Constitutive models

Part 1

Background and terminology

Elasticity

An initial excuse

- One of the difficulties of the presentation of a difficult theory is that the subject will not be fully appreciated until it is studied in detail
- Difficulties of this sort are encountered frequently in teaching a mature subject
- The wholeness of the subject can rarely be communicated quickly

Material behaviour – models

- **Linear elastic**
- **Nonlinear elastic**
- **Hyperelastic**
- **Hypoelastic**
- Elastoplastic
- Creep
- Viscoplasticity

Preliminaries

- Invariants
- Strain energy
- Continuum mechanics background

PRINCIPAL AXIS AND INVARIANTS OF SECOND ORDER TENSORS

STRAIN INVARIANTS

Two sets of invariants are defined and used in continuum mechanics

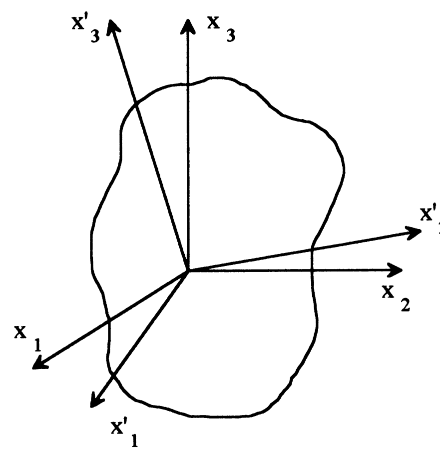
1) Associated with characteristic equation of tensor standard transformation

$$[\varepsilon'] = [A]\{\varepsilon\}[A]^T$$

We are looking for such $[A]$ which gives $[\varepsilon']$ of diagonal form

Standard eigenvalue problem for a generic strain tensor

$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$



is defined by $([\varepsilon] - \lambda[I])\{a\} = 0$ and has nontrivial solution only if

$$\det([\varepsilon] - \lambda[I]) = 0$$

which given characteristic equation of strain tensor in the form

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

where

λ_i are principal strains and

$\bar{I}_1 = \varepsilon_{ii}$

$$\bar{I}_2 = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{vmatrix} + \begin{vmatrix} \varepsilon_{11} & \varepsilon_{13} \\ \varepsilon_{13} & \varepsilon_{33} \end{vmatrix} + \begin{vmatrix} \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{23} & \varepsilon_{33} \end{vmatrix}$$

$\bar{I}_3 = |\varepsilon_{ij}|$

are the first, second , and third strain invariants respectively.

2)Associated with the tensor through its property of trace and different orders of trace (tr . . . trace . . . sum of diagonal t.)

$$I_1 = \mathrm{tr}([\varepsilon])$$

$$I_2 = \mathrm{tr}\left([\varepsilon]^2\right) = \tfrac{1}{2}\varepsilon_{ij}\varepsilon_{ji}$$

$$I_3 = \mathrm{tr}([\varepsilon]^3) = \tfrac{1}{3}\varepsilon_{ik}\varepsilon_{km}\varepsilon_{mi}.$$

STRESS INVARIANTS

similarly as before for strains

1) $\bar{J}_1 = \sigma_{ii}$

$$\bar{J}_2 = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix}$$

$\bar{J}_3 = |\sigma_{ij}|$

2) $J_1 = tr([\sigma]) = \sigma_{ii}$

$J_2 = tr\left([\sigma]^2\right) = \frac{1}{2}\sigma_{ij}\sigma_{ji}$

$J_3 = tr\left([\sigma]^3\right) = \frac{1}{3}\sigma_{ik}\sigma_{km}\sigma_{mi}$

Sometimes hydrostatic
or spherical part of stress tensor

DECOMPOSITION OF STRESS INTO VOLUMETRIC AND DEVIATORIC COMPONENTS

$\sigma_{ij} = v_{ij} + s_{ij}$

where

v_{ij}... volumetric part (sometimes mean), changes of volume only

s_{ij} deviatoric part of stress responsible for changes of shape

$$\sigma_{ij} = \begin{bmatrix} \sigma_m & & \\ & \sigma_m & \\ & & \sigma_m \end{bmatrix} + \begin{bmatrix} \sigma_{11} - \sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma_m & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma_m \end{bmatrix}$$

$\sigma_m = \frac{1}{3}\sigma_{ii}$ volumetric stress tensor, also: mean, hydrostatic, spherical

recall $J_1 = \sigma_{ii}$

$s_{ij} = \sigma_{ij} - \sigma_m \delta_{ij}$ deviatoric stress

INVARIANTS OF DEVIATORIC STRESS TENSOR

$J_{D1} = s_{ii} = \sigma_{ii} - \frac{1}{3}\sigma_{kk}\delta_{ii}$

$J_{D2} = \frac{1}{2}s_{ij}s_{ji} = \frac{1}{2}tr\left([s]^2\right) = J_2 - \frac{1}{6}J_1^2$

$J_{D3} = \frac{1}{3}s_{im}s_{mk}s_{ki} = \frac{1}{3}tr\left([s]^3\right) = J_3 - \frac{2}{3}J_1J_2 + \frac{2}{27}J_1^3$

Note

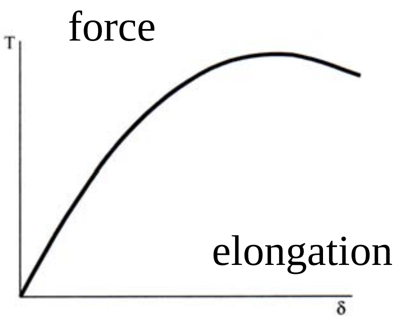
-these invariants are invariant with respect to the choice of coordinate system,

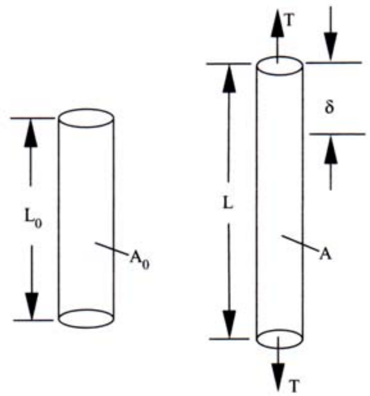
-each particle has its own invariants,

-a multiple of an invariant is an invariant as well (see characteristic equation)

Interpretation of 1D tensile test

L … current length
A … current area





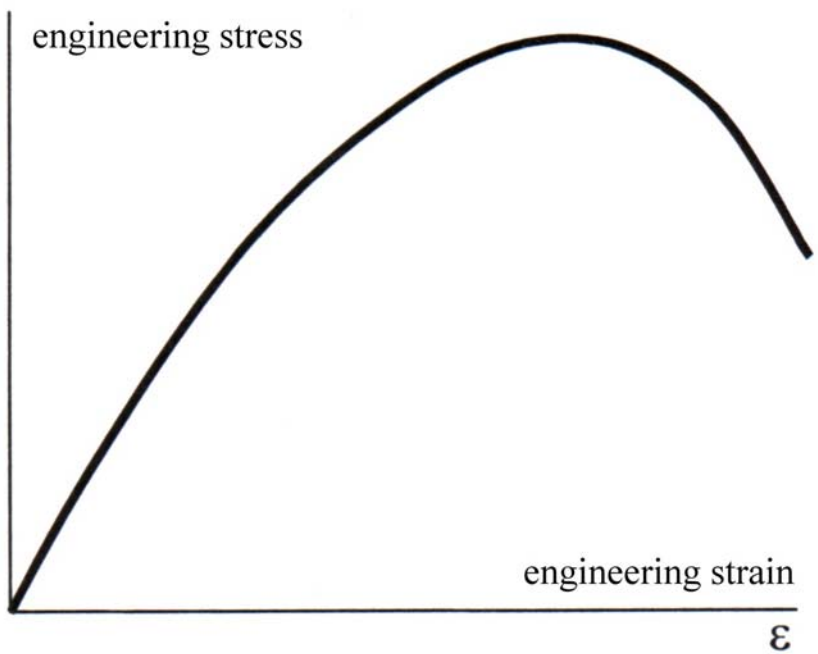
stretch $\lambda_x = L / L_0, \quad L = L_0 + \delta$

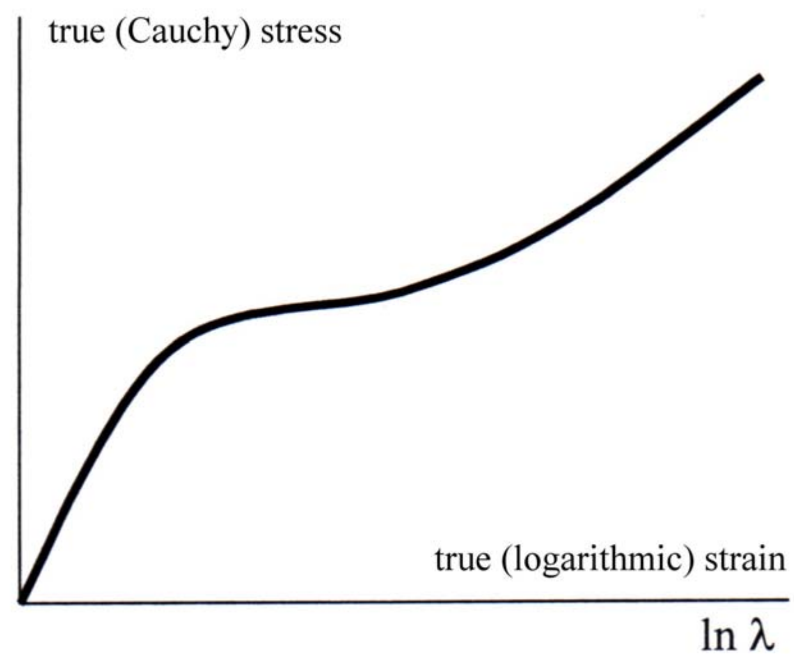
$\sigma_x^{\text{eng}} = T / A_0, \quad \varepsilon_x^{\text{eng}} = \delta / L_0 = \lambda_x - 1$

$\sigma_x^{\text{true}} = T / A$

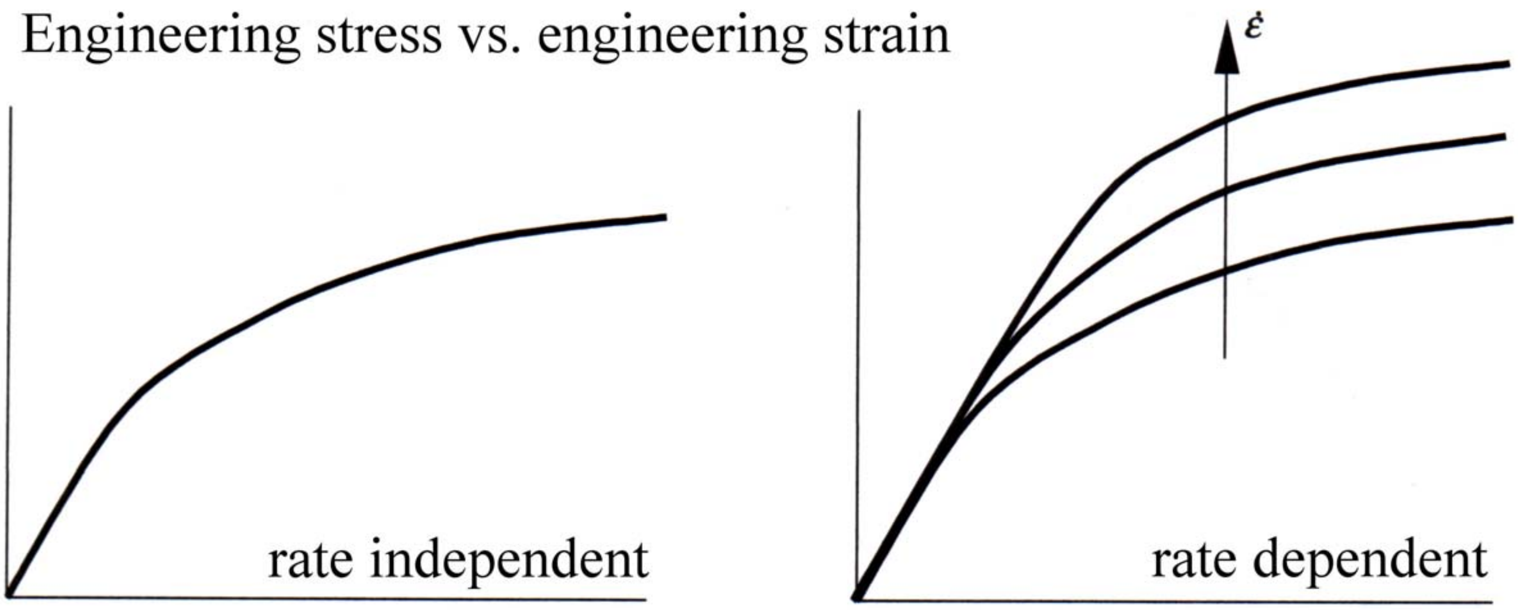
true strain increment $\mathrm{d}\varepsilon_x^{\text{true}} = \mathrm{d}L / L$

true strain $\varepsilon_x^{\text{true}} = \int_{L_0}^{L} dL / L = \ln(L / L_0) = \ln \lambda_x$





Rate dependent properties of material



engineering stress rate is $\dot{\varepsilon}_x^{\text{eng}} = \dot{\delta} / L_0$

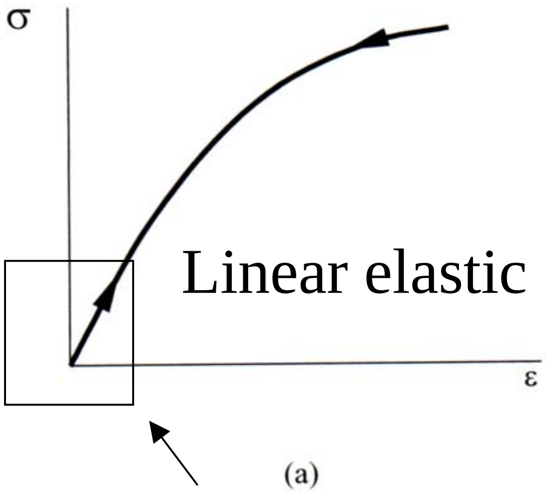
since $\dot{\delta} = \dot{L}$ and $\dot{\delta} / L_0 = \dot{L} / L_0 = \dot{\lambda}_x$

$\Rightarrow \dot{\varepsilon}_x^{\text{eng}} = \dot{\lambda}_x$

So the engineering stress rate is equivalent to the rate of stretching

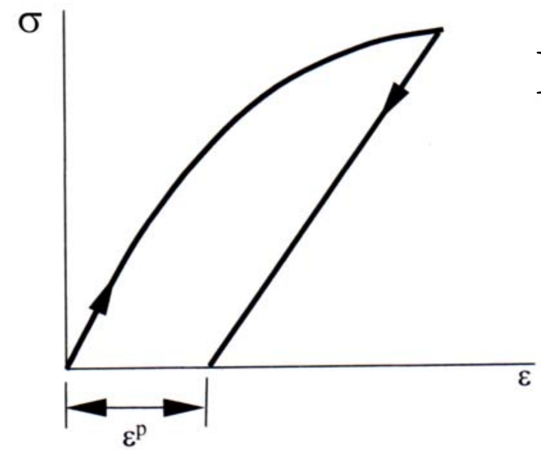
Loading, unloading – different models of material behaviour

Non-linear elastic



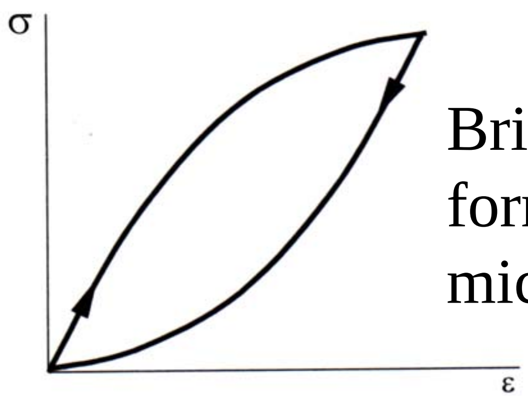
(a)

0.2% for steel



(b)

Elastic plastic



(c)

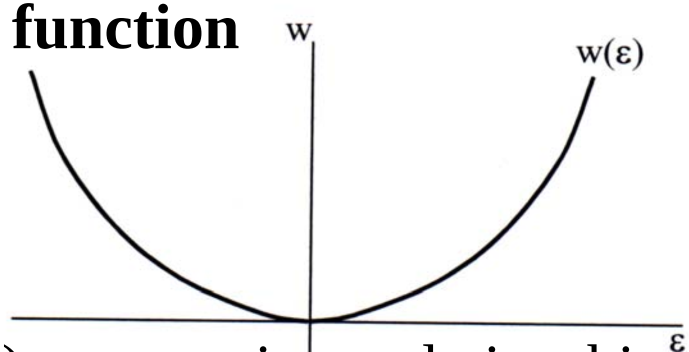
Brittle material which is damaged due to formation of microcracks during loading, microcracks are closed upon removal of the load

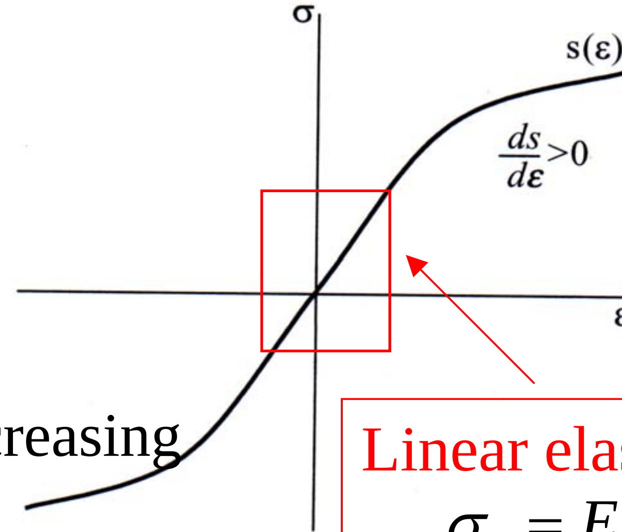
Theory of elasticity

- There is a unique relationship between stress and strain
- Elastic actually means no hysteresis, it does not mean ‘linear’ or ‘force proportional to displacement’.
- Could be linear or nonlinear.
- Strains are said to be reversible.
- Elastic material is rate independent
- It is purely mechanical theory – no thermodynamic effects – are considered.

Nonlinear elasticity, 1D, small strains 1/2

Convex function





$\sigma_x = s(\varepsilon_x)$ $\cdots$ unique relationship

if $\mathrm{d}s/\mathrm{d}\varepsilon_x > 0 \ldots s(\varepsilon_x)$ monotonically increasing

(i.e. strain hardening)

if not, material is said to exhibit strain softening

and its response is then unstable

Linear elasticity

$$\sigma_x = E\varepsilon_x$$

$$W = \tfrac{1}{2}E\varepsilon_x^2$$

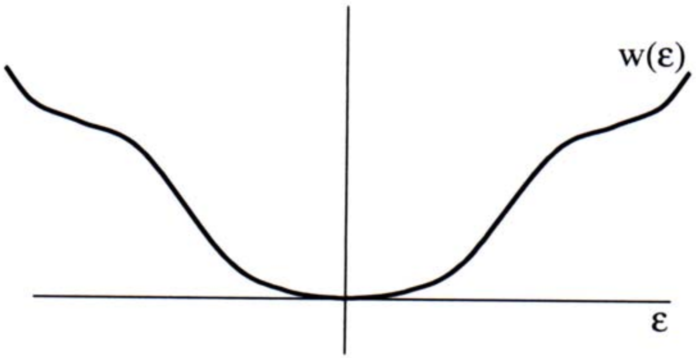
Increment of strain energy density $\mathrm{d}W(\varepsilon_x) = \sigma_x \,\mathrm{d}\varepsilon_x$

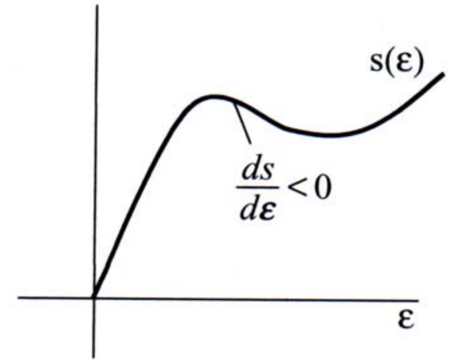
strain energy density $W(\varepsilon_x) = \int_0^{\varepsilon_x} \sigma_x \,\mathrm{d}\varepsilon_x$ ... is a potential

$$\sigma_x = \frac{\mathrm{d}W(\varepsilon_x)}{\mathrm{d}\varepsilon_x}$$

Nonlinear elasticity, 1D, small strains 2/2

Non-convex function





if $\mathrm{d}s / \mathrm{d}\varepsilon_x < 0$ (locally) then $s(\varepsilon_x)$ is not motonically increasing

strain energy function is non-convex

material is said to exhibit strain softening

and its response is then unstable

1D behaviour of elastic material is characterized by

- path independence
- reversibility
- non-dissipativeness

Nonlinear elasticity, 1D, large strains 1/2

$$S_x = \frac{\mathrm{d}W}{\mathrm{d}E_x} \quad \cdots \quad \text{2nd Piola - Kirchhoff stress, Green - Lagrange strain}$$

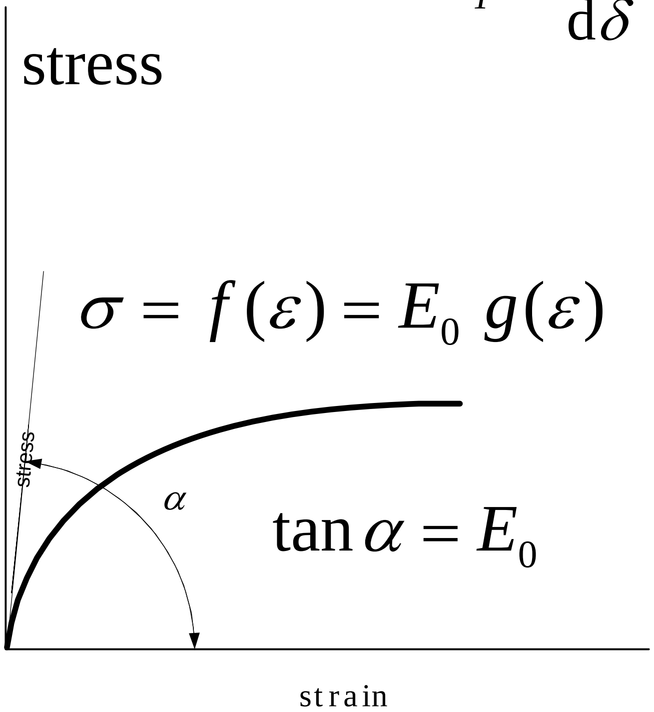
$$W = W(E_x)$$

Elastic stress-strain relationships in which the stress can be obtained from a potential function of the strains are called hyperelastic

- path independence
- reversibility
- non-dissipativeness

1D element – small strains, small rotations, no hysteresis

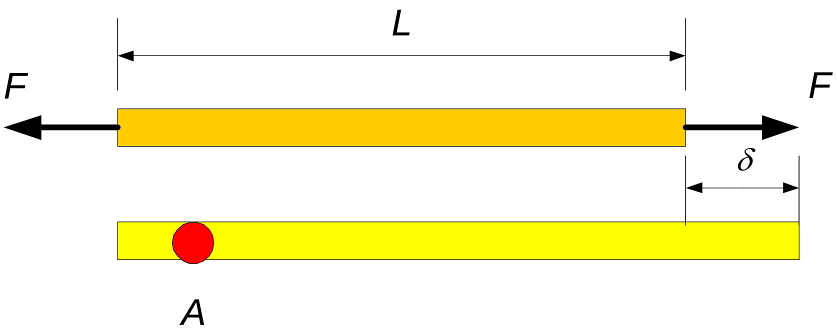
$$k_T = \frac{\mathrm{d}F}{\mathrm{d}\delta} = E_0\ A \frac{\mathrm{d}g(\varepsilon)}{\mathrm{d}\delta} = E_0\ A \frac{\mathrm{d}g(\varepsilon)}{\mathrm{d}\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}\delta} = E_0\ A\, g'(\varepsilon)\,\frac{1}{L}$$



$$\sigma = f(\varepsilon) = E_0\ g(\varepsilon)$$

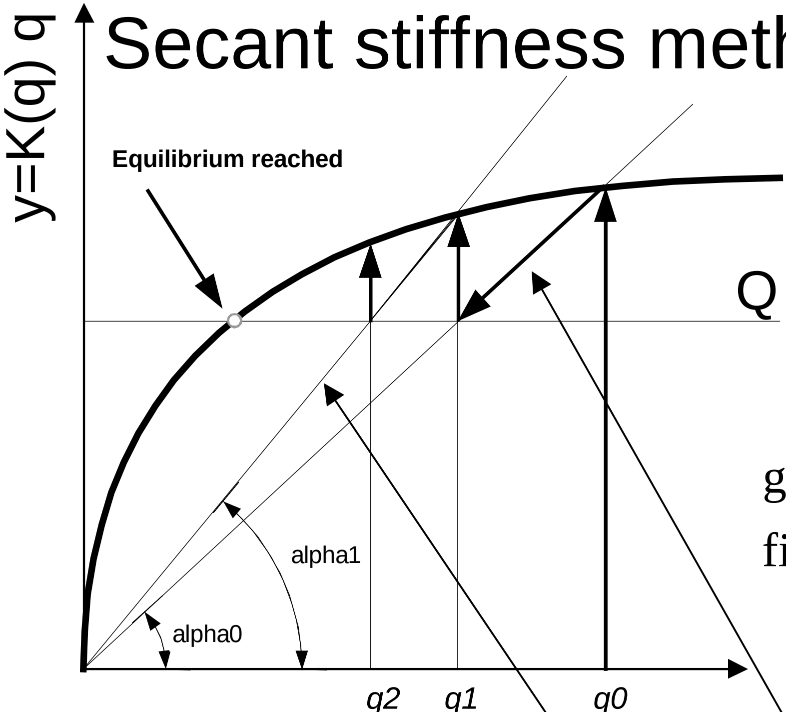
$$\tan\alpha = E_0$$

$$\mathbf{k} = \frac{E_0 A}{L}\, g'(\varepsilon)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



$$F = E_0\ A\, g(\varepsilon) = E_0\ A\, g(\frac{\delta}{L})$$

Secant stiffness method



given Q and initial quess ... q_0

first step

$$k_0 = \tan\alpha_0 = \frac{K(q_0)\, q_0}{q_0} = K(q_0)$$

$y = k_0 q$ and $y = Q$ gives $q_1 = Q / k_0 = [K(q_0)]^{-1}\, Q$

second step

$$k_1 = \tan\alpha_1 = \frac{K(q_1)\, q_1}{q_1} = K(q_1)$$

$y = k_1 q$ and $y = Q$ gives $q_2 = Q / k_1 = [K(q_1)]^{-1}\, Q$

Linear elastic model

- Good for many materials under reasonable temperatures provided that the stresses and strains are small
- Examples
  - Steel
  - Cast iron
  - Glass
  - Rock
  - Wood

Linear elastic model, Hooke's law $\sigma_{ij} = C_{ijkl}\ \varepsilon_{kl}$

- **Linear**
  - Strain is proportional to stress
- **Homogeneous** $\lim \Delta m = \lim \rho\ \Delta V = \rho \lim \Delta V$
  - Material properties are independent of the size of specimen – corpuscular structure of matter is disregarded
- **Anisotropic**
  - Stress-strain coefficients in **C** depend on direction
- Fourth tensor **C** is constant and has 81 components
  - Generally, however, there are 21 independent elastic constants
  - Orthotropic material … 9
  - Cubic anisotropy … 3
  - Isotropic material … 2 (Young modulus, Poisson ratio)

Hooke's law – Voigt's notation

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \mathbf{E} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \qquad \mathbf{E} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 & 0 & 0 \\ \mu & 1-\mu & \mu & 0 & 0 & 0 \\ \mu & \mu & 1-\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

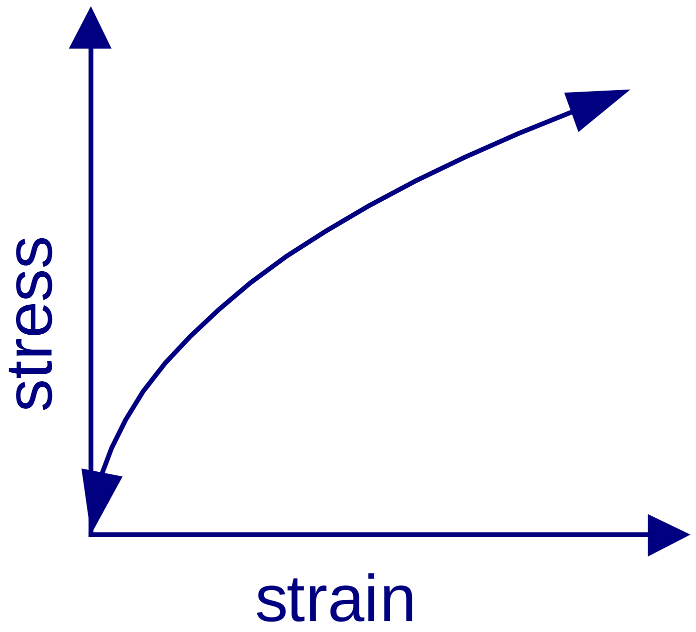
$$\mathbf{D} = \mathbf{E}^{-1} = \frac{1}{E} \begin{bmatrix} 1 & -\mu & -\mu & 0 & 0 & 0 \\ -\mu & 1 & -\mu & 0 & 0 & 0 \\ -\mu & -\mu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\mu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\mu) \end{bmatrix}$$

Nonlinear elastic

$$\sigma_{ij} = C_{ijkl}\ \varepsilon_{kl} = f(\varepsilon_{kl})$$

Tensor **C** is a function of strain

There is no hysteresis



For large displacements (but small strains)

engineering strain (Cauchy) is replaced by **Green-Lagrange strain** and engineering stress is replaced by **2PK stress**

and **TL** or **UL** formulations should be used

Hyperelastic

- Stress is calculated from strain energy functional by

$$S_{ij} = \frac{\partial W}{\partial E_{ij}}$$

- Good for rubberlike materials
- W is assumed by
  - Mooney-Rivlin model,
  - Ogden model,
  - Etc.
- Path-independent and fully reversible

Hypoelastic

- Path-dependent
- Stress increments are calculated from strain increments

$$\dot{\sigma}_{ij} = C_{ijkl}\,\dot{\varepsilon}_{kl}$$

$$C_{ijkl} = f\,(stress, strain,\ fracture\ criteria, loading, unloading, ...)$$

- Models for concrete