$

which appear with the monomials $\mathbf{u}'^{\mathbf{p}}$.

If the invariant $I(a, b, \ldots)$ from (10) is a homogeneous function of order k in the coordinates of the tensors, then substituting (19) and (20) we obtain the following theorem.

Theorem 11 (Generalized Fundamental Theorem of Moment Invariants) *If the n-ary algebraic forms have under transformation* (9) *invariant as a homogeneous function of order k*

$$I(a', b', \dots) = \Lambda I(a, b, \dots)$$

then the same function of moments of orders corresponding to the orders of substituted coefficients (denoted schematically) is invariant

$$I(m_a[f'], m_b[f'], \dots) = \Lambda |J|^k I(m_a[f], m_b[f], \dots)$$

with the factor multiplied by $|J|^k$, where $|J|$ is the absolute value of the Jacobian of transformation (9).

It should be pointed out that in the first presentation [39], this theorem contained an error. The first correct version was published in a local journal [46]. However, the result did not become well known and the FTMI was independently revised again by Flusser and Suk [47] and Reiss [48]. The FTMI was then generalized to N-D by [45].

Fundamental Theorem for Moment Blur Invariants

It is natural to investigate, what is the corresponding result for blur invariants $c_{\mathbf{k}}$. We proceed analogously to the derivation of the FTMI.

Calculating tangent of phase of both sides of (16)

$$\tan\left[\mathrm{ph}F[f](\mathbf{u})\right] = \tan\left[\mathrm{ph}F[f'](\mathbf{u}')\right] \tag{21}$$

removes the factor $1/|J|$. Application of the Theorem 7 to both sides of (21) gives

$$\sum_{\mathbf{0}\le\mathbf{k}} c_{\mathbf{k}}[f]\mathbf{u}^{\mathbf{k}} = \sum_{\mathbf{0}\le\mathbf{k}} c_{\mathbf{k}}[f']\mathbf{u}'^{\mathbf{k}}. \tag{22}$$

One particular order $k = |\mathbf{k}|$ is again extracted from (22). However, we substitute (7) to get relation

$$\sum_{|\mathbf{k}|=k} \frac{k!}{\mathbf{k}!} Q_{\mathbf{k}}[f]\mathbf{u}^{\mathbf{k}} = \sum_{|\mathbf{k}|=k} \frac{k!}{\mathbf{k}!} Q_{\mathbf{k}}[f']\mathbf{u}'^{\mathbf{k}} \tag{23}$$

in terms of the new set of blur invariants $Q_{\mathbf{p}}$. The coefficients corresponding to the monomials $\mathbf{u}^{\mathbf{p}}$, resp. $\mathbf{u}'^{\mathbf{p}}$ are

$$a_{\mathbf{p}} = Q_{\mathbf{p}}[f], \tag{24}$$

resp.

$$a'_{\mathbf{p}} = Q_{\mathbf{p}}[f']. \tag{25}$$

Substituting (24) and (25) to (10), we can formulate an analogous theorem to the Theorem 11.

Theorem 12 *If the n-ary algebraic forms have under transformation* (9) *invariant*

$$I(a', b', \dots) = \Lambda I(a, b, \dots),$$

then the same function of moment forms Q of orders corresponding to the orders of substituted coefficients is invariant under transformation (9)

$$I(Q_a[f'], Q_b[f'], \dots) = \Lambda I(Q_a[f], Q_b[f], \dots).$$

Of course, this result is really useful when only nontrivial blur invariants are used (i.e. of odd orders, see (8)). Otherwise we could get either trivial results, or expressions, which are not defined (e.g. division by zero).

6 Simplification with Moment Tensors

There is a substantial difference between the Theorems 11 and 12. The latter theorem does not contain the factor $\frac{1}{|J|}$ in (18) any more. We can proceed further.

From (18) it follows for $k = 0$ that

$$m_{\mathbf{0}}[f] = \frac{1}{|J|} m_{\mathbf{0}}[f']. \tag{26}$$

In both (26) and (18), the same transformation (9) is used. Therefore the equalities can be divided and we get

$$\sum_{|\mathbf{k}|=k} \frac{k!}{\mathbf{k}!} \frac{m_{\mathbf{k}}[f]}{m_{\mathbf{0}}[f]} \mathbf{u}^{\mathbf{k}} = \sum_{|\mathbf{k}|=k} \frac{k!}{\mathbf{k}!} \frac{m_{\mathbf{k}}[f']}{m_{\mathbf{0}}[f']} \mathbf{u}'^{\mathbf{k}}. \tag{27}$$

From comparison of (13) and (27) we get equivalence

$$a_{\mathbf{p}} = \frac{m_{\mathbf{k}}[f]}{m_{\mathbf{0}}[f]}, \tag{28}$$

and

$$a'_{\mathbf{p}} = \frac{m_{\mathbf{k}}[f']}{m_{\mathbf{0}}[f']}. \tag{29}$$

Now we could have formulated a theorem analogous to the Theorem 12 and dedicated to moments.

Instead of comparing the forms (13), (23) and (27), we can find the same results differently. As in case of the form (13), we can look at forms (23) and (27) as at invariants under the transformation A. In both forms, $\mathbf{u}$ is transformed as a covariant vector. Objects $m_{\mathbf{p}}/m_{\mathbf{0}}$ and $Q_{\mathbf{p}}$ are symmetric in all indices (in tensor notation). Then using the same conclusion as before, these objects must be contravariant symmetric tensors. We will use notation

$$\tau_{\mathbf{k}}[f] \equiv \frac{m_{\mathbf{k}}[f]}{m_{\mathbf{0}}[f]} \tag{30}$$

and we will call it a moment tensor, as other authors.

The fact that $\tau_{\mathbf{p}}$'s and $Q_{\mathbf{p}}$'s are coordinates of tensors can be seen immediately after substitution of the transformation coefficients. However, the presented derivation is probably more illustrative.

At this moment, the way how to treat $\tau_{\mathbf{p}}$'s and $Q_{\mathbf{p}}$'s is determined: they are objects equivalent to $a_{\mathbf{p}}$'s. Therefore, the same methods can be applied to calculate invariants on them. Provided, we have a function of coordinates of one of these three tensors, we can use it for the other tensors as well.

The FTMI shows that moments $m_{\mathbf{p}}$ are transforming almost in the same way as $a_{\mathbf{p}}$. Due to presence of the additional factor $\frac{1}{|J|}$, simple substitution

$$a_{\mathbf{p}} \longrightarrow m_{\mathbf{p}}$$

is not possible. The FTMI investigates the consequences of this substitution. However, such a treatment is not necessary—the moment tensor can be used.

Normalizing by $m_{\mathbf{0}}$ is not the only possibility to make a tensor from a moment. It is just the most straightforward choice. Other scalars transforming under the transformation (9) according to

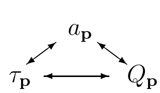
$$\text{scalar}[f'] = |J|\text{scalar}[f] \tag{31}$$

can be found. Every ratio of relative moment invariants in the forms of homogeneous polynomials fulfills this condition provided that: (1) the weights of both invariants are the same; (2) order of the polynomial in the nominator minus order of the polynomial in the denominator equals one.

7 Practical Consequences

The results derived in the last section can be formulated in a theorem.

Theorem 13 *Given there is an invariant function of coordinates of one of the tensors a, τ, Q, we can use it for all these tensors. In terms of their coefficients, we can perform all the possible substitutions*:



In practice, invariants built on moments, not on the moment tensors, are often at disposal. Therefore, it would be desirable to have a recipe, how to make blur invariants from them—in the best case, by a simple substitution. Before composing the recipe, the next section discusses the question of order of applying the blur and geometric degradations.

Another important question concerns independence of the invariants. This paper elaborates on adding blur invariance to already existing geometrical invariants. Discussion of the independence of the geometrical invariants was a topic of several earlier papers and therefore we do not deal with this issue here. More information about this generally unresolved problem can be found in [41, 49, 50]. Blur invariants $Q_{\mathbf{p}}$ of odd orders $|\mathbf{p}|$ are independent, as mentioned in Sect. 3. Therefore, as far as only invariants containing odd order moments are used, dependence/independence of the combined invariants would remain the same as of the original geometrical invariants. Usage of invariants with even order moments would lead to trivial combined invariants. Here we assume that the geometrical invariants are so-called irreducible, i.e. they cannot be expressed as a linear combination or a product of other invariants.

7.1 Order of the Blur and Geometric Degradations

Model expressed by (1) describes most of the practical degradation problems, which have been studied in the literature. This model assumes, that the geometric degradation is applied before the blur degradation. This is the case of out-of-focus blur, atmospheric blur and most of the blurs caused by motion. Omitting noise, (1) is

$$g = \tau(f) * h. \tag{32}$$

The combined invariants described in this paper are constructed so, that blur invariant forms are used to add an additional geometric invariance. Influence of the blur degradation, which is the second degradation in (32), is immediately removed by using the blur invariants instead of moments. Since the blur invariants are geometrically analogous to moments, influence of the remaining geometric degradation can be also easily removed.

Certain degradations could be modeled by reverse order of the degradations

$$g = \tau(f * h). \tag{33}$$

One example could be an observation of a surface diffusion process. However, other examples are difficult to find.

The degradations (32) and (33) are not equivalent—convolution and geometric transformation are generally not commutative. Fortunately, the second model can be expressed as the first model, using a different convolution kernel:

$$\tau(f * h) = \tau(f) * \frac{1}{|J|}\tau(h), \tag{34}$$

where J is the Jacobian of the transformation τ.

This result shows, that the combined invariants reviewed in Sect. 3 can be used also for the model (33). However, symmetry of the convolution kernel has to be kept in mind. Kernels with the central symmetry keep the property (34) after a linear transformation and they can be used safely.

7.2 Invariants to Brightness Non-preserving Blur

Given an arbitrary function of moments is an invariant to the transformation (9). Then it is possible to rewrite it as

$$I(m_{\mathbf{p}},\ldots) = I\left(\frac{m_{\mathbf{p}}}{m_{\mathbf{0}}}m_{\mathbf{0}},\ldots\right) = I(\tau_{\mathbf{p}}m_{\mathbf{0}},\ldots). \tag{35}$$

If we want to make (35) invariant also to blur (for the model (1)), we must substitute blur invariant objects for $\tau_{\mathbf{p}}$ and $m_{\mathbf{0}}$, which are transforming in the same way under the geometric transformation. According to the Theorem 12, we substitute $Q_{\mathbf{p}}$ for $\tau_{\mathbf{p}}$. Unfortunately, there is apparently no scalar, transforming like (31), which would be invariant also to brightness non-preserving blur. Thus, there is no object, which could be simply substituted for $m_{\mathbf{0}}$.

There are two possibilities how to proceed:

- *$m_{\mathbf{0}}$ is eliminated in* (35).
  It can happen when the invariant is a rational function. Equation (35) is then immediately also a blur invariant.
- *a power of $m_{\mathbf{0}}$ can be factored out of the invariant function.*
  The factor can be simply omitted. This step will change the Λ in (10) (it can destroy an absolute invariance, see [41]). However, the resulting relative invariants can be combined to get a new absolute invariant. If a k-th power of $m_{\mathbf{0}}$ is omitted, then the new Λ equals the old Λ divided by $|J|^k$.

7.3 Invariants to Brightness Preserving Blur

Another possibility is to leave the formerly formulated requirement of blur invariance. Instead of it, brightness preserving blur can be taken into account:

$$\int_{R^N} h(\mathbf{x})\mathbf{dx} = 1. \tag{36}$$

This is a very common condition in practice (motion blur, de-focus, atmospheric blur, dispersion). Thus, it is very useful to investigate this case. Under this condition, the moment form denoted as

$$M_{\mathbf{p}} = m_{\mathbf{0}} Q_{\mathbf{p}} \tag{37}$$

is also blur invariant. We can reveal transformational properties of these forms by multiplying (26) and (23). This can be done, because the same transformation (9) is used in both of them and we get

$$\sum_{|\mathbf{k}|=k} \frac{k!}{\mathbf{k}!} M_{\mathbf{k}}[f]\mathbf{u}^{\mathbf{k}} = \frac{1}{|J|} \sum_{|\mathbf{k}|=k} \frac{k!}{\mathbf{k}!} M_{\mathbf{k}}[f']\mathbf{u}'^{\mathbf{k}}. \tag{38}$$

By comparison with (18), the moment blur invariants $M_{\mathbf{p}}$ are transforming in the same way as moments $m_{\mathbf{p}}$. Therefore, we can formulate the following theorem.

Theorem 14 *Under the condition* (36), *a moment blur invariant can be constructed from the plain moment invariant by substitution of the blur invariants $M_{\mathbf{p}}$ for the moments $m_{\mathbf{p}}$ for* $|\mathbf{p}| > 0$:

$$m_{\mathbf{p}} \longleftrightarrow M_{\mathbf{p}}.$$

7.4 Invariants to Geometric Transformation with $|J| = 1$

The last proposed simplification stays with the $Q_{\mathbf{p}}$'s and with the brightness non-preserving blur. On the other hand, it gives up generality concerning the types of geometric transformations. If the geometric transformation is orthogonal, i. e. $|J| = 1$, then analogously to the previous discussion, comparing the equalities (18) and (23), we get the consequence: the moment blur invariants $Q_{\mathbf{p}}$ are transforming in the same way as moments $m_{\mathbf{p}}$. Therefore, if we are interested only in orthogonal transformations, we can substitute $Q_{\mathbf{p}}$ for $m_{\mathbf{p}}$. We need also a blur invariant, transforming like (31) under condition $|J| = 1$. We can simply use 1 and formulate the following theorem.

Theorem 15 *Given a function of moments $m_{\mathbf{p}}$ is an invariant to orthogonal transformation of spatial coordinates. Under the condition $|J| = 1$, a combined invariant can be constructed from the plain moment invariant by substitutions*:

$$m_{\mathbf{p}} \longrightarrow Q_{\mathbf{p}},$$
$$m_{\mathbf{0}} \longrightarrow 1.$$

7.5 Shift Invariance

Finally, possible shift invariance should be discussed. Although the geometric transformations mentioned so far have been linear, the additional shift invariance can be obtained simply by substitution of the central moments $\mu_{\mathbf{p}}$ to the ordinary moments:

$$m_{\mathbf{p}} \longrightarrow \mu_{\mathbf{p}}.$$

Two points must be clarified to justify this step. Coordinates of the center of gravity are spatial coordinates. Thus, the shift to the center of gravity does not change transformational properties of moments. Secondly, the center of gravity is a blur invariant (it equals the vector of $Q_{\mathbf{p}}$'s for $|\mathbf{p}| = 1$).

Analogously, it is also possible to use the substitutions proposed in this section, when invariants containing central moments are at disposal. We must simply use central moment counterparts of $M_{\mathbf{p}}$'s and $Q_{\mathbf{p}}$'s.

8 Examples

In this section, numerous examples illustrate how to apply the rules derived in this paper. It is demonstrated, that many results, published in the literature, could have been obtained more easily. Using the derived rules, other applications will hopefully appear. We will not demonstrate invariance and discriminability of the combined invariants. Instead, we provide several links to papers, where these experiments have already been performed. Numerical stability of the blur invariants in 2-D, combined blur-rotation invariants and combined blur-affine invariants was explicitly studied in [22, 33, 38, 51, 52].

8.1 Invariants to Rotation and Blur

8.1.1 2-D Rotation

One of the famous Hu's invariants published in [39] is

$$\phi_3 = (m_{30} - 3m_{12})^2 + (3m_{21} - m_{03})^2.$$

Since this is a form of moments of the third order, corresponding blur invariants Q are non-trivial. However, if the central moments are used, they equal the corresponding moments and the resulting combined invariant is $\phi'_3 = \phi_3$.

If an invariant of order five is used, the result is more interesting. Given the following invariant (calculated according to [49])

$$I = (m_{50} - 10m_{32} + 5m_{14})^2 + (5m_{41} - 10m_{23} + m_{05})^2,$$

Q's can be substituted to obtain a combined invariant, which is not identical to I, but which is not listed here due to its length.

Similar approach to obtain combined invariants to blur and rotation is undertaken in [32], although another proof of invariance is provided. The authors evaluate usefulness of the combined invariants for pattern recognition tasks.

8.1.2 3-D Rotation

Conclusions of this paper can be applied to any number of dimensions. In [38], group representation theory is applied to derive combined invariants to blur and rotation in arbitrary number of dimensions. Particularly, a few 3-D invariants are derived there and their applicability to 3-D image registration is demonstrated (see also [53] and [54]).

The same results can be obtained by finding 3-D moment invariants and substituting the blur invariants Q for the moments. For example, in [53], the following invariant is derived:

$$\begin{aligned} I = \big(&{m_{003}}^2 + 6{m_{012}}^2 + 6{m_{021}}^2 + {m_{030}}^2 \\ &+ 6{m_{102}}^2 + 15{m_{111}}^2 \\ &- 3m_{102}m_{120} + 6{m_{120}}^2 \\ &- 3m_{021}m_{201} + 6{m_{201}}^2 \\ &- 3m_{003}(m_{021} + m_{201}) \\ &- 3m_{030}m_{210} + 6{m_{210}}^2 \\ &- 3m_{012}(m_{030} + m_{210}) \\ &- 3m_{102}m_{300} - 3m_{120}m_{300} + {m_{300}}^2\big)/{m_{000}}^2. \end{aligned}$$

If the central moments are used, the combined invariant I' is identical, i.e. $I' = I$.

8.2 Invariants to Affine Transformation and Blur

In the literature, two approaches can be found to obtain invariants to affine transformation. The first approach [47, 55] finds invariant forms of moments, like that of (10). The second approach [55] sequentially imposes constraints on image moments so, that they are finally independent of the transformation parameters.

8.2.1 Moment Invariants

Reference [33] is an example of a paper using moment invariants in the form of (10). A few invariants to affine transformation are listed there. The authors gave a proof, that after substitution of M_{pq} to m_{pq}, additional blur invariance is obtained. However, this fact is obvious from Theorem 14.

On the following examples, the rules derived in this paper are demonstrated. The following affine invariant is listed in [33]:

$$I_1 = (\mu_{30}^2\mu_{03}^2 - 6\mu_{30}\mu_{21}\mu_{12}\mu_{03} + 4\mu_{30}\mu_{12}^3 + 4\mu_{21}^3\mu_{03} - 3\mu_{21}^2\mu_{12}^2)/\mu_{00}^{10}.$$

We can factor out $\frac{1}{\mu_{00}^6}$ and omit it. This changes the weight to $\Lambda = |J|^6$. Then we can substitute the corresponding Q's to get the relative combined invariant

$$I'_1 = (\mu_{30}^2\mu_{03}^2 - 6\mu_{30}\mu_{21}\mu_{12}\mu_{03} + 4\mu_{30}\mu_{12}^3 + 4\mu_{21}^3\mu_{03} - 3\mu_{21}^2\mu_{12}^2)/\mu_{00}^{4}.$$

In case of brightness-preserving blur, M's can be used and the combined absolute invariant is identical to I_1. For the other invariants listed in [33], the corresponding combined invariants would be more difficult. However, the formulae are complex. The authors [33] experimentally verified the combined invariance both on simulated and real data. They also revealed the discrimination power of the invariants.

Table 1 Survey of publications suitable as sources of invariants for practical calculations. Geometric transformation is specified for each contribution, as well as number of dimensions. When the dimension is arbitrary, the results are not limited to a particular dimension. However, dimension of the examples is specified in parentheses.

Publication	Geometric tr.	Dimension	Invariants
Suk, Flusser [33]	shift	2	blur invariants
Flusser et al. [37]	shift	arbitrary (3)	blur invariants
Flusser [49]	rotation	2	moments
Lo, Don [53]	rotation	3	moments
Guo [54]	rotation	3	moments
Flusser et al. [47]	affine	2	moments
Suk, Flusser [50]	affine	2	moments
Gurevich [41]	affine	arbitrary (2,3)	algebraic invariants
Reiss [55]	affine	arbitrary (2,3)	algebraic invariants
Mamistvalov [45]	affine	arbitrary (2,3)	algebraic invariants
Hilbert [40]	affine	arbitrary (2,3)	algebraic invariants
Rothe et al. [18]	affine	2	moments, normalization
Shen, Ip [19]	affine	2	moments, normalization
Flusser et al. [56]	affine	2	moments, normalization
Zhang et al. [25]	affine	2	combined inv., norm.

8.2.2 Image Normalization

Methods of moment (or image) normalization to obtain invariants to affine transformation of 2-D images can be found in [18, 19, 56]. Affine transformation is always decomposed into several simpler transformations. For example in [56], affine transformation is decomposed into horizontal and vertical translation, scaling, first rotation, stretching, second rotation and mirror reflection. Constraints are chosen to normalize image moments to all these transformations, sequentially.

Since it is known from Theorem 14 that moments m_{pq} are transforming in the same way as blur invariants M_{pq}, we can use M_{pq} instead of m_{pq}. If the deformation corresponds to the model (32), it is natural to calculate the blur invariants M_{pq} first to remove the dependence on blur. Normalization can be done afterwards, where all the procedures of [56] would be analogous. Usage of blur invariants would insure that the final normalized values are invariant both to affine transformation and blur.

A paper has already been published, using the method of normalization [25]. As the normalization constraints, the authors use the central moments of the third order, which are also invariant to blur. These constraints can be used also on images degenerated by different degrees of blur. After the moments are normalized to affine transformation, everything built on them stays affine invariant, as the authors argue. Therefore, blur invariants are calculated from the normalized moments, resulting in combined invariants.

However, the sequence of steps could have been different. Blur invariants M_{pq} can be again used immediately instead of m_{pq}. These M's can be then normalized to the remaining affine degradation. The resulting moment forms would be equivalent. The authors [25] proved the combined invariance of their results, as well as their discriminability.

It should be pointed out, that the method presented in [25] works only for the centrally symmetrical blur, as claimed. Although the method is tested on the degradation (32), affine degradation is normalized first. Initial assumptions allow such a procedure, see also (34).

8.3 Literature Containing Lists of Invariants

Table 1 contains links to several contributions, presenting lists of algebraic or moment invariants, their derivation and other information, which is relevant to this paper. The table cannot be regarded as complete. Our aim was to find papers, which could serve, together with the provided rules, as recipes how to build blur invariants easily and fast.

9 Conclusions

Contribution of this paper is two-fold. First, it can serve as a review for those familiar with the field of combined invariants. The way of explanations in this paper and all the context should contribute to better orientation in this field also for developers, which would readily like to use this kind of invariants.

For the first time, the substitution rules for straightforward construction of the combined invariants are published here. The rules simplify the calculations significantly. Numerous examples are given, giving a unified explanation to several already published results. We hope that this contribution will encourage a broader and more creative usage of combined invariance.

The goals of this contribution are mainly theoretical and unifying. However, as it was pointed out in the introduction, the resulting combined invariants have already been tested. They proved to be robust and at the same time discriminative features, suitable for usage in many image processing tasks.

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