Continuum Mechanics, part 22

Kinematics_2

- Small deformation theory
- Right and left Cauchy-Green deformation tensors
- Stretch, polar decomposition
- Rate of polar rotation

SMALL DEFORMATION THEORY

Let's start with relation between material and spatial displacement gradients $[Z]$ and $[\bar{Z}]$.

From

$$u_i = x_i - a_i$$

We obtain by differentiation

$$\frac{\partial u_i}{\partial x_j} = \delta_{ij} - \frac{\partial a_i}{\partial x_j} \qquad \frac{\partial u_i}{\partial a_j} = \frac{\partial x_i}{\partial a_j} - \delta_{ij}$$

$$[\bar{Z}] = [I] - [F]^{-1} \qquad [Z] = [F] - [I]$$

Let's arrange the left equation

$$[\bar{Z}] = [I] - [F]^{-1} = [I] - \Big([I] + \underbrace{\big([F]-[I]\big)}_{[Z]}\Big)^{-1} =$$

$$= [I] - \big([I] - [Z]\big)^{-1}$$

If $[Z]$ and $[\bar{Z}]$ are 'small' compared to $[I]$ then we can write

$$[\bar{Z}] \doteq [I] - \big([I] - [Z] + [Z]^2 - [Z]^3 + \dots\big) =$$

$$= [Z] - [Z]^2 + [Z]^3 - \dots$$

Remember

$$\bar{z} = 1 - f^{-1} = 1 - \frac{1}{f} = 1 - \big(1 + (f-1)\big)^{-1} = 1 - \frac{1}{1+(f-1)} =$$

$$= 1 - \frac{1}{1+z} = 1 - \big(1 - z + z^2 - z^3 + \dots\big) = z - z^2 + z^3 - \dots$$

Therefore to the first order displacement gradients

$$[\bar{Z}] \doteq [Z] \quad \text{or} \quad \frac{\partial u_i}{\partial x_j} \doteq \frac{\partial u_i}{\partial a_j} .$$

So in this particular case it is immaterial whether the displacement gradients are formed by differentiation with respect to material or spatial coordinates.

To this order of approximation it follows that

$$A_{ij} = E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right)$$

The infinitesimal Cauchy strain tensor known from linear elasticity is defined as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} \right) \quad \text{or}$$

$$[\varepsilon] = \frac{1}{2}\left([Z] + [Z]^T \right) = \frac{1}{2}\left([F] - [I] + [F]^T - [I] \right) =$$

$$= \frac{1}{2}\left([F] + [F]^T \right) - [I]$$

It should be noticed that this relation is exact and involves no approximation

PARADOX

For a rigid-body motion $[F] = [Q]$ and

$$[\varepsilon] = \frac{1}{2}\left([Q]^T + [Q] \right) - [I] \neq [\emptyset]$$

This is however quite a false information. We cannot use small deformation theory for rigid-body motion!

the material displacement gradient,

$$[Z] = \left[\frac{\partial u_i}{\partial a_j} \right],$$

as any other second order tensor, can be decomposed into symmetric and antisymmetric parts

$$[Z] = [Z^s] + [Z^{as}]$$

where

$$[Z^s] = \frac{1}{2}\left([Z] + [Z]^T\right), \quad [Z^{as}] = \frac{1}{2}\left([Z] - [Z^T]\right)$$

If we recall that $[Z]$ operates on

$$\{du\} = [Z]\{da\}$$

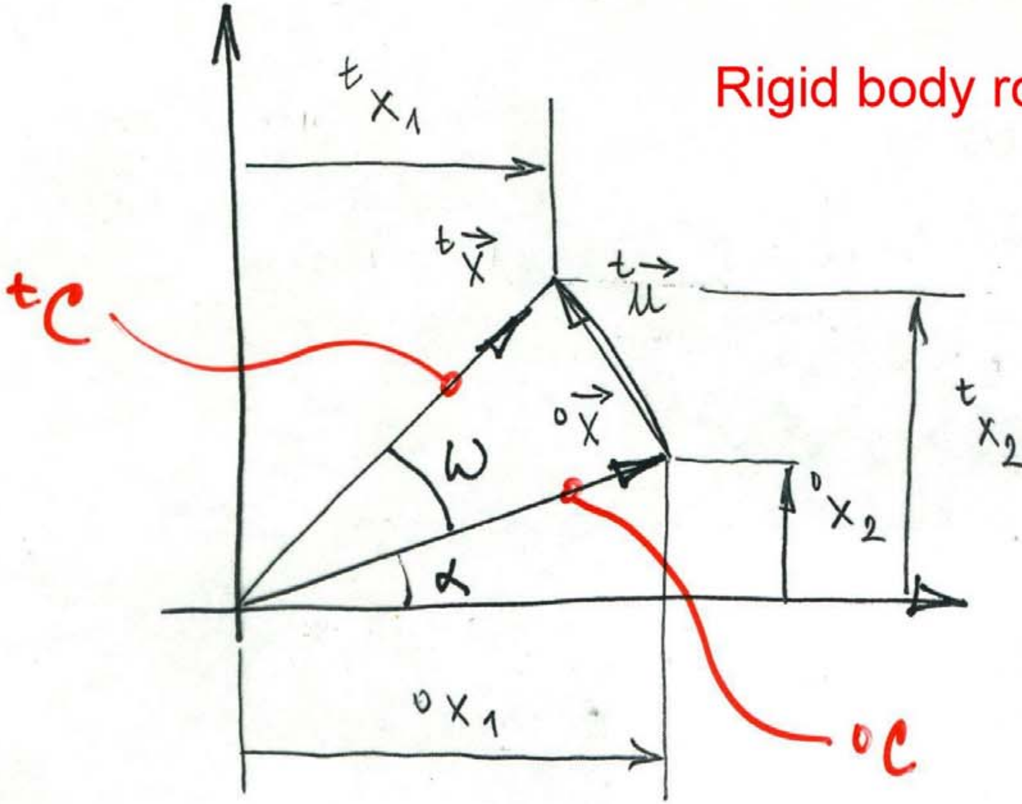
then

$$\{du\} = \Big[\underbrace{\tfrac{1}{2}\left([Z] + [Z]^T \right)}_{[\varepsilon]} + \underbrace{\tfrac{1}{2}\left([Z] - [Z]^T \right)}_{[\Omega]} \Big] \{da\}$$

The first part gives the infinitesimal Cauchy strain tensor ε_{ij}, while the second part Ω_{ij} is so called infinitesimal rotation tensor or Lagrangian rotation tensor

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} - \frac{\partial u_j}{\partial a_i} \right)$$

Rigid body rotation, an example



$$|{}^0\vec{X}| = r$$

$$|{}^t\vec{X}| = r$$

$${}^t\vec{u} = {}^t\vec{X} - {}^0\vec{X}$$

$${}^t u_1 = {}^t x_1 - {}^0 x_1 = r\cos(\omega+\alpha) - r\cos\alpha$$

$${}^t u_2 = {}^t x_2 - {}^0 x_2 = r\sin(\omega+\alpha) - r\sin\alpha$$

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$$ {}^0x_1 = r\cos\alpha \qquad {}^0x_2 = r\sin\alpha $$

$$ \cos(\omega+\alpha) = \cos\omega\cos\alpha - \sin\omega\sin\alpha $$

$$ \sin(\omega+\alpha) \qquad \sin\omega\cos\alpha + \cos\omega\sin\alpha $$

$$ {}^t u_1 = r\cos\omega\cos\alpha - r\sin\omega\sin\alpha - r\cos\alpha $$

$$ {}^t u_1 = {}^0x_1\cos\omega - {}^0x_2\sin\omega - {}^1x_0 $$

$$ {}^t u_1 = {}^0x_1(\cos\omega - 1) - {}^0x_2\sin\omega $$

$$ {}^t u_2 = {}^0x_1\sin\omega + {}^0x_2(\cos\omega - 1) $$

Displacement gradient

$$ {}^{t}_{0}Z = {}^{t}_{0}Z_{ij} = \frac{\partial^t u_i}{\partial^0 x_j} = \begin{bmatrix} \frac{\partial^t u_1}{\partial^0 x_1} & \frac{\partial^t u_1}{\partial^0 x_2} \\ \frac{\partial^t u_2}{\partial^0 x_1} & \frac{\partial^t u_2}{\partial^0 x_2} \end{bmatrix} = $$

$$ = \begin{bmatrix} \cos\omega - 1 & -\sin\omega \\ \sin\omega & \cos\omega - 1 \end{bmatrix} $$

Cauchy infinitesimal strain

$$ \varepsilon_{ij} = \frac{1}{2}\left(Z_{ij} + Z_{ji}\right) = \frac{1}{2}\left([Z] + [Z]^T \right) = $$

$$ = \frac{1}{2}\left(\begin{bmatrix} \cos\omega - 1 & -\sin\omega \\ \sin\omega & \cos\omega - 1 \end{bmatrix} + \begin{bmatrix} \cos\omega - 1 & \sin\omega \\ -\sin\omega & \cos\omega - 1 \end{bmatrix} \right) = $$

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So, the Cauchy strain depends on rigid-body motion

$$= \frac{1}{2}\begin{bmatrix} 2\cos\omega - 2 & 0 \\ 0 & 2\cos\omega - 2 \end{bmatrix} = \begin{bmatrix} \cos\omega - 1 & 0 \\ 0 & \cos\omega - 1 \end{bmatrix}$$

If $\omega \to 0$ then $\varepsilon_{ij} \to 0$

$$E_{ij} = \frac{1}{2}\left(Z_{ij} + Z_{ji} + Z_{ik} Z_{kj} \right) =$$

$$= \begin{bmatrix} \cos\omega - 1 & 0 \\ 0 & \cos\omega - 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 - 2\cos\omega & 0 \\ 0 & 2 - 2\cos\omega \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

While the Green-Lagrange strain is quite insensitive to it

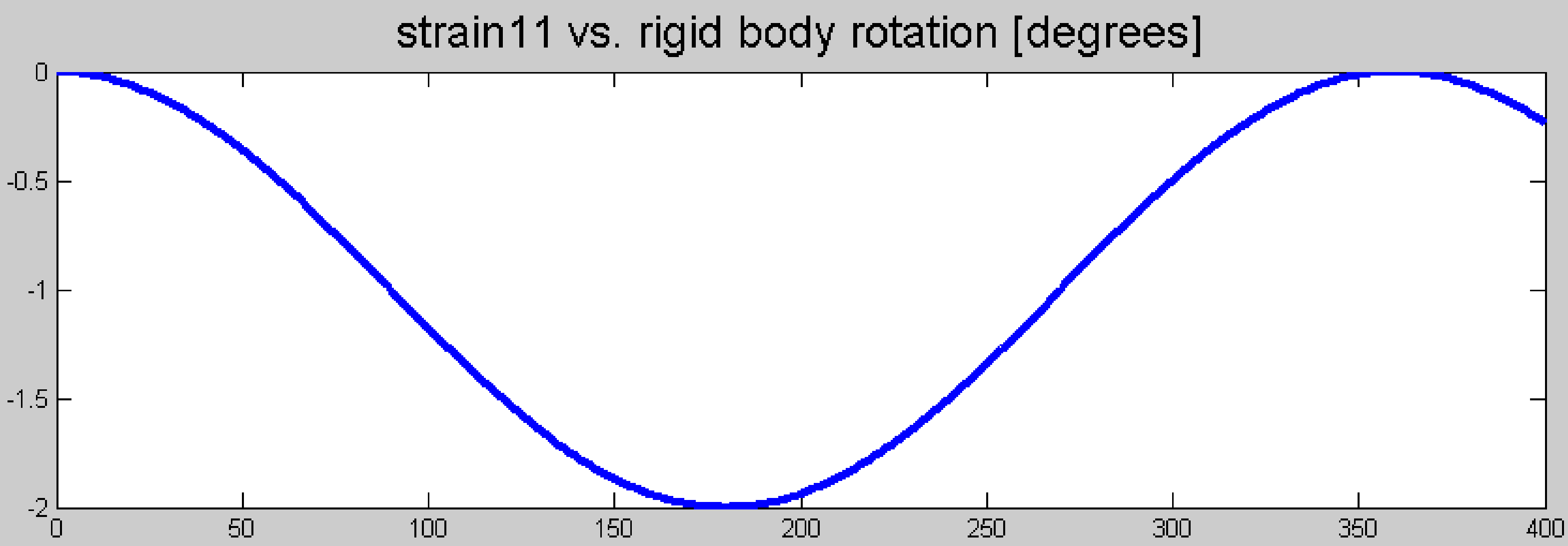
The plane stress induced by Cauchy strain is

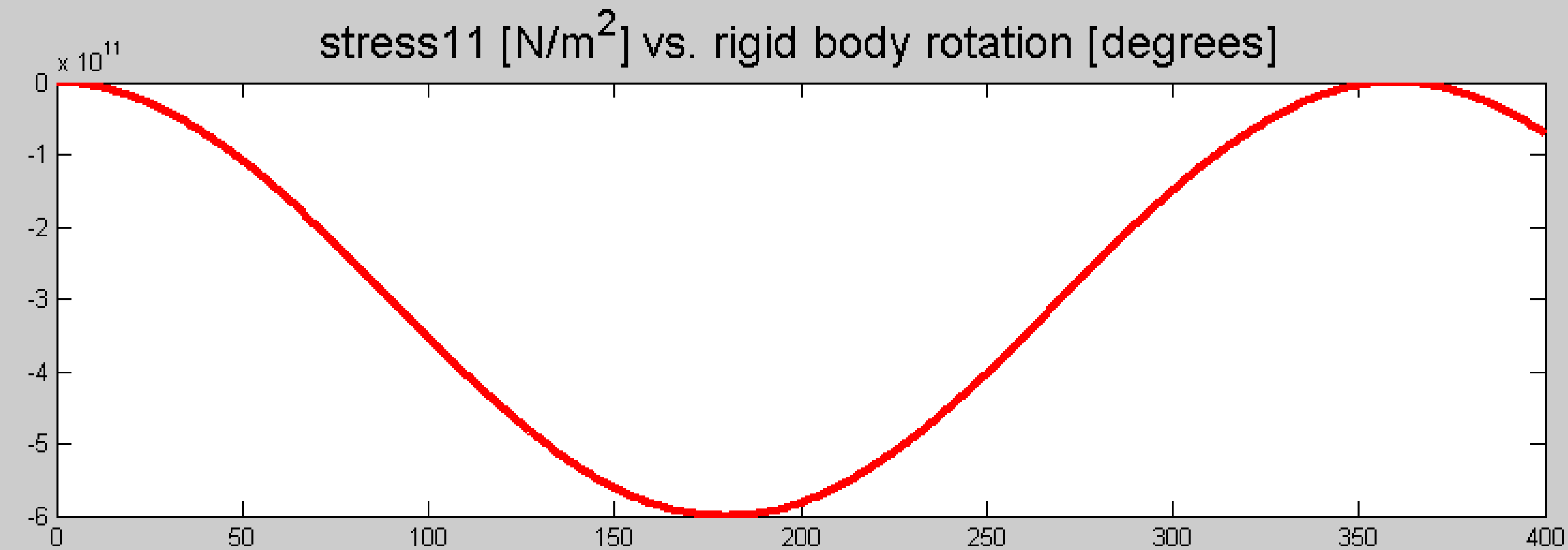
$$\{\sigma\} = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & (1-\mu)/2 \end{bmatrix}\{\varepsilon\}$$

% rigrot.m % str

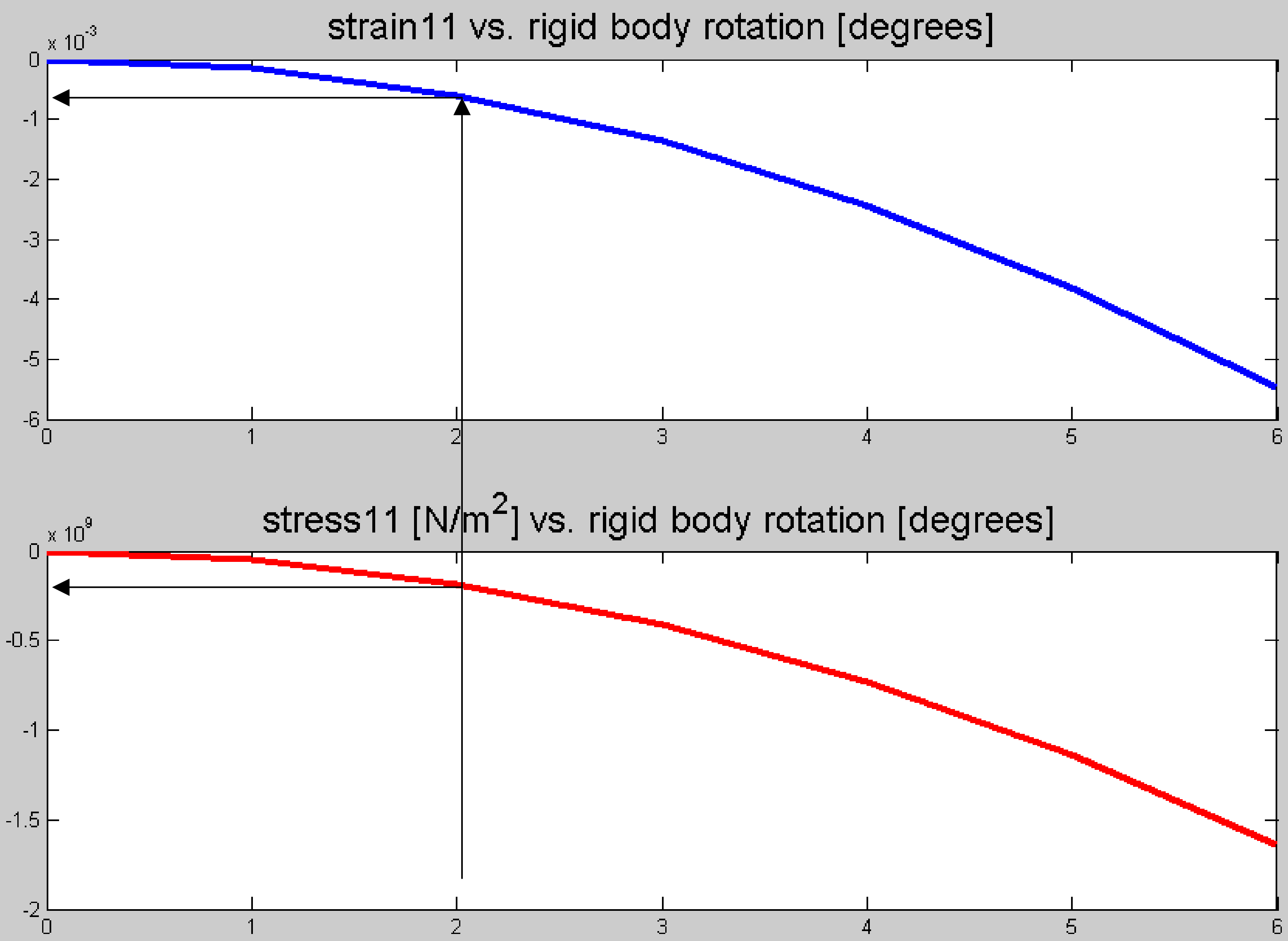
```
% rigrot.m
% strain and stress "induced" due to a rigid body rotation
clear
ommax=6;
omrange=1:ommax;
ss=zeros(3,ommax); ee=zeros(3,ommax);
splot=zeros(ommax,1); eplot=zeros(ommax,1);
mu=0.3;
E=2.1e11;
const=E/(1-mu^2);
d=const*[1 mu 0; mu 1 0; 0 0 (1-mu)/2];
for om=omrange
  om;
  omr=om*pi/180; e1=cos(omr)-1;
  eplot(om)=e1;       ep=[e1 e1 0];
  ee(:,om)=ep'; sigma=d*ep';
  splot(om)=sigma(1,1); ss(:,om)=sigma;
end

figure(1)
subplot(2,1,1);plot(0:ommax,[0 eplot']);
title('strain11 vs. rigid body rotation [degrees]');
subplot(2,1,2);plot(0:ommax,[0 splot']);
title('stress11 [N/m^2] vs. rigid body rotation [degrees]')
% end of rigrot
```

False strain and stress due to rigid-body motion





As before, but for smaller values of rotation angle



THE STRETCH or EXTENSION RATIO

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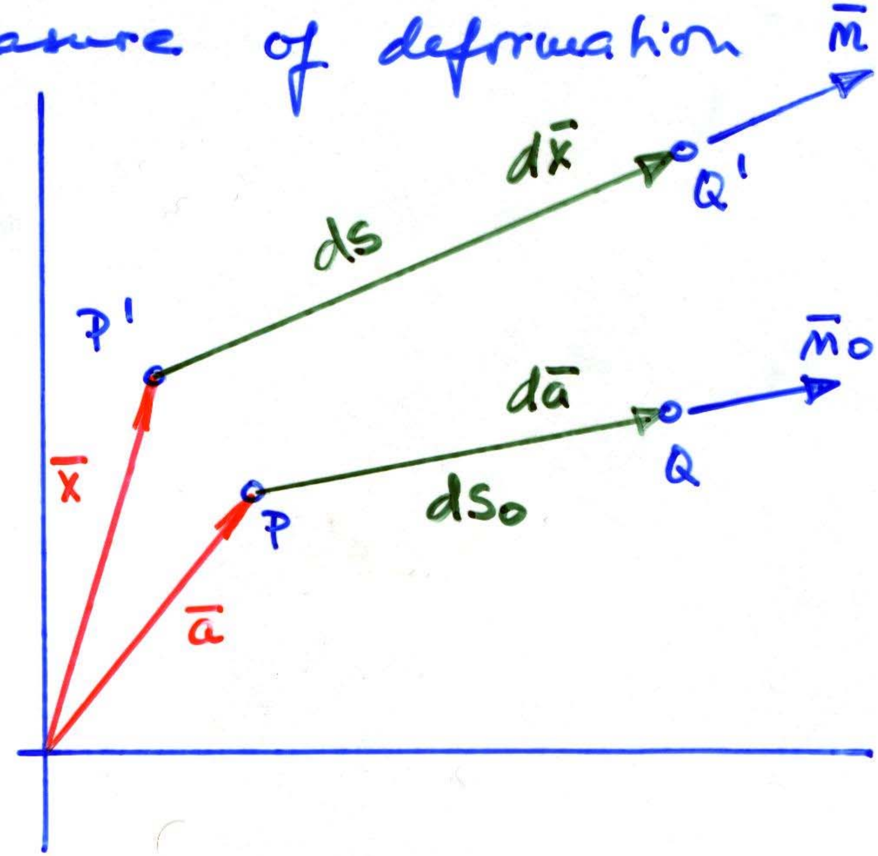
is another suitable measure of deformation

If

$\bar{n}$ and $\bar{n}_0$ are unit vectors alligned with line segments $\overline{PQ}$ and $\overline{P'Q'}$

then

$$\bar{n}_0 = \frac{d\bar{a}}{ds_0} \, , \quad \bar{n} = \frac{d\bar{x}}{ds}$$



Positional vectors of particles P, Q which are moved into P' and Q' respectively

P: $\bar{a}$ $\quad a_i$ cm0114b

Q: $\bar{a} + d\bar{a} = \bar{a} + ds_0 \bar{n}_0$; $a_i + da_i = a_i + ds_0\, n_{0i}$

P': $\bar{x}$ $\quad x_i$

Q': $\bar{x} + dx = \bar{x} + ds\bar{n}$; $x_i + dx_i = x_i + ds\, n_i$

Using Taylor theorem for Q' coordinates

$$\cancel{x_i} + dx_i = \cancel{x_i} + \frac{\partial x_i}{\partial a_k} da_k + \ldots \doteq \cancel{x_i} + F_{ik}\, da_k$$

to cancel out

$$\Rightarrow \quad ds\, n_i = F_{ik}\, ds_0\, n_{0k}$$

$$n_i \underbrace{\frac{ds}{ds_0}}_{\tilde{\lambda}} = F_{ik}\, n_{0k} \qquad \{n\}\tilde{\lambda} = [F]\{n_0\} \quad (*)$$

... this is known as STRETCH, stretch ratio or stretch ratio

So the stretch is defined as

$$\tilde{\lambda} = ds/ds_0$$

Rewrite (*)

$$\tilde{\lambda} n_i = F_{ik} n_{0k} \qquad \tilde{\lambda}\{n\} = [F]\{n_0\} \quad (**)$$

By squaring each side we get

$$(\tilde{\lambda})^2 \underbrace{\{n\}^T\{n\}}_{1} = \{n_0\}^T \underbrace{[F]^T[F]}_{[C]} \{n_0\}$$

right Cauchy-Green def. tens.

$$\Rightarrow \qquad \tilde{\lambda}^2 = \{n_0\}^T [C] \{n_0\}$$

Rewrite (**)

$$\{n_0\} = \tilde{\lambda}\,[F]^{-1}\{n\}$$

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By squaring: $\underbrace{\{n_0\}^T\{n_0\}}_{1} = \tilde{\lambda}^2 \{n\}^T \underbrace{[F]^{-T}[F]^{-1}}_{[\bar{C}]^{-1}}\{n\}$

inverse of left Cauchy-Green def. tensor

$\Rightarrow \dfrac{1}{\tilde{\lambda}^2} = \{n\}[\bar{C}]^{-1}\{n\}$

this quantity is sometimes called Green def. tensor and denoted $[G]$

STRETCH TENSORS, ROTATION TENSOR

Arbitrary nonsingular second order tensor may be decomposed into product of a positive symmetric second-order tensor with an orthogonal second-order tensor ... this is so called polar decomposition.

When applied to deformation gradient

a) $[F] \rightarrow [R][U]$ right decomposition

b) $[F] \rightarrow [V][R]$ left

$[R]^T = [R]^{-1}$ orthogonality

$[U] = [U]^T, \; [V] = [V]^T$ symmetry

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Physical interpretation

the considered deformation consists of sequential

a) stretching [U] followed by rotation [R]

b) rotation [R] followed by stretching [V] cm0117

ALGORITHM FOR POLAR DECOMPOSITION

1) $[F]$ for ${}^{0}C \rightarrow {}^{t}C$. — deformation gradient

2) $[C] = [F]^T [F]$ — right Cauchy-Green def. t.

3) $([\phi], [\Lambda]) = eig([C])$

4) $[U] = [\phi][\Lambda]^{1/2}[\phi]^T$ — right stretch tensor

5) $[R] = [F][U]^{-1}$ — rotation tensor

6) $[V] = [R][U][R]^T$ — left stretch tensor

Proof

If $F = RU$

then $R = FU^{-1}$

and $R^T = U^{-T}F^T = U^{-1}F^T$. (symmetry of U)

So $R^TR = U^{-1}F^TFU^{-1} = I$ (orthogonality of R)

But $C = F^TF$ ... right Cauchy-Green def. tensor

$$\Rightarrow \quad U^{-1}CU^{-1} = I$$

$$C = U^2,$$

Solving the standard eigenvalue problem

$$(\phi, \Lambda) = eig(C), \quad C - \text{symmetric}$$

We get the orthogonal matrix of eigenvectors $[\phi]^{-1} \equiv [\phi]^T$

$$[\phi] = [\{\varphi\}^{(1)} \ \{\varphi\}^{(2)} \ \{\varphi\}^{3}], \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix},$$

and real eigenvalues

So that

$$\mathbf{\Phi}^{\mathrm{T}}\,\mathbf{C}\,\mathbf{\Phi} = \mathbf{\Lambda}$$

The above equation also represents the diagonalization of $\mathbf{C}$

$\mathbf{C}' = \mathbf{\Phi}^{\mathrm{T}}\,\mathbf{C}\,\mathbf{\Phi}$ with an obvious conclusion that $\mathbf{C}' = \mathbf{\Lambda}$.

Now, we can transform $\mathbf{F}$ to the primed coordinate system

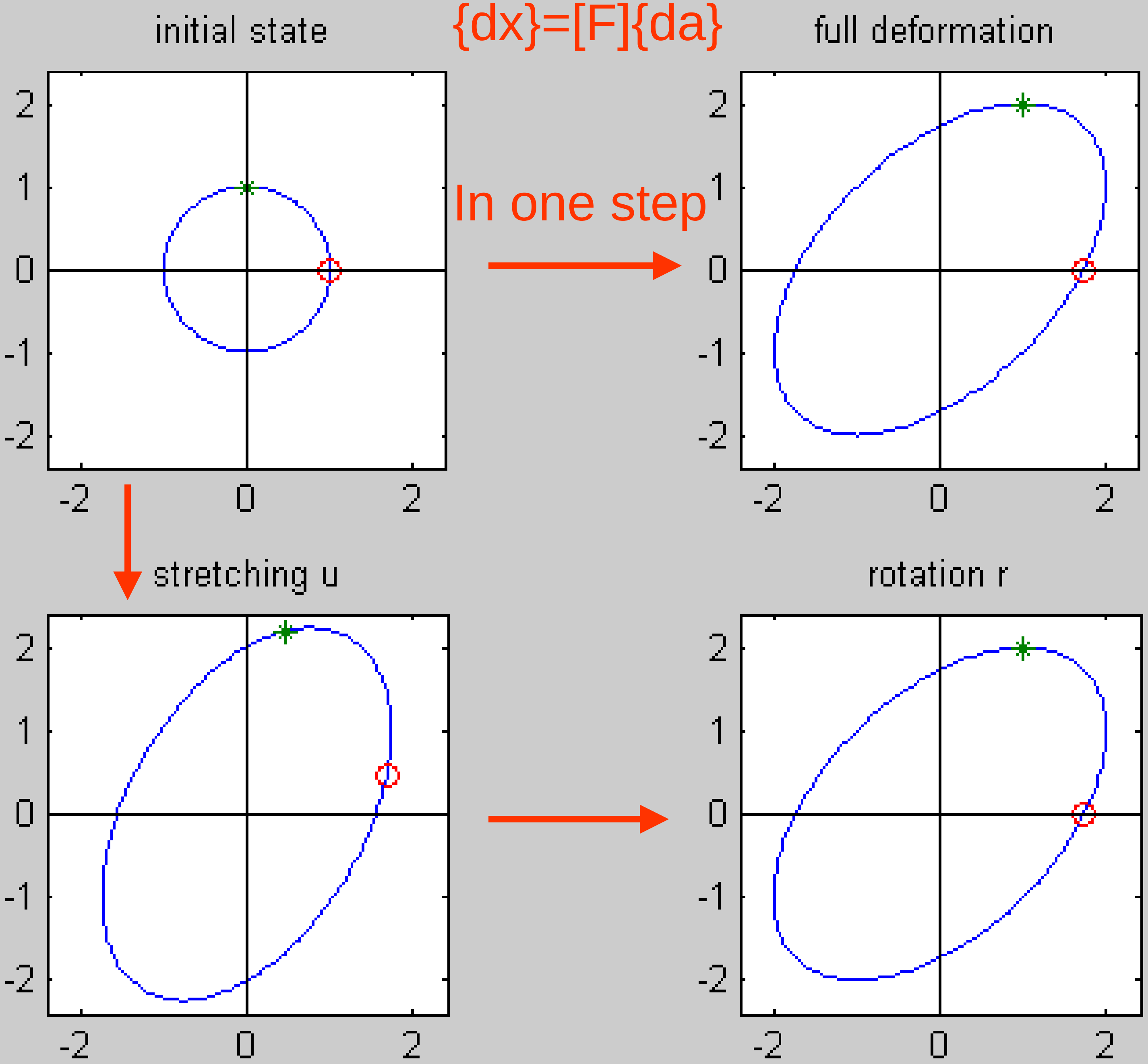
$\mathbf{F}' = \mathbf{\Phi}^{\mathrm{T}}\,\mathbf{F}\,\mathbf{\Phi}$ and $\mathbf{F}' = \mathbf{R}'\,\mathbf{U}'$

and similarly

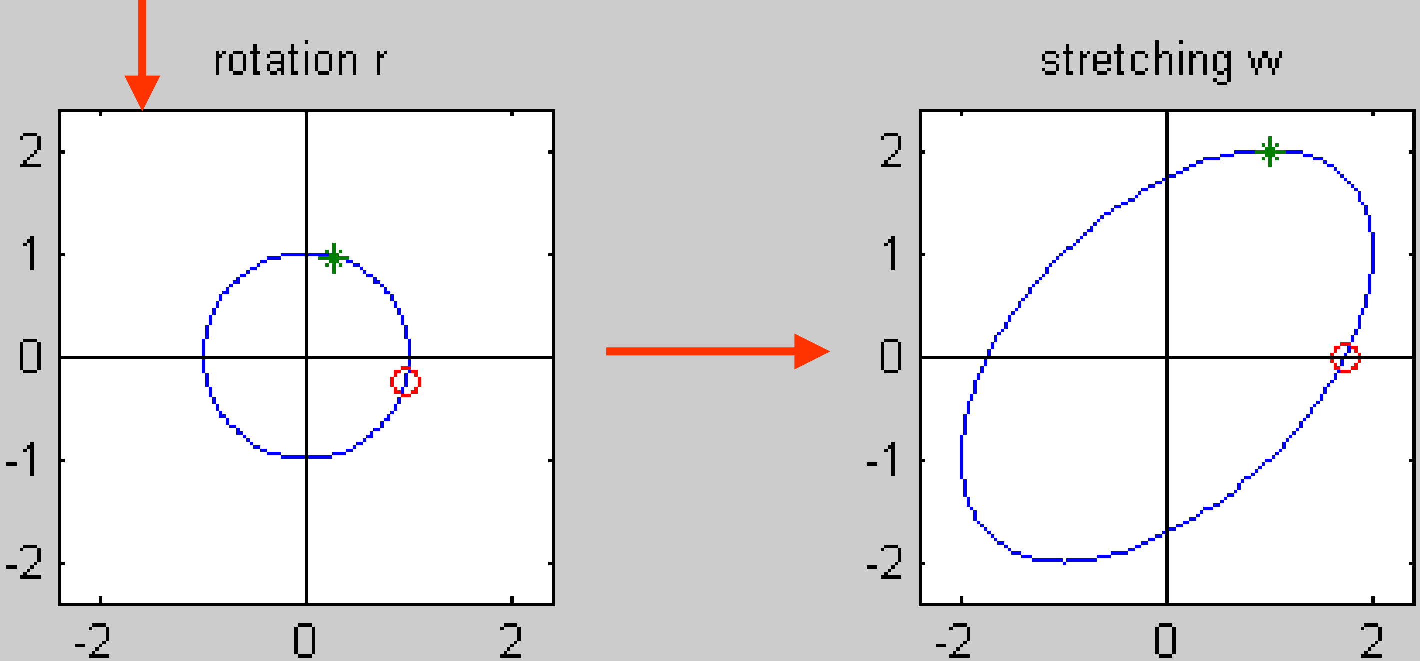
$\mathbf{U}' = \mathbf{\Phi}^{\mathrm{T}}\,\mathbf{U}\,\mathbf{\Phi}$ with $\mathbf{C}' = \left(\mathbf{U}'\right)^2$

$\Rightarrow \mathbf{U} = \mathbf{\Phi}\,\mathbf{U}'\,\mathbf{\Phi}^{\mathrm{T}}$ but we have seen that $\mathbf{U}' = \sqrt{\mathbf{C}'}$ since $\mathbf{C}' = \mathbf{\Lambda}$ is diagonal

$\Rightarrow$ so $\mathbf{U} = \mathbf{\Phi}\,\sqrt{\mathbf{\Lambda}}\,\mathbf{\Phi}^{\mathrm{T}}$, which solves the problem.

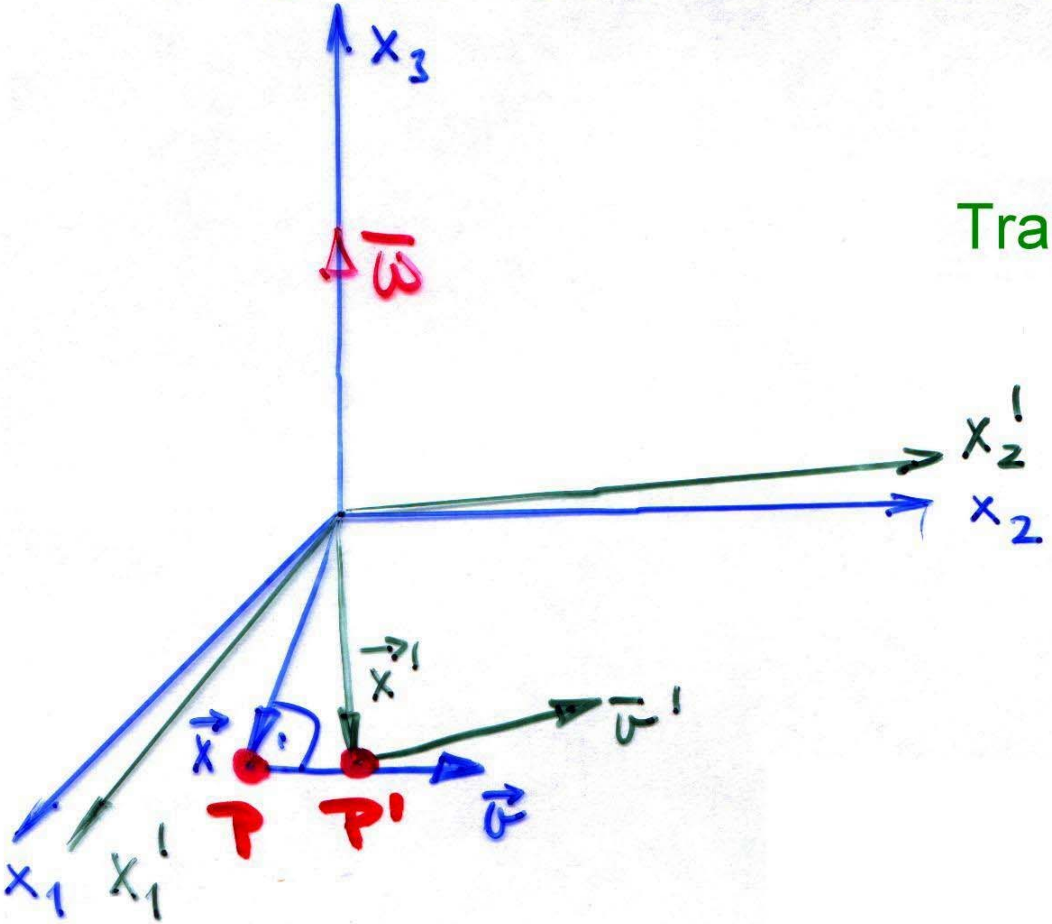


Right polar decomposition F = RU



Left polar decomposition F = VR

All processes are time dependent. Quantities in rate form are often used. Rate of polar rotation is one of them and we will use it in the text which follows.



Transformation of coordinates

$$\{x'\} = [R]\{x\} \qquad (1)$$

and its time derivative

$$\{\dot{x}'\} = [\dot{R}]\{x\}$$

stationary syst

$\Rightarrow \{x\}$ is cons

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Time derivative of a vector is the velocity of its end point $\{v'\} = [\dot{R}]\{x\}$ (2)

Rotation can be expressed in both coordinate systems

$\vec{v} = \vec{\omega} \times \vec{x}$ or $\{v\} = [\Omega^R]\{x\}$

$$\Omega_{ij}^{\mathrm{R}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$\{v'\} = [\Omega^R]\{x'\}$ (3)

substituting (1) to (3)

and comparing with (2) gives

$[\dot{R}]\{x\} = [\Omega^R][R]\{x\}$ which must hold

for any $\{x\}$ =>

$[\dot{R}] = [\Omega^R][R]$ or ([R] orthogonal)

$[\Omega^R] = [\dot{R}][R]^T$

The quantity

$$[\Omega^R] = [\dot{R}][R]^T$$

Rate of polar rotation

is usually called only RATE of ROTATION and denoted $[\Omega]$ only.

DON'T MIXE IT WITH Infinitesimal rotation tensor defined as

$$[\Omega] = \frac{1}{2}\left([Z] - [Z]^T\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} - \frac{\partial u_j}{\partial a_i}\right)$$

DON'T BE CONFUSED EASILY ! SINCE IT WILL BE SHOWN THAT FOR LINEAR ELASTICITY THE RATE OF INFINITESIMAL ROTATION TENSOR or SIMPLY RATE OF ROTATION IS SO CALLED SPIN TENSOR

$$[W] = [\dot{\Omega}]$$