Graph Method for Generating Affine Moment Invariants

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Abstract

A general method of systematic derivation of affine moment invariants of any weights and orders is introduced. Each invariant is expressed by its generating graph. Techniques for elimination of reducible invariants and dependent invariants are discussed. This approach is illustrated on the set of all affine moment invariants up to the weight ten.

1. Introduction

Recognition of objects and patterns that are deformed in various ways has been a goal of much recent research. The degradations use to be introduced during the image acquisition process by such factors as imaging geometry, systematic and random sensor errors, illumination changes, object occlusion, etc. Finding a set of invariant descriptors is a key step to recognizing degraded objects regardless of the particular deformations.

Moment invariants have become a powerful tool for recognizing objects regardless of their particular position, orientation, viewing angle, and other variations. There is a well elaborated theory on rotation moment invariants [1], [2], which have been successfully used in numerous applications. In practice, however, we often face object deformations that are beyond the rotation-translation-scaling model. An exact model of photographing a planar scene by a pinhole camera whose optical axis is not perpendicular to the scene is *projective transform*. For small objects and large camera-to-scene distance is the perspective effect negligible and the projective transform can be well approximated by *affine transform*. Thus, having powerful affine moment invariants for object description and recognition is in great demand.

A pioneer work on this field was done independently by Reiss [3] and Flusser and Suk [4], who introduced affine moment invariants (AMI's) and proved their applicability in simple recognition tasks. In their papers, the derivation of the AMI's originated from the classical theory of algebraic invariants [5]. They derived only few invariants in explicit forms and they did not study the problem of their mutual independence. However, independence of features is a crucial issue in pattern recognition because dependent features do not contribute to the discrimination power of the system at all and may even cause object misclassifications due to the "curse of dimensionality".

In this paper we present a general method how to systematically derive arbitrary number of the AMI's of any weights and any orders. The proposed method is based on representation of the AMI's by graphs. Considerable attention is paid to the elimination of reducible invariants and dependent invariants. Contrary to the above mentioned papers, we do not employ the theory of algebraic invariants.

2. Basic Terms

By *image function* (or *image*) we understand any real function $f(x, y)$ having a bounded support and a finite nonzero integral.

Geometric moment m_{pq} of the image $f(x, y)$, where p, q are non-negative integers and $(p + q)$ is called the *order* of the moment, is defined as

$$m_{pq} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^p y^q f(x, y) \mathrm{d}x \mathrm{d}y. \qquad (1)$$

Corresponding *central moment* μ_{pq} is defined analogously by substituting central coordinates for the absolute ones.

Affine transformation is a transformation of spatial coordinates (x, y) into new coordinates (x', y') defined by the equations

$$\begin{aligned} x' &= a_0 + a_1 x + a_2 y, \\ y' &= b_0 + b_1 x + b_2 y. \end{aligned} \qquad (2)$$

By *Affine moment invariant* (AMI) we understand any function I of moments such that $I(f(x,y)) = I(f(x',y'))$ for any image f and arbitrary affine transformation.

In the following text, we will not explicitly consider the translation parameters a_0, b_0 because we will construct the AMI's solely from central moments, which are invariant to arbitrary translation. We will also assume that all considered affine transforms are regular, i.e. the Jacobian $J = a_1 b_2 - a_2 b_1$ is nonzero.

3. Construction of the AMI's

Let us consider an image f and two arbitrary points (x_1, y_1), (x_2, y_2) from its support. Let us denote the "cross-product" of these points as C_{12}:

$$C_{12} = x_1 y_2 - x_2 y_1.$$

After an affine transform it holds $C'_{12} = J \cdot C_{12}$. Now we consider various numbers of points and we integrate their cross-products (or some powers of their cross-products) on the support of f. These integrals can be expressed in terms of moments and, after eliminating the Jacobian by proper normalization, they yield affine invariants.

More precisely, having N points ($N \geq 2$) we define functional I depending on N and on non-negative integers n_{kj} as

$$I(f) = \int_{-\infty}^{\infty} \prod_{k,j=1}^{N} C_{kj}^{n_{kj}} \cdot \prod_{i=1}^{N} f(x_i, y_i) dx_i dy_i. \quad (3)$$

Note that it is meaningful to consider only $j > k$, because $C_{kj} = -C_{jk}$ and $C_{kk} = 0$. After an affine transform, I becomes

$$I' = J^w |J|^N \cdot I,$$

where $w = \sum_{k,j} n_{kj}$ is called the *weight* of the invariant and N is called the *degree* of the invariant.

If I is normalized by μ_{00}^{w+N} we get a desirable affine invariant

$$\left(\frac{I}{\mu_{00}^{w+N}}\right)' = \left(\frac{I}{\mu_{00}^{w+N}}\right)$$

(if w is odd and $J < 0$ there is an additional factor -1).

We illustrate the general formula (3) on two simple invariants. First, let $N = 2$ and $n_{12} = 2$. Then

$$\begin{aligned} I(f) = \int_{-\infty}^{\infty} & (x_1 y_2 - x_2 y_1)^2 f(x_1, y_1) f(x_2, y_2) \\ & dx_1 dy_1 dx_2 dy_2 = 2(m_{20} m_{02} - m_{11}^2). \end{aligned} \quad (4)$$

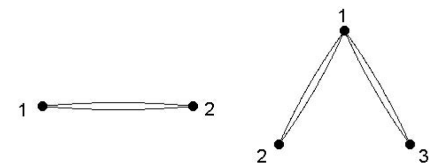


Figure 1. The graphs corresponding to the invariants (4) (left) and (5) (right).

Similarly, for $N = 3$ and $n_{12} = 2, n_{13} = 2, n_{23} = 0$ we get

$$\begin{aligned} I(f) &= \int_{-\infty}^{\infty} (x_1 y_2 - x_2 y_1)^2 (x_1 y_3 - x_3 y_1)^2 f(x_1, y_1) \\ & \quad f(x_2, y_2) f(x_3, y_3) dx_1 dy_1 dx_2 dy_2 dx_3 dy_3 \\ &= m_{20}^2 m_{04} - 4 m_{20} m_{11} m_{13} + 2 m_{20} m_{02} m_{22} \\ & \quad + 4 m_{11}^2 m_{22} - 4 m_{11} m_{02} m_{31} + m_{02}^2 m_{40}. \end{aligned} \quad (5)$$

4. The Graph Method

Each invariant generated by formula (3) can be represented by a graph, where each point (x_k, y_k) corresponds to one node and each cross-product C_{kj} corresponds to one edge of the graph. If $n_{kj} > 1$, the respective term $C_{kj}^{n_{kj}}$ corresponds to n_{kj} edges connecting k-th and j-th nodes. Thus, the number of nodes equals the degree of the invariant and the total number of the graph edges equals the weight w of the invariant. From the graph one can also learn about the orders of the moments the invariant is composed from and about its structure. The number of edges originating from each node equals the order of the moments involved, each invariant of the form (3) is in fact a sum where each term is a product of certain number of moments. This number is constant for all terms of one invariant and is equal to the total number of the graph nodes.

Particularly, for the invariants (4) and (5) the corresponding graphs are shown in Fig. 1.

Now one can see that the problem of derivation of the AMI's up to the given weight w is equivalent to generating all graphs with at least two nodes and at most w edges.

5. Irreducibility and Independence of the AMI's

Generating of all graphs as specified above is a combinatorial task with exponential complexity but formally easy to implement. Unfortunately, most resulting graphs are useless because they generate invariants, which are dependent on some other invariants. Identifying and discarding dependent invariants is very important but very complicated task.

There might be various kinds of dependencies in the set of all AMI's (i.e. in the set of all graphs). Let us categorize them into four groups and explain how they can be eliminated.

Identical invariants. Isomorphic graphs (and rarely some non-isomorphic graphs) generate identical invariants. Elimination is done by comparing the invariants term-wise.

Zero invariants. Some AMI's might equal identically zero regardless of the image they are calculated from. If there are one or more nodes with one adjacent edge only, then all terms of the invariants contain first-order moment(s). When using central moments, all first-order moments are zero by definition and, consequently, such invariants are zero, too. However, also some other graphs may generate zero invariants because of pair-wise cancellation of individual terms.

Products. Some invariants might be equal to products of other invariants. Elimination of these is done by incremental exhaustive search. All possible pairs, triples, quadruples, etc. of the admissible invariants (the sum of their individual weights must not exceed w) are multiplied and the independence of the result is checked.

Linear combinations. Some invariants might be equal to linear combinations of other invariants or of products of other invariants. All possible combinations of the admissible invariants (all terms must have the same numbers of moments of the same order and the sum of the individual weights must not exceed w) are calculated and their independence is checked.

After eliminating all the above dependencies, we get a set of so-called *irreducible* invariants. However, "irreducible" does not mean "independent" – there may be higher-order polynomial dependencies among irreducible invariants. Each such a relationship indicates that the invariants involved are dependent and one of them should be removed.

6. Irreducible AMI's up to the Weight 10

Using the graph method and by means of algorithms for eliminating reducible invariants, we derived the AMI's up to the weight $w = 10$ in explicit forms. After discarding identical invariants and zero invariants, we get 1519 AMI's. Then we applied the algorithms eliminating other reducible invariants (products and linear combinations) which led to 362 remaining invariants.

In the theory of algebraic invariants, the number of irreducible affine invariants of a given weight can be calculated by means of Cayley-Sylvester Theorem (see [5], Lecture XV, for instance). For all weights less or equal 10 this number is exactly 362, which proves the excellent performance of our method.

However, it is not clear what irreducible invariants (and how many of them) should be used for practical pattern recognition applications. Certainly not all of them because of hidden polynomial dependencies. One feels intuitively that the number of actually independent invariants is much lower. If the number of moments up to the 10-th order is 66 and affine transform has 6 degrees of freedom, then, because the moments themselves are independent, the number of independent invariants should be $66 - 6 = 60$. (This intuitive reasoning is not generally true but provides a good estimate.)

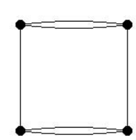
At the moment we do not have an automatic method for selection of all independent invariants. It should consist on exhaustive search in the space of all possible polynomial dependencies among the irreducible invariants. This is an extremely complex task for several reasons. We have to test a huge number of possible dependencies and, moreover, we have to identify and discard multiple dependencies found among the same invariants. As an example, let us consider two invariants I_a, I_b such that $I_a^2 = I_b^3$. In addition to this relationship, the algorithm would find also dependencies $I_a^4 = I_b^6$, $I_a^3 = I_b^3 I_a$ etc. After eliminating these multiple dependencies, each dependency relation among the irreducible invariants indicates that one invariant is dependent on the others and can be removed. In our set of 362 irreducible invariants, we have found explicitly 64 "independent" dependencies. Since $362 - 60 - 64 = 238$, there should be other 238 dependencies whose identification will be a topic of our future research.

Below we present a list of 10 independent affine moment invariants up to the 5th order (including normalization factors) along with their corresponding graphs. We recommend to use them (or just some of them) in practical object recognition applications. They provide discrimination power which should be sufficient in most cases.

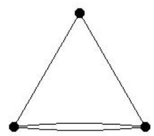
All 362 irreducible invariants (printed in explicit forms, shown in forms of corresponding graphs, and implemented in MATLAB ready to use) along with the 64 dependency relationships can be downloaded from our FTP-site [6].

$$I_1 = (\mu_{20}\mu_{02} - \mu_{11}^2)/\mu_{00}^4$$

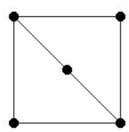
$$I_2 = (-\mu_{30}^2\mu_{03}^2 + 6\mu_{30}\mu_{21}\mu_{12}\mu_{03} - 4\mu_{30}\mu_{12}^3 - 4\mu_{21}^3\mu_{03} + 3\mu_{21}^2\mu_{12}^2)/\mu_{00}^{10}$$



$$\begin{aligned} I_3 = {} & (\mu_{20}\mu_{21}\mu_{03} - \mu_{20}\mu_{12}^2 - \mu_{11}\mu_{30}\mu_{03} + \mu_{11}\mu_{21}\mu_{12} \\ & +\mu_{02}\mu_{30}\mu_{12} - \mu_{02}\mu_{21}^2)/\mu_{00}^7 \end{aligned}$$

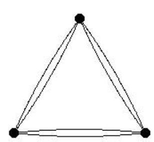


$$\begin{aligned} I_4 = {} & (-\mu_{20}^3\mu_{03}^2 + 6\mu_{20}^2\mu_{11}\mu_{12}\mu_{03} - 3\mu_{20}^2\mu_{02}\mu_{12}^2 \\ & -6\mu_{20}\mu_{11}^2\mu_{21}\mu_{03} - 6\mu_{20}\mu_{11}^2\mu_{12}^2 \\ & +12\mu_{20}\mu_{11}\mu_{02}\mu_{21}\mu_{12} - 3\mu_{20}\mu_{02}^2\mu_{21}^2 \\ & +2\mu_{11}^3\mu_{30}\mu_{03} + 6\mu_{11}^3\mu_{21}\mu_{12} - 6\mu_{11}^2\mu_{02}\mu_{30}\mu_{12} \\ & -6\mu_{11}^2\mu_{02}\mu_{21}^2 + 6\mu_{11}\mu_{02}^2\mu_{30}\mu_{21} - \mu_{02}^3\mu_{30}^2)/\mu_{00}^{11} \end{aligned}$$

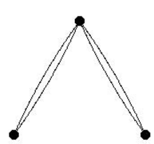


$$I_5 = (\mu_{40}\mu_{04} - 4\mu_{31}\mu_{13} + 3\mu_{22}^2)/\mu_{00}^6$$

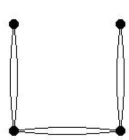
$$I_6 = (\mu_{40}\mu_{22}\mu_{04} - \mu_{40}\mu_{13}^2 - \mu_{31}^2\mu_{04} + 2\mu_{31}\mu_{22}\mu_{13} - \mu_{22}^3)/\mu_{00}^9$$



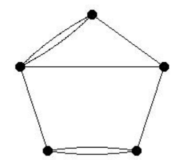
$$\begin{aligned} I_7 = {} & (\mu_{20}^2\mu_{04} - 4\mu_{20}\mu_{11}\mu_{13} + 2\mu_{20}\mu_{02}\mu_{22} + 4\mu_{11}^2\mu_{22} \\ & -4\mu_{11}\mu_{02}\mu_{31} + \mu_{02}^2\mu_{40})/\mu_{00}^7 \end{aligned}$$



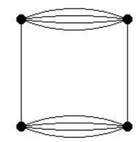
$$\begin{aligned} I_8 = {} & (\mu_{20}^2\mu_{22}\mu_{04} - \mu_{20}^2\mu_{13}^2 - 2\mu_{20}\mu_{11}\mu_{31}\mu_{04} \\ & +2\mu_{20}\mu_{11}\mu_{22}\mu_{13} + \mu_{20}\mu_{02}\mu_{40}\mu_{04} \\ & -2\mu_{20}\mu_{02}\mu_{31}\mu_{13} + \mu_{20}\mu_{02}\mu_{22}^2 + 4\mu_{11}^2\mu_{31}\mu_{13} \\ & -4\mu_{11}^2\mu_{22}^2 - 2\mu_{11}\mu_{02}\mu_{40}\mu_{13} + 2\mu_{11}\mu_{02}\mu_{31}\mu_{22} \\ & +\mu_{02}^2\mu_{40}\mu_{22} - \mu_{02}^2\mu_{31}^2)/\mu_{00}^{10} \end{aligned}$$



$$\begin{aligned} I_9 = {} & (\mu_{30}^2\mu_{12}^2\mu_{04} - 2\mu_{30}^2\mu_{12}\mu_{03}\mu_{13} + \mu_{30}^2\mu_{03}^2\mu_{22} \\ & -2\mu_{30}\mu_{21}^2\mu_{12}\mu_{04} + 2\mu_{30}\mu_{21}^2\mu_{03}\mu_{13} \\ & +2\mu_{30}\mu_{21}\mu_{12}^2\mu_{13} - 2\mu_{30}\mu_{21}\mu_{03}^2\mu_{31} \\ & -2\mu_{30}\mu_{12}^3\mu_{22} + 2\mu_{30}\mu_{12}^2\mu_{03}\mu_{31} + \mu_{21}^4\mu_{04} \\ & -2\mu_{21}^3\mu_{12}\mu_{13} - 2\mu_{21}^3\mu_{03}\mu_{22} + 3\mu_{21}^2\mu_{12}^2\mu_{22} \\ & +2\mu_{21}^2\mu_{12}\mu_{03}\mu_{31} + \mu_{21}^2\mu_{03}^2\mu_{40} - 2\mu_{21}\mu_{12}^3\mu_{31} \\ & -2\mu_{21}\mu_{12}^2\mu_{03}\mu_{40} + \mu_{12}^4\mu_{40})/\mu_{00}^{13} \end{aligned}$$



$$\begin{aligned} I_{10} = {} & (-\mu_{50}^2\mu_{05}^2 + 10\mu_{50}\mu_{41}\mu_{14}\mu_{05} - 4\mu_{50}\mu_{32}\mu_{23}\mu_{05} \\ & -16\mu_{50}\mu_{32}\mu_{14}^2 + 12\mu_{50}\mu_{23}^2\mu_{14} - 16\mu_{41}^2\mu_{23}\mu_{05} \\ & -9\mu_{41}^2\mu_{14}^2 + 12\mu_{41}\mu_{32}^2\mu_{05} + 76\mu_{41}\mu_{32}\mu_{23}\mu_{14} \\ & -48\mu_{41}\mu_{23}^3 - 48\mu_{32}^3\mu_{14} + 32\mu_{32}^2\mu_{23}^2)/\mu_{00}^{14} \end{aligned}$$



7. Conclusion

We presented a general method how to automatically generate the AMI's of any weights and orders. The method is based on representation of the AMI's by graphs. We developed an algorithm for eliminating all reducible invariants and we also discussed how to identify polynomial dependencies among irreducible invariants. As an example, we applied the proposed technique to derive all irreducible AMI's up to weight 10 in explicit forms. Some of them are presented here, the complete list can be downloaded from our FTP server.

References

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[6] ftp.utia.cas.cz/pub/income/suk, anonymous login.

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