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Banach spaces with projectional skeletons ⋆

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Abstract

A projectional skeleton in a Banach space is a σ-directed family of projections onto separable subspaces, covering the entire space. The class of Banach spaces with projectional skeletons is strictly larger than the class of Plichko spaces (i.e. Banach spaces with a countably norming Markushevich basis). We show that every space with a projectional skeleton has a projectional resolution of the identity and has a norming space with similar properties to Σ-spaces. We characterize the existence of a projectional skeleton in terms of elementary substructures, providing simple proofs of known results concerning weakly compactly generated spaces and Plichko spaces.

We prove a preservation result for Plichko Banach spaces, involving transfinite sequences of projections. As a corollary, we show that a Banach space is Plichko if and only if it has a commutative projectional skeleton.

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1 Introduction

It is well known that a Banach space in which every closed subspace is complemented is necessarily isomorphic to a Hilbert space (Lindenstrauss & Tzafriri [30]). On the other hand, there exist Banach spaces in which only finite-dimensional and co-finite-dimensional subspaces are complemented (Gowers & Maurey [11]). There even exist (necessarily, non-separable) $\mathcal{C}(K)$ spaces with no nontrivial bounded projections (Koszmider [21] and Plebanek [33]). In the positive direction, one has to mention the work of Heinrich & Mankiewicz [15] where, using substructures of ultrapowers of Banach spaces, the authors show in particular the existence of non-trivial bounded projections in every dual Banach space of density greater than the continuum (see [38] for an elementary proof).

We are interested in Banach spaces which have "many" projections onto separable subspaces. Perhaps one of the most general classes of this sort would be the class of Banach spaces with the *separable complementation property* (SCP): a space X has the SCP if, by definition, for every countable set $S \subseteq X$ there is a bounded projection $P\colon X \to X$ such that $\operatorname{im} P$ is separable and $S \subseteq \operatorname{im} P$. It turns out that this property is general enough in order to include somewhat pathological spaces. For a survey on the complementation property and its variations we refer to [37].

Another possible class of Banach spaces with many projections are spaces with the *projectional resolution of the identity* (PRI), notion introduced by Lindenstrauss [28,29], defined to be a well ordered continuous chain of projections onto smaller subspaces. This property together with transfinite induction allows proving various properties of a non-separable Banach space, e.g. a locally uniformly rotund/Kadec renorming [42,6] and the existence of a linear injection into $c_0(\Gamma)$ [1,41]. See [10] or [7] for more information concerning the PRI method. Unfortunately, the existence of a PRI in a Banach space is good enough only for density $\aleph_1$, otherwise it does not even imply the SCP.

We propose a natural class of Banach spaces which have a family of projections onto separable subspaces, indexed by a σ-directed partially ordered set and satisfying some natural conditions, similar to a PRI. We call it a "projectional skeleton". It turns out that a Banach space with a projectional skeleton looks "almost" like a Plichko space, i.e. a Banach space with a countably norming Markushevich basis (see [34,35,36] and [40]). In fact, we essentially know only one basic example distinguishing those two classes: the space $\mathcal{C}(K)$, where K is the ordinal $\omega_2 + 1$, endowed with the interval topology. This was shown by Kalenda in [19]. Banach spaces with a projectional skeleton can be characterized by a property involving norming sets and countable elementary submodels. We explain how to use elementary submodels of (some initial part of) set theory for constructing bounded projections and we show that the existence of a projectional skeleton is equivalent to some natural model-theoretic property of a suitable norming space, similar to the existence of the so-called *projectional generator*. As an application, we get short proofs of some well known results, like the existence of a PRI in every weakly compactly generated space. Our characterization allows us to show that the class of Banach spaces with a projectional skeleton of norm one is stable under arbitrary c_0- and ℓ_p-sums ($1 \leqslant p < \infty$). We apply elementary submodels for constructing a PRI from a projectional skeleton.

The main part (Section 5) is devoted to Plichko spaces. We prove a preservation result for inductive limits of certain projective sequences of Plichko spaces, similar in spirit to Gul′ko's results on subspaces of Σ-products [12,13,14]. As an application, we show that a Banach space is Plichko if and only if it has a commutative projectional skeleton.

Finally, we discuss retractional skeletons in compact spaces – a notion dual to projectional skeleton. We characterize this class of compacta by means of elementary submodels and we state a preservation property for Valdivia compacta, dual to the corresponding result for Banach spaces. Retractional skeletons were introduced in [25]. It is proved there that Valdivia compacta are precisely those compact spaces which have a commutative retractional skeleton. For more information and recent results concerning Valdivia compacta and their spaces of continuous functions we refer to [17,20,25,26,3,23].

2 Preliminaries

We shall consider Banach spaces over the field of real numbers, although the results are true also for the complex case. Below we recall most relevant notions, definitions and notation.

By a *projection* in a Banach space X we mean a bounded linear operator $P\colon X \to X$ such that $P \circ P = P$. In this case $\operatorname{im} P = \{x \in X\colon x = Px\}$ and $\ker P = \{x - Px\colon x \in X\} = \operatorname{im}(\mathrm{id}_X - P)$, where id_X is the identity map. Recall that a space X has the *separable complementation property* (*SCP* for short) if for every countable set $S \subseteq X$ there is a projection $P\colon X \to X$ such that PX is a separable space containing S. Given $B \subseteq X^*$, we write ${}^\perp(B) = \{x \in X\colon (\forall\, b \in B)\ b(x) = 0\}$. The right annihilator $(A)^\perp$ is defined similarly.

Let λ be a limit ordinal. A *projectional sequence* of length λ in a Banach space X is a sequence of projections $\{P_\xi\}_{\xi<\lambda}$ satisfying the following conditions:

(1) $\xi < \eta \implies P_\xi = P_\eta \circ P_\xi = P_\xi \circ P_\eta$,
(2) $P_\delta X = \operatorname{cl}(\bigcup_{\xi<\delta} P_\xi X)$ for every limit ordinal $\delta < \lambda$,
(3) $X = \operatorname{cl}(\bigcup_{\xi<\lambda} P_\xi X)$.

A special case is a *projectional resolution of the identity* (*PRI*): this is a projectional sequence $\{P_\xi\}_{\xi<\lambda}$ such that $\|P_\xi\| = 1$ and $\operatorname{dens}(P_\xi X) \leqslant |\xi| + \aleph_0$ for each $\xi < \lambda$, where $|\xi|$ denotes the cardinality of ξ.

All topological spaces are assumed to be completely regular. The closure of a set A in a space X will be denoted by $\mathrm{cl}(A)$ or, more precisely, by $\mathrm{cl}_X(A)$. If X is a dual to a Banach space then cl_* will denote the *weak** closure, i.e. the closure with respect to the *weak** topology on X.

A space X is *countably tight* if for every $A \subseteq X$ and for every $p \in \mathrm{cl}\, A$ there exists $A_0 \in [A]^{\aleph_0}$ with $p \in \mathrm{cl}\, A_0$. Let Γ be a set. Given $x \in \mathbb{R}^\Gamma$, we denote by $\mathrm{suppt}(x)$ the *support* of x, i.e. $\mathrm{suppt}(x) = \{\alpha \in \Gamma \colon x(\alpha) \neq 0\}$. The set $\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma \colon |\mathrm{suppt}(x)| \leqslant \aleph_0\}$ is called a Σ-*product*.

A *Valdivia compact* is a compact space homeomorphic to $K \subseteq [0,1]^\kappa$ satisfying $K = \mathrm{cl}(K \cap \Sigma(\kappa))$. A *Corson compact* is, by definition, a compact subset of $\Sigma(\kappa)$ for some κ. Given a compact K, $D \subseteq K$ is called a Σ-*subset* of K if there is a homeomorphic embedding $h \colon K \to [0,1]^\kappa$ such that $D = h^{-1}[\Sigma(\kappa)]$.

Let $\langle X, \|\cdot\|\rangle$ be a Banach space. A set $D \subseteq X^*$ is *norming* if the formula

$$(*) \qquad |x| = \sup\{|\varphi(x)|/\|\varphi\| \colon \varphi \in D \setminus \{0\}\}$$

defines an equivalent norm on X. More precisely, we say that D is r-*norming* if $\|x\| \leqslant r|x|$ for every $x \in X$. D is 1-*norming* if $|\cdot| = \|\cdot\|$. The following fact is well known.

Proposition 1 *Let D be a norming subset of X^*. Then D is 1-norming with respect to the norm $|\cdot|$ defined by equation (*).*

PROOF. Let D' be the linear span of D and let $D_1 := \{\varphi \in D' \colon \|\varphi\| \leqslant 1\}$. Then $|x| = \sup_{\varphi \in D_1} |\varphi(x)|$. Thus, it remains to notice that $D_1 = \{\varphi \in D' \colon |\varphi| \leqslant 1\}$. Indeed, if $\varphi \in D_1$ then $|\varphi(x)| \leqslant 1$ whenever $|x| \leqslant 1$, so $|\varphi| \leqslant 1$. Clearly, $|x| \leq \|x\|$ for $x \in X$ and therefore for dual norms we have the inverse inequality, i.e., $\|\varphi\| \leq |\varphi|$ for $\varphi \in X^*$.

Recall that a Banach space X is called *Plichko* if there exists a one-to-one *weak** continuous linear operator $T \colon X^* \to \mathbb{R}^\kappa$ such that $T^{-1}[\Sigma(\kappa)]$ is norming for X. Equivalently: there are a linearly dense set $A \subseteq X$ and a norming set $D \subseteq X^*$ such that for every $y \in D$ the set $\{a \in A \colon y(a) \neq 0\}$ is countable. If additionally D is linear, we shall say that $\langle X, D\rangle$ is a *Plichko pair*. More generally, we say that $\langle Y, D\rangle$ is a Plichko pair in a space X if Y is a closed linear subspace of X and $\langle Y, D'\rangle$ is a Plichko pair, where $D' = \{y \restriction Y \colon y \in D\}$. Given $A \subseteq X$, the set $\mathrm{suppt}(y, A) = \{a \in A \colon y(a) \neq 0\}$ will be called the A-*support* of $y \in X^*$. The space $\overline{D} = \{y \in X^* \colon |\mathrm{suppt}(y, A)| \leqslant \aleph_0\}$ is called a Σ-*space*.

A particularly interesting subclass of Plichko spaces is the class of weakly Lindelöf determined spaces, introduced by Valdivia [39]. A Banach space X is *weakly Lindelöf determined* (*WLD* for short) if $\langle X, X^*\rangle$ is a Plichko pair, i.e. X^* is a Σ-space. This is equivalent to saying that the dual unit ball with the *weak** topology is Corson compact (see [31, Prop. 4.1]).

3 Elementary submodels and projections

In this section we introduce the method of elementary submodels, which will be used extensively throughout the paper. In the context of retractions – topological counterparts of linear projections – elementary submodels turned out to be an important tool in [23,25]. We refer to the survey article of Dow [8], where several applications of elementary submodels in set-theoretic topology are explained.

Let N be a fixed set. The pair $\langle N, \in \rangle$, where $\in$ is restricted to $N \times N$, is a structure of the language of set theory. Given a formula $\varphi(x_1, \dots, x_n)$ with all free variables shown and given $a_1, \dots, a_n \in N$ one defines the relation "$\langle N, \in \rangle$ satisfies $\varphi(a_1, \dots, a_n)$" (briefly "$\langle N, \in \rangle \models \varphi(a_1, \dots, a_n)$" or just "$N \models \varphi(a_1, \dots, a_n)$") in the usual way, by induction on the length of the formula. Namely, $N \models a_1 \in a_2$ iff $a_1 \in a_2$ and $N \models a_1 = a_2$ iff $a_1 = a_2$. It is clear how "satisfaction" is defined for conjunction, disjunction and negation. Finally, if φ is of the form $(\exists\ y)\psi(x_1, \dots, x_n, y)$ then $N \models \varphi(a_1, \dots, a_n)$ iff there exists $b \in N$ such that $N \models \psi(a_1, \dots, a_n, b)$.

As an example, if $s = \{a, b, c\}$ and $s, a, b \in N$ while $c \notin N$, then N satisfies "s has at most two elements", because for every $x \in N$ if $x \in s$ then either $x = a$ or $x = b$.

Instead of the above definition, some authors use relativization, see e.g. Kunen's book [27]. Given a formula φ, the *relativization* of φ to N is a formula φ^N which is built from φ by replacing each quantifier of the form "$\forall\ x$" by "$\forall\ x \in N$" and each quantifier of the form "$\exists\ x$" by "$\exists\ x \in N$". By this way, $N \models \varphi(a_1, \dots, a_n)$ iff $\varphi^N(a_1, \dots, a_n)$ holds (of course, $a_1, \dots, a_n$ must be elements of N).

Given a set x, we define the *transitive closure* of x to be $\operatorname{tc}(x) = \bigcup_{n\in\omega} \operatorname{tc}_n(x)$, where $\operatorname{tc}_1(x) = x \cup \bigcup x$ and $\operatorname{tc}_{n+1}(x) = \operatorname{tc}_1(\operatorname{tc}_n(x))$. In other words: $y \in \operatorname{tc}(x)$ iff there are $x_0 \in x_1 \in \dots \in x_k$ such that $y \in x_0$ and $x_k \in x$. Thanks to the Axiom of Regularity, these two definitions of transitive closure are equivalent and every set of the form $\operatorname{tc}(x)$ is *transitive*, i.e. $y \in \operatorname{tc}(x)$ implies $y \subseteq \operatorname{tc}(x)$.

Given a cardinal θ, we denote by $H(\theta)$ the class of all sets whose transitive closure has cardinality $< \theta$. It is well known that $H(\theta)$ is a set, not a proper class. It is clear that $H(\theta)$ is transitive. We shall consider elementary substructures of $\langle H(\theta), \in \rangle$. It is well known that for a regular uncountable cardinal θ, the structure $\langle H(\theta), \in \rangle$ satisfies all the axioms of set theory, except possibly the Power Set Axiom, see [27, IV.3].

Recall that a substructure M of $\langle H(\theta), \in \rangle$ is called *elementary* if for every formula

$\varphi(x_1, \dots, x_n)$ with all free variables shown, for every $a_1, \dots, a_n \in M$, we have that

$$M \models \varphi(a_1, \dots, a_n) \iff H(\theta) \models \varphi(a_1, \dots, a_n).$$

The fact that M is an elementary submodel of $\langle H(\theta), \in \rangle$ is denoted by $M \preceq \langle H(\theta), \in \rangle$.

In order to illustrate elementarity, let us come back to the simple example described above: let $s = \{a, b, c\} \in N$, $a, b \in N$ and now assume that $N \preceq H(\theta)$ and that a, b, c are pairwise distinct. Since $N \subseteq H(\theta)$, we see that $s \in H(\theta)$ and consequently also $c \in H(\theta)$. Clearly, $H(\theta) \models c \in s$, therefore $H(\theta)$ satisfies "there is $x \in s$ such that neither $x = a$ nor $x = b$ ". By elementarity, N satisfies the same statement, which means that there exists $d \in N$ such that $N \models (d \in s \wedge d \neq a \wedge d \neq b)$. This is a conjunction of atomic formulas and their negations, so indeed $d \in s$ and $d \notin \{a, b\}$. But we have assumed that $s \cap N = \{a, b\}$, which is a contradiction. This example shows that elementary substructures of $H(\theta)$ "keep" elements of a finite set. In general, if $N \preceq H(\theta)$ then $s \in N$ does not necessarily imply that $s \subseteq N$, unless s is countable or N contains a sufficiently big initial interval of ordinals, see Proposition 2(c) below or [8, Thm. 1.6].

The reason for using elementary submodels of $H(\theta)$ is that these structures satisfy most of the axioms of set theory: if $\theta > \aleph_0$ is regular then $H(\theta)$ satisfies all the axioms except possibly the power-set, because it may happen that $2^\lambda > \theta$ for some $\lambda < \theta$. Moreover, in practice it is usually easy to point out a cardinal θ such that $H(\theta)$ satisfies given finitely many formulas with parameters, needed for applications. Another useful feature of $H(\theta)$ with θ regular is the fact for every formula $\varphi(x_1, \dots, x_n)$ in which all quantifiers are bounded (i.e. of the form "$\forall\, x \in y$" or "$\exists\, x \in y$") and for every $a_1, \dots, a_n \in H(\theta)$, $\varphi(a_1, \dots, a_n)$ holds if and only if $H(\chi) \models \varphi(a_1, \dots, a_n)$. For more information, see [27, IV.3]. Since in most cases we indeed use formulas with bounded quantifiers, one can simply "check" their validity by looking at a sufficiently large $H(\theta)$.

One can also use *Reflection Principle*, which says that given a formula of set theory $\varphi(x_1, \dots, x_n)$ and given sets $a_1, \dots, a_n$ such that $\varphi(a_1, \dots, a_n)$ holds, there exists θ such that the structure $\langle H(\theta), \in \rangle$ satisfies $\varphi(a_1, \dots, a_n)$. In some cases θ may not be regular, although it may be arbitrarily big and it may have arbitrarily big cofinality. More precisely: the class of cardinals θ with the above property is closed and unbounded. Thus, when considering finitely many formulas and parameters, we can "check" their validity by restricting attention to $H(\theta)$, where θ is a "big enough" cardinal, meaning that the cofinality of θ is greater than a prescribed cardinal and all relevant formulas are satisfied in $\langle H(\theta), \in \rangle$.

Summarizing: assume we would like to use in our arguments formulas $\varphi_1, \dots, \varphi_n$ and parameters from a finite set S. We then find a cardinal θ so that $S \subseteq H(\theta)$ and, by Reflection Principle, all valid formulas $\varphi_1, \dots, \varphi_n$ with suitable parameters are satisfied in $H(\theta)$. Finally, we shall use elementary substructures of $H(\theta)$ which contain S. If the formulas $\varphi_1, \ldots, \varphi_n$ have only bounded quantifiers (which happens in most cases), then we do not really need to use Reflection Principle, since the formulas will be satisfied in every $H(\theta)$ with θ "big enough", i.e. every regular θ greater than some fixed cardinal θ_0.

A particular case of the Löwenheim-Skolem Theorem (for the language of set theory) says that for every infinite set $S \subseteq H(\theta)$ there exists $M \preceq \langle H(\theta), \in \rangle$ such that $|M| = |S|$. This theorem can be viewed as the "ultimate" closing-off argument and its typical proof indeed proceeds by "closing-off" the given set S, by adding elements which witness "satisfaction" of all suitable formulas of the form $(\exists\, x)\ \psi$.

Important for applications is the fact that, thanks to the Löwenheim-Skolem theorem, we may consider *countable* elementary substructures of an arbitrarily large $H(\theta)$.

Proposition 2 *Let θ be an uncountable cardinal and let $M \preceq \langle H(\theta), \in \rangle$.*

(a) Assume $u \in H(\theta)$, $a_1, \ldots, a_n \in M$ and $\varphi(y, x_1, \ldots, x_n)$ is a formula such that u is the unique element of $H(\theta)$ for which $H(\theta) \models \varphi(u, a_1, \ldots, a_n)$. Then $u \in M$.

(b) Let $s \subseteq M$ be a finite set. Then $s \in M$.

(c) If $S \in M$ is a countable set then $S \subseteq M$.

PROOF. (a) Since $H(\theta) \models (\exists\, u)\ \varphi(u, a_1, \ldots, a_n)$, by elementarity there exists $v \in M$ such that $M \models \varphi(v, a_1, \ldots, a_n)$. Using elementarity backwards, we see that $H(\theta) \models \varphi(v, a_1, \ldots, a_n)$. By uniqueness, $u = v$.

(b) Let $s = \{a_1, \ldots, a_n\} \subseteq M$. Since θ is infinite, $s \in H(\theta)$. Thus s is the unique element of $H(\theta)$ satisfying the formula $\varphi(s, a_1, \ldots, a_n)$, where $\varphi(x, y_1, \ldots, y_n)$ is

$$(\forall\, t)\ t \in x \iff t = y_1 \vee t = y_2 \vee \cdots \vee t = y_n.$$

Applying (a), we see that $s \in M$.

(c) By induction and by (a), we see that all natural numbers are in M. Also by (a), the set of natural numbers ω is an element of M, being uniquely defined as the minimal infinite ordinal. Notice that $H(\theta)$ satisfies "there exists a surjection from ω onto S". By elementarity, there exists $f \in M$ such that M satisfies "f is a surjection from ω onto S". Again using (a), we see that $f(n) \in M$ for each $n \in \omega$. Finally, it suffices to observe that f is indeed a surjection, i.e. for every $x \in S$ there is n such that $x = f(n)$. This follows from elementarity, because assuming $f[\omega] \neq S$, the formula "$(\exists\ x \in S)(\forall\ n \in \omega)\quad x \neq f(n)$" would be satisfied in M, contradicting that f is a surjection.

Fix a Banach space X and choose a cardinal θ, so that $X \in H(\theta)$. Take an elementary substructure M of $\langle H(\theta), \in \rangle$ such that $X \in M$. Note that M may be countable, by the

Löwenheim-Skolem Theorem. What can we say about the set $X\cap M$? In order to answer this question, we need to say more precisely what we mean by saying "a Banach space X is an element of M". Traditionally, we have in mind structure of the form $\langle X,+,\ \cdot\ ,\|\cdot\|\rangle$, although we omit predicates $+,\ \cdot$ and the norm symbol $\|\cdot\|$, assuming implicitely that they are fixed. Thus, saying "$X\in M$" we really mean "$\langle X,+,\ \cdot\ ,\|\cdot\|\rangle\in M$". By this reason, we conclude that $X\cap M$ is closed under addition, because given $u,v\in X\cap M$, the vector $u+v$ is uniquely defined and parameters $u,v,+$ are elements of M. Similarly, $t\ \cdot\ u\in X\cap M$ whenever $t\in\mathbb{R}\cap M$ and $u\in X\cap M$. Notice that, by Proposition 2(a), the field of rationals is contained in M, therefore $X\cap M$ is a $\mathbb{Q}$-linear subspace of X. More precisely, notice that $\mathbb{R}\in M$ as a uniquely determined object; hence $\mathbb{R}\cap M$ is a subfield of $\mathbb{R}$ and $X\cap M$ is an $\mathbb{R}\cap M$-linear space. Consequently, the norm closure of $X\cap M$ is a Banach subspace of X. In particular, the weak closure of $X\cap M$ equals the norm closure of $X\cap M$. We shall write X_M instead of $\operatorname{cl}(X\cap M)$ and we shall call X_M the *subspace induced by* M.

In case of some typical Banach spaces, we can describe the subspace X_M. For instance, let $X=\ell_p(\Gamma)$, where $1\leqslant p<\infty$ and Γ is an uncountable set. Then X_M can be identified with $\ell_p(\Gamma\cap M)$. Indeed, identify $x\in\ell_p(\Gamma\cap M)$ with its extension $x'\in\ell_p(\Gamma)$ defined by $x'(\alpha)=0$ for $\alpha\in\Gamma\setminus M$. Let $x\in X\cap M$. Then $\operatorname{suppt}(x)=\{\alpha\in\Gamma\colon x(\alpha)\neq 0\}$ is a countable set and hence, by elementarity, it belongs to M. By Proposition 2(c), $\operatorname{suppt}(x)\subseteq M$. Thus $x\in\ell_p(\Gamma\cap M)$. On the other hand, if $x\in\ell_p(\Gamma\cap M)$ then arbitrarily close to x we can find $y\in\ell_p(\Gamma\cap M)$ such that $s=\operatorname{suppt}(y)$ is finite. Moreover, we may assume that $y(\alpha)\in\mathbb{Q}$ for $\alpha\in s$. By Proposition 2(b), $y\restriction s\in M$ and consequently also $y\in M$. Hence $x\in\operatorname{cl}(X\cap M)=X_M$.

Given a compact space $K\in H(\theta)$ and $M\preceq\langle H(\theta),\in\rangle$, define the following equivalence relation $\sim_M$ on K:

$$x\sim_M y\iff(\forall\ f\in\mathcal{C}(K)\cap M)\ f(x)=f(y).$$

We shall write K/M instead of $K/{\sim_M}$ and we shall denote by q^M (or, more precisely, q^M_K) the canonical quotient map. It is not hard to check that K/M is a compact Hausdorff space of weight not exceeding the cardinality of M. This construction has been used by Bandlow [4,5] for characterizing Corson compacta in terms of elementary substructures.

Lemma 3 *Let K be a compact space, let θ be a big enough cardinal and let $M\preceq\langle H(\theta),\in\rangle$ be such that $K\in M$. Then*

$$\operatorname{cl}(\mathcal{C}(K)\cap M)=\{\varphi\circ q^M\colon\varphi\in\mathcal{C}(K/M)\},$$

where cl *denotes the norm closure in the above formula.*

PROOF. Let Y denote the set on the right-hand side. Then Y is a closed linear subspace of $\mathcal{C}(K)$. Given $\psi \in \mathcal{C}(K) \cap M$, by the definition of $\sim_M$, there exists a (necessarily continuous) function ψ' such that $\psi = \psi' \circ q^M$. Thus $\mathcal{C}(K) \cap M \subseteq Y$. Let $R = \{\varphi \in \mathcal{C}(K/M) : \varphi \circ q^M \in M\}$. Then R is a subring of $\mathcal{C}(K/M)$ which separates points and which contains all rational constants. By the Stone-Weierstrass Theorem, R is dense in $\mathcal{C}(K/M)$, which implies that $\mathcal{C}(K) \cap M$ is dense in Y.

Observe that, under the assumptions of the above Lemma, the norm closure of $\mathcal{C}(K) \cap M$ is pointwise closed. Indeed, if $f \in \mathcal{C}(K) \setminus \mathrm{cl}(K \cap M)$ then there are $x, y \in K$ such that $x \sim_M y$ while $f(x) \neq f(y)$. Consequently, $V = \{g \colon g(x) \neq g(y)\}$ is a neighborhood of f in the pointwise convergence topology, disjoint from $\mathrm{cl}(K \cap M)$.

3.1 Projections induced by elementary substructures

Next we show how to use elementary submodels for constructing bounded projections. This idea has already been applied, in case of WCG spaces, by Koszmider [22].

Lemma 4 *Assume X is a Banach space, $D \subseteq X^*$ is r-norming and M is an elementary substructure of a big enough $\langle H(\theta), \in \rangle$ such that $X, D \in M$. Then*

(a) $X_M \cap {}^{\perp}(D \cap M) = \{0\}$;

(b) the canonical projection $P \colon X_M \oplus {}^{\perp}(D \cap M) \to X_M$ has norm $\leqslant r$.

PROOF. Fix $x \in X \cap M$, $y \in {}^{\perp}(D \cap M)$ and fix $\varepsilon > 0$. Since D is r-norming, there exists $d \in D$ such that $r|d(x)|/\|d\| \geqslant \|x\| - \varepsilon$. Since $x \in M$, by elementarity we may assume that $d \in M$. Thus $d \in D \cap M$ and $d(y) = 0$. It follows that

$$\|x\| \leqslant r|d(x)|/\|d\| + \varepsilon = r|d(x+y)|/\|d\| + \varepsilon \leqslant r\|x+y\| + \varepsilon.$$

By continuity, we see that $\|x\| \leqslant r\|x+y\|$ whenever $x \in X_M$ and $y \in {}^{\perp}(D \cap M)$. In particular, $X_M \cap {}^{\perp}(D \cap M) = \{0\}$, because if $x \in X_M \cap {}^{\perp}(D \cap M)$ then $-x \in {}^{\perp}(D \cap M)$ and $\|x\| \leqslant r\|x - x\| = 0$.

Note that, in the above lemma, the subspace $X_M \oplus {}^{\perp}(D \cap M)$ is closed in X.

It may happen that ${}^{\perp}(D \cap M) = 0$ (consider $X = \ell_\infty$) and in that case the above statement is meaningless. We are going to discuss Banach spaces for which Lemma 4 provides a way to construct full projections.

We demonstrate the use of elementary submodels for finding projections in weakly compactly generated spaces. Recall that a Banach space is *weakly compactly generated* (briefly: *WCG*) if it contains a linearly dense weakly compact set.

Proposition 5 *Let X be a weakly compactly generated Banach space and let θ be a big enough cardinal. Further, let $M \preceq \langle H(\theta), \in\rangle$ be such that $X \in M$. Then there exists a norm one projection $P_M \colon X \to X_M$ such that* $\ker(P_M) = {}^{\perp}(X^* \cap M)$.

PROOF. Let K be a linearly dense weakly compact subset of X. By Lemma 4, it suffices to check that $X_M \cup {}^{\perp}(X^* \cap M)$ is linearly dense in X.

Suppose $\varphi \in X^* \setminus \{0\}$ is such that $(X \cap M) \subseteq \ker(\varphi)$ and ${}^{\perp}(X^* \cap M) \subseteq \ker(\varphi)$. The latter inclusion implies that $\varphi \in \mathrm{cl}_*(X^* \cap M)$, because $X^* \cap M$ is $\mathbb{Q}$-linear. Fix $p \in K$ such that $\varphi(p) \neq 0$. Let $U_0, U_1 \subseteq \mathbb{R}$ be disjoint open rational intervals such that $0 \in U_0$ and $\varphi(p) \in U_1$. Let K_0 be the weak closure of $K \cap M$. Note that $\varphi \restriction K_0 = 0$, because φ is weakly continuous. Using the fact that $\varphi \in \mathrm{cl}_*(X^* \cap M)$, for each $x \in K_0$ choose $\psi_x \in X^* \cap M$ such that $\psi_x(x) \in U_0$ and $\psi_x(p) \in U_1$. By compactness, there are $x_0, x_1, \ldots, x_{n-1} \in K_0$ such that

$$(*) \qquad K_0 \subseteq \bigcup_{i<n} \psi_{x_i}^{-1}[U_0].$$

Note that $U_0, U_1 \in M$. Let $\Psi = \{\psi_{x_i} \colon i < n\}$. By Proposition 2(b), also $\Psi \in M$. Further, $p \in K$ witnesses the validity of

$$(\exists\, x \in K)(\forall\, \psi \in \Psi)\ \ \psi(x) \in U_1.$$

All parameters in the above formula are in M, therefore by elementarity there exists $x \in K \cap M \subseteq K_0$ such that $\psi_{x_i}(x) \in U_1$ for each $i < n$. This contradicts $(*)$.

Fix a Banach space X and a norming set $D \subseteq X^*$. We shall say that D *generates projections in* X if there exists θ_0 such that for every cardinal $\theta \geqslant \theta_0$, for every countable elementary substructure $M \preceq \langle H(\theta), \in\rangle$ with $D \in M$ it holds that

$$X = X_M \oplus {}^{\perp}(D \cap M).$$

Note that if $D \in M$ then also $X \in M$, being the common domain of all functionals from D. We shall say that the projection $P \colon X \to X$ such that $\operatorname{im} P = X_M$ and $\ker P = {}^{\perp}(D \cap M)$ is *induced by* the triple $\langle X, D, M\rangle$ (X is actually uniquely determined by D, but it is convenient to emphasize it). Proposition 5 says that X^* generates projections in X whenever X is WCG. The class of Banach spaces X with the property that X^* generates projections in X is well known: these are precisely weakly Lindelöf determined spaces. Below we prove the easier implication.

Proposition 6 *Let $\langle X, D\rangle$ be a Plichko pair. Then D generates projections in X.*

A similar statement was proved by Koszmider [22, Lemma 4.1], using the existence of a countably 1-norming Markushevich basis.

PROOF. Fix $M \preceq \langle H(\theta), \in\rangle$ so that $D \in M$. Suppose $\varphi \in X^*$ is such that $X \cap M \subseteq \ker \varphi$ and ${}^{\perp}(D \cap M) \subseteq \ker \varphi$. The latter fact means that $\varphi \in \mathrm{cl}_*(D \cap M)$. Let $G \subseteq X$ be a linearly dense set such that $\mathrm{suppt}(y, G)$ is countable for each $y \in D$. By elementarity, we may assume that $G \in M$. Suppose $\varphi \neq 0$ and fix $u \in G$ such that $\varphi(u) \neq 0$. Since φ is in the *weak** closure of $D \cap M$, we may find $\psi \in D \cap M$ such that $\psi(u) \neq 0$. Thus $u \in \mathrm{suppt}(\psi, G)$. On the other hand, $\mathrm{suppt}(\psi, G) \in M$, because $\psi, G \in M$. By Proposition 2(c), $\mathrm{suppt}(\psi, G) \subseteq M$. In particular $u \in X \cap M$ and hence $\varphi(u) = 0$, a contradiction.

The above result says in particular that X^* generates projections in X whenever X is WLD. The converse implication will be proved in Section 5.

3.3 Projectional generators

Projectional generators were introduced by Orihuela and Valdivia [32] as a tool for showing that certain Banach spaces have a PRI. In fact, first projectional generators were implicitely constructed by Lindenstrauss [28,29]. We refer to Chapter 6 of Fabián's book [10] for more information. Let us recall the definition.

Let X be a Banach space. A pair $\langle D, \Phi\rangle$ is a *projectional generator* in X if

(1) $D \subseteq X^*$ is a norming $\mathbb{Q}$-linear subspace,
(2) $\Phi \colon D \to [X]^{\leqslant \aleph_0}$,
(3) $(\bigcup \Phi[B])^{\perp} \cap \mathrm{cl}_*(B) = 0$, whenever $B \subseteq D$ is $\mathbb{Q}$-linear.

Below we show how projectional generators together with elementary submodels induce projections.

Proposition 7 *Let X be a Banach space which has a projectional generator with domain $D \subseteq X^*$. Then D generates projections in X.*

PROOF. Fix $M \preceq \langle H(\theta), \in\rangle$ with $D \in M$, where θ is big enough so that $H(\theta)$ satisfies "there exists a projectional generator on X with domain D". By elementarity, there is $\Phi \in M$ such that M satisfies "$\langle D, \Phi\rangle$ is a projectional generator on X". It suffices to check that the only $\psi \in X^*$ which vanishes on $X_M \cup {}^{\perp}(D \cap M)$ is the zero functional.

Let $B := D \cap M$. By elementarity, B is $\mathbb{Q}$-linear because D is assumed to be $\mathbb{Q}$-linear. By the definition of a projectional generator, $(\bigcup \Phi[B])^\perp \cap \mathrm{cl}_*(B) = 0$. Thus, if $\psi \in X^*$ is such that $X_M \cup {}^\perp(D \cap M) \subseteq \ker \psi$ then $\psi \in (\bigcup \Phi[B])^\perp$. This is because $\Phi(b) \subseteq M$ whenever $b \in B$ (by Proposition 2(c)). It follows that $\psi = 0$.

3.4 Bandlow's Property Ω

A result of Bandlow [5, Thm. 5.6] says that K is Corson compact iff $\mathcal{C}_p(K)$ has Property Ω, which says that given a big enough cardinal θ, for every countable $M \preceq \langle H(\theta), \in \rangle$ with $K \in M$, for every $f \in \mathcal{C}_p(K)$ there exists $g \in \mathrm{cl}(\mathcal{C}_p(K) \cap M)$ such that $f \restriction (K \cap M) = g \restriction (K \cap M)$ (in other words: $f - g \in {}^\perp(K \cap M)$). According to Bandlow's definition, cl means here the pointwise closure, however by Lemma 3 we can replace it by the norm closure.

A natural generalization of the above condition is *Property* Ω for a pair $\langle X, D \rangle$, where D is a norming set in the dual of X: given a suffciently big θ, for every countable elementary substructure M of $H(\theta)$ such that $D \in M$, for every $p \in X$ there is $q \in \mathrm{cl}(X \cap M)$ such that $p - q \in {}^\perp(D \cap M)$. Recall that in this definition we do not have to assume that $X \in M$, because it is implied by the fact that $D \in M$.

Proposition 8 *Let X be a Banach space and let $D \subseteq X^*$ be a norming set. The following statements are equivalent.*

(a) $\langle X, D \rangle$ has Property Ω.
(b) D generates projections in X.

PROOF. (a) $\Longrightarrow$ (b) Fix $M \preceq \langle H(\theta), \in \rangle$ such that $D \in M$ and fix $\varphi \in X^*$ such that $(X \cap M) \cup {}^\perp(D \cap M) \subseteq \ker \varphi$. Fix $p \in X$. Applying Property Ω, find $q \in \mathrm{cl}(X \cap M)$ such that $p - q \in {}^\perp(D \cap M)$. Then $\varphi(q) = 0$ and $\varphi(p - q) = 0$, thus also $\varphi(p) = 0$. This shows that $\varphi = 0$. Since φ was arbitrary, we get $X = \mathrm{cl}(X \cap M) \oplus {}^\perp(D \cap M)$.

(b) $\Longrightarrow$ (a) Fix $M \preceq \langle H(\theta), \in \rangle$ with $D \in M$ and fix $p \in X$. By (b), there is a projection $P \colon X \to X$ satisfying $\mathrm{im}\, P = \mathrm{cl}(X \cap M)$ and $\ker P = {}^\perp(D \cap M)$. Let $q = P(p)$. Then $p - q \in {}^\perp(D \cap M)$. This is exactly Property Ω.

Thus, in our language, Bandlow's result says that K is Corson compact if and only if K generates projections in $\mathcal{C}(K)$, where K is naturally identified with a suitable subset of $\mathcal{C}(K)^*$. Recall that consistently there exists a Corson compact K for which $\mathcal{C}(K)$ is not WLD, see [2].

4 Projectional skeletons

In this section we define the crucial notion of this work. Recall that a partially ordered set Γ is *directed* if for every $s_0, s_1 \in \Gamma$ there is $t \in \Gamma$ such that $s_0 \leqslant t$ and $s_1 \leqslant t$. Γ is *σ-complete* if every sequence $s_0 < s_1 < \dots$ has the least upper bound in Γ. A subset A of Γ is *closed* if $\sup_{n\in\omega} s_n \in A$ whenever $\{s_n \colon n \in \omega\} \subseteq A$ is such that $s_0 < s_1 < \dots$. A set $A \subseteq \Gamma$ is *cofinal* if for every $s \in \Gamma$ there exists $t \in A$ with $s \leqslant t$.

4.1 Definition and basic properties

Let X be a Banach space. A *projectional skeleton* in X is a family $\{P_s\}_{s\in\Gamma}$ of bounded projections of X indexed by a directed partially ordered set Γ, satisfying the following conditions

(1) $X = \bigcup_{s\in\Gamma} P_s X$ and each $P_s X$ is separable.
(2) $s \leqslant t \implies P_s = P_s \circ P_t = P_t \circ P_s$.
(3) If $s_0 < s_1 < s_2 < \dots$ then $t = \sup_{n\in\omega} s_n$ exists in Γ and $P_t X = \operatorname{cl}(\bigcup_{n\in\omega} P_{s_n} X)$.

Condition (3) says in particular that the poset Γ is σ-complete. We have not assumed so far that the projections P_s are uniformly bounded. Note that for every closed cofinal set $\Gamma' \subseteq \Gamma$ the restriction $\{P_s\}_{s\in\Gamma'}$ is again a projectional skeleton in X. The notion of a projectional skeleton makes sense for non-separable Banach spaces only: in a separable Banach space X the family $\{\mathrm{id}_X\}$ is a projectional skeleton.

The next observation was obtained jointly with Ondřej Kalenda.

Proposition 9 *Let $\{P_s\}_{s\in\Gamma}$ be a projectional skeleton in a Banach space X. Then there exists a closed cofinal set $\Gamma' \subseteq \Gamma$ such that*

$$\sup_{s\in\Gamma'} \|P_s\| < +\infty.$$

PROOF. For each $n \geqslant 1$ define $G_n = \{s \in \Gamma \colon \|P_s\| \leqslant n\}$. We claim that one of these sets is cofinal in Γ. Suppose otherwise and for each $n \in \omega$ choose t_n such that $\|P_s\| > n$ whenever $t_n \leqslant s$. Using the directedness of Γ, construct a sequence $s_1 < s_2 < \dots$ such that $t_n \leqslant s_n$ for $n \in \omega$. Let $s_\infty = \sup_{n\in\omega} s_n$. Then $\|P_{s_\infty}\| = +\infty$, a contradiction.

Fix $k \geqslant 1$ such that $\Gamma' := G_k$ is cofinal in Γ. We claim that Γ' is also closed. For fix $s_0 < s_1 < \dots$ in Γ' and let $t = \sup_{n\in\omega} s_n$. We need to show that $\|P_t\| \leqslant k$.

Suppose this is not true and fix $x \in X$ with $\|x\| = 1$ and $\|P_t x\| = r > k$. Let $\varepsilon = (r-k)/2$. Using the second part of (3), find $m \in \omega$ and $y \in P_{s_m} X$ such that $\|P_t x - y\| < \varepsilon/k$. Note that $P_{s_m} = P_{s_m} \circ P_t$. Using the fact that $\|P_{s_m}\| \leqslant k$, we get

$\|P_{s_m}x\| \leqslant k$ and

$$\|y - P_{s_m}x\| = \|P_{s_m}(y - P_t x)\| \leqslant k\|y - P_t x\| < \varepsilon.$$

Thus

$$\|P_t x\| \leqslant \|P_t x - y\| + \|y - P_{s_m}x\| + \|P_{s_m}x\| < \varepsilon/k + \varepsilon + k \leqslant 2\varepsilon + k = r = \|P_t x\|.$$

This contradiction completes the proof.

By the above proposition, we shall always assume that a projectional skeleton $\{P_s\}_{s\in\Gamma}$ satisfies the condition

(4) $\sup_{s\in\Gamma}\|P_s\| < +\infty$.

We shall say that $\{P_s\}_{s\in\Gamma}$ is an *r-projectional skeleton* if it is a projectional skeleton such that $\|P_s\| \leqslant r$ for every $s \in \Gamma$. The remaining part of this section will be devoted to proving basic properties of projectional skeletons.

Lemma 10 *Let $\{P_s\}_{s\in\Gamma}$ be a projectional skeleton in X and let $s_0 < s_1 < \dots$ be such that $t = \sup_{n\in\omega} s_n$ in Γ. Then*

$$P_t x = \lim_{n\to\infty} P_{s_n}x$$

for every $x \in X$.

PROOF. Let $r = \sup_{s\in\Gamma}\|P_s\|$ and fix $x \in X$, $\varepsilon > 0$. By the second part of (3), find $y \in \bigcup_{n\in\omega} P_{s_n}X$ such that $\|P_t x - y\| < \varepsilon/(1+r)$. Choose k such that $y \in P_{s_k}X$. Note that $P_t y = y$ and $P_{s_n}y = y$ for $n \geqslant k$. Thus, given $n \geqslant k$, we have

$$\begin{aligned}\|P_t x - P_{s_n}x\| &\leqslant \|P_t x - y\| + \|y - P_{s_n}x\| < \varepsilon/(1+r) + \|P_{s_n}(y - P_t x)\| \\ &\leqslant \varepsilon/(1+r) + r\|y - P_t x\| < \varepsilon/(1+r) + r\varepsilon/(1+r) = \varepsilon.\end{aligned}$$

This shows that $\lim_{n\to\infty}\|P_t x - P_{s_n}x\| = 0$.

Lemma 11 *Let $\{P_s\}_{s\in\Gamma}$ be a projectional skeleton in X and let $T \subseteq \Gamma$ be a directed subset of Γ. Then the formula*

$$P_T x = \lim_{s\in T} P_s x$$

well defines a bounded projection of X onto $\mathrm{cl}(\bigcup_{s\in T} P_s X)$.

PROOF. It is enough to show that $\{P_s x\}_{s\in T}$ is a Cauchy net for every $x \in X$.

Suppose this is not the case and fix $x \in X$ and $\varepsilon > 0$ such that for each $s \in T$ there are $t_1, t_2 \in T$, $s \leqslant t_1$, $s \leqslant t_2$ with $\|P_{t_1}x - P_{t_2}x\| > 2\varepsilon$. As T is directed, we have that for each $s \in T$ there are $t_1, t_2 \in T$ with $s \leqslant t_1 \leqslant t_2$ and $\|P_{t_1}x - P_{t_2}x\| > \varepsilon$. Hence, by induction we can construct a sequence $t_1 \leqslant t_2 \leqslant t_3 \leqslant t_4 \leqslant \dots$ in T such that $\|P_{t_{2k-1}}x - P_{t_{2k}}x\| > \varepsilon$ for $k \in \mathbb{N}$. This contradicts Lemma 10.

4.2 Projectional resolutions of the identity

Theorem 12 *Every Banach space with* 1*-projectional skeleton has a projectional resolution of the identity.*

PROOF. Let $\kappa = \operatorname{dens} X$ and let $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ be a projectional skeleton in X such that $\|P_s\| = 1$ for every $s \in \Gamma$. Fix a continuous chain $\{T_\alpha\}_{\alpha<\kappa}$ of up-directed subsets of Γ satisfying $|T_\alpha| \leqslant \alpha + \aleph_0$ for each α and such that

$$E = \bigcup \{P_s X \colon s \in T_\alpha,\ \alpha < \kappa\}$$

is dense in X. Continuity of the chain means that $T_\delta = \bigcup_{\xi<\delta} T_\xi$ whenever δ is a limit ordinal. Let $X_\alpha = \operatorname{cl}(\bigcup_{s\in T_\alpha} P_s X)$. Then $\{X_\alpha\}_{\alpha<\kappa}$ is a chain of closed subspaces of X, the density of X_α does not exceed $|\alpha| + \aleph_0$ and $X_\delta = \operatorname{cl}(\bigcup_{\xi<\delta} X_\xi)$ for every limit ordinal δ. By Lemma 11, formula $P_\alpha x = \lim_{s\in T_\alpha} P_s x$ defines a projection of X onto X_α. Clearly, $\|P_\alpha\| = 1$, because $\|P_s\| = 1$ for $s \in \Gamma$. Fix $\alpha < \beta$. Then $T_\alpha \subseteq T_\beta$, therefore $P_\beta \circ P_\alpha x = P_\alpha$. Observe that $P_s \circ P_\beta = P_s$ for every $s \in T_\alpha$. Indeed, given $s \in T_\alpha$, the set $A = \{t \in T_\beta \colon t \geqslant s\} \subseteq T_\beta$ is cofinal in T_β, so

$$P_s P_\beta x = \lim_{t\in T_\beta} P_s P_t x = \lim_{t\in A} P_s P_t x = P_s x.$$

It follows that $P_\alpha P_\beta x = \lim_{s\in T_\alpha} P_s P_\beta x = \lim_{s\in T_\alpha} P_s x = P_\alpha x$. Thus, $\{P_\alpha\}_{\alpha<\kappa}$ is a PRI on X.

Corollary 13 *Given a Banach space X of density $\aleph_1$, the following properties are equivalent.*

(a) X has a 1*-projectional skeleton.*
(b) X has a projectional resolution of the identity.

PROOF. Implication (a) $\Longrightarrow$ (b) follows from the above theorem. In case of density $\aleph_1$, every PRI is a 1-projectional skeleton.

4.3 Norming space induced by a projectional skeleton

We shall now look at the dual of a space with a projectional skeleton. Let $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ be a projectional skeleton in a Banach space X. The set

$$D = \bigcup_{s\in\Gamma} P_s^* X^*$$

is clearly a linear subspace of X. Notice that $P_s^* X^* \cap \overline{\mathrm{B}}_{X^*}$ endowed with the *weak** topology is second countable, because $P_s^* X^*$ is linearly homeomorphic to the dual of $P_s X$. Let $r = \sup_{s\in\Gamma} \|P_s\|$. Given $x \in \mathrm{S}_X$, there is $s \in \Gamma$ such that $x = P_s x$; choose $\varphi \in X^*$ such that $\varphi(x) = 1 = \|\varphi\|$. Then $(P_s^*\varphi)x = \varphi(P_s x) = \varphi(x) = 1$ and $\|P_s^*\varphi\| \leqslant r$. This shows that the space D is r-norming. We shall say that D is the dual norming subspace *induced by* $\mathfrak{s}$ and we shall denote it by $D(\mathfrak{s})$.

Let $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ and $\mathfrak{t} = \{Q_t\}_{t\in\Pi}$ be projectional skeletons in the same Banach space X. We say that $\mathfrak{s}$ and $\mathfrak{t}$ are *equivalent* if they induce the same norming subspace, i.e. $\bigcup_{s\in\Gamma} \operatorname{im} P_s^* = \bigcup_{t\in\Pi} \operatorname{im} Q_t^*$. It turns out that, with help of elementary submodels, a projectional skeleton can be recovered (up to equivalence) from the norming space.

Lemma 14 *Let $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ be a projectional skeleton in a Banach space X and let $D \subseteq D(\mathfrak{s})$ be norming for X. Further, let θ be a big enough regular cardinal and let $M \preceq \langle H(\theta), \in\rangle$ be countable and such that $\mathfrak{s} \in M$. Then the projection induced by $\langle X, D, M\rangle$ equals P_t, where $t = \sup(\Gamma \cap M)$.*

PROOF. First notice that, by elementarity, the set $\Gamma \cap M$ is directed. Indeed, if $t_0, t_1 \in \Gamma \cap M$ then M satisfies the formula "$(\exists\ s \in \Gamma)\ t_0 \leqslant s \wedge t_1 \leqslant s$", therefore there is $t \in M$ such that $t \in \Gamma$ and $t_0 \leqslant t$, $t_1 \leqslant t$. Since M is countable, the supremum of $\Gamma \cap M$ indeed exists. Now observe that

$$\bigcup_{s\in\Gamma\cap M} \operatorname{im} P_s \subseteq \operatorname{cl}(X \cap M).$$

This is because, given $s \in \Gamma \cap M$, by elementarity there exists a countable set $A \in M$ which is dense in $\operatorname{im} P_s$. By Proposition 2(c), $A \subseteq X \cap M$, therefore $\operatorname{im} P_s \subseteq \operatorname{cl}(X \cap M)$. It follows that $\operatorname{im} P_t = \operatorname{cl}(\bigcup_{s\in\Gamma\cap M} \operatorname{im} P_s) \subseteq \operatorname{cl}(X \cap M)$. On the other hand, given $x \in X \cap M$, by elementarity there is $s \in \Gamma \cap M$ such that $x \in \operatorname{im} P_s$; thus $x \in \operatorname{im} P_t$. Hence $\operatorname{im} P_t = X_M$. Notice that, again by elementarity, $P_t^*\varphi = \varphi$ whenever $\varphi \in D \cap M$. Thus $\ker P_t \subseteq {}^\perp(D \cap M)$, because if $P_t x = 0$ and $\varphi \in D \cap M$ then $\varphi(x) = (P_t^*\varphi)x = \varphi(P_t x) = 0$. It follows that $\ker P_t = {}^\perp(D \cap M)$, because $X = X_M \oplus \ker P_t$ and $X_M \cap {}^\perp(D \cap M) = \{0\}$ (Lemma 4(a)).

Theorem 15 *Let X be a Banach space and let $D \subseteq X^*$ be an r-norming set ($r \geqslant 1$). The following properties are equivalent.*
(a) X has an r-projectional skeleton $\mathfrak{s}$ such that $D \subseteq D(\mathfrak{s})$.
(b) D generates projections in X.

PROOF. Implication (a) $\Longrightarrow$ (b) follows from Lemma 14 – take θ so that $D \in H(\theta)$ and $H(\theta)$ satisfies "there exists an r-projectional skeleton $\mathfrak{s}$ such that $D \subseteq D(\mathfrak{s})$.
(b) $\Longrightarrow$ (a) Fix a big enough regular cardinal θ and let Γ be the family of all countable elementary substructures M of $\langle H(\theta), \in \rangle$ such that $D \in M$. Endow Γ with inclusion. Clearly, Γ is a σ-directed poset. Fix $M \in \Gamma$ and let P_M be the projection onto $\operatorname{cl}(X \cap M)$ with $\ker(P_M) = {}^{\perp}(D \cap M)$. By definition, P_M is defined on the entire space. Further, $\|P_M\| \leqslant r$, by Lemma 4. Given $M \subseteq N$ in Γ, we have that $\operatorname{cl}(X \cap M) \subseteq \operatorname{cl}(X \cap N)$ and ${}^{\perp}(D \cap M) \supseteq {}^{\perp}(D \cap N)$. The first inclusion shows that $P_N \circ P_M = P_M$ and the latter one shows that $P_M \circ P_N = P_M$. Finally, given a sequence $M_0 \subseteq M_1 \subseteq \dots$ in Γ, the union $M = \bigcup_{n\in\omega} M_n$ is an elementary substructure of $H(\theta)$ such that $\operatorname{cl}(X \cap M)$ is the closure of $\bigcup_{n\in\omega} \operatorname{cl}(X \cap M_n)$. It follows that $\mathfrak{s} = \{P_M\}_{M\in\Gamma}$ is a projectional skeleton in X. It is clear that $D \subseteq D(\mathfrak{s})$, because $D \cap M \subseteq \operatorname{im} P_M^*$.

Corollary 16 *Let X be a Banach space with a projectional skeleton. Then there exists a renorming of X under which X has an equivalent 1-projectional skeleton.*

PROOF. Let D be the dual norming subspace induced by a fixed projectional skeleton in X. By Lemma 14, D generates projections in X. Consider a renorming of X after which D becomes 1-norming (see Proposition 1). By Theorem 15, X has a 1-projectional skeleton.

As an application of Theorem 15, we prove that the class of Banach spaces with a 1-projectional skeleton is stable under arbitrary c_0- and ℓ_p-sums.
Theorem 17 *Let $\{X_\alpha\}_{\alpha<\kappa}$ be a collection of Banach spaces and let $X = \bigoplus_{\alpha<\kappa} X_\alpha$ be endowed either with the c_0-norm or with ℓ_p-norm ($1 \leqslant p < \infty$). Further, assume that for each $\alpha < \kappa$, $D_\alpha \subseteq X_\alpha$ is 1-norming and generates projections in X_α. Then the set*

$$D = \{\varphi \in X^* \colon (\forall\, \alpha)\ \varphi \restriction X_\alpha \in D_\alpha \ \textit{ and } \ |\{\alpha \colon \varphi \restriction X_\alpha \neq 0\}| \leqslant \aleph_0\}$$

is 1-norming and generates projections in X.

PROOF. The fact that D is 1-norming follows from the properties of the c_0-sum and the ℓ_p-sum. Define $\operatorname{suppt}(\varphi) = \{\alpha \colon \varphi \restriction X_\alpha \neq 0\}$. Fix $M \preceq \langle H(\theta), \in \rangle$ with a big enough θ, so that $D \in M$. Let $S = \kappa \cap M$. Note that $\operatorname{suppt}(\varphi) \subseteq S$ whenever

$\varphi \in D \cap M$. Suppose $X \neq X_M \oplus {}^{\perp}(D \cap M)$ and fix $\psi \in X^*$ satisfying $X \cap M \subseteq \ker \psi$ and ${}^{\perp}(D \cap M) \subseteq \ker \psi$. Then ψ is in the *weak** closure of the linear hull of $D \cap M$, therefore $\operatorname{suppt}(\psi) \subseteq S$. Assuming $\psi \neq 0$, there is $\alpha \in S$ such that $\psi_\alpha := \psi \restriction X_\alpha \neq 0$. Note that $X_\alpha \cap M \subseteq \ker \psi_\alpha$. If $x \in {}^{\perp}(D_\alpha \cap M)$ and $\varphi \in D \cap M$ then $\varphi \restriction X_\alpha \in D_\alpha \cap M$ so $\varphi(x) = 0$. It follows that ${}^{\perp}(D \cap M) \cap X_\alpha = {}^{\perp}(D_\alpha \cap M)$. Thus ${}^{\perp}(D_\alpha \cap M) \subseteq \ker \psi$. On the other hand, $X_\alpha = \operatorname{cl}(X_\alpha \cap M) \oplus {}^{\perp}(D_\alpha \cap M)$, because $D_\alpha \in M$. This is a contradiction.

We finish this section by exhibiting a topological property of norming spaces induced by a projectional skeleton.

Theorem 18 *Let X be a Banach space with a projectional skeleton $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ and let $D \subseteq X^*$ be the norming space induced by $\mathfrak{s}$, endowed with the weak* topology. Then:*

(a) The closure in X^* of every countable bounded subset of D is metrizable and contained in D.

(b) D is countably tight.

PROOF. Part (a) is trivial: every countable subset of D is contained in $Y_s = P_s^* X^*$ for some $s \in \Gamma$ and every bounded subset of Y_s with the *weak** topology is second countable.

(b) Let $A \subseteq D$ and $p \in \operatorname{cl}_*(A) \cap D$ be given. Replacing A by $A - p$, we may assume that $p = 0$. Fix a big enough regular cardinal θ and a countable elementary substructure M of $\langle H(\theta), \in \rangle$ such that $X, \mathfrak{s}, A \in M$. We claim that $0 \in \operatorname{cl}_*(A \cap M)$.

Let $t = \sup(\Gamma \cap M)$ and let $Y_t = P_t^* X^*$. Fix a *weak** neighborhood U of p. We may assume that $U = \bigcap_{i<n} U(x_i, \varepsilon)$, where $U(x, \varepsilon) := \{y \in X^* \colon |y(x)| < \varepsilon\}$, $x_0, \dots, x_{n-1} \in X$ and $\varepsilon > 0$ is rational. By Lemma 14, $P_t X = \operatorname{cl}(X \cap M)$. By Banach's Open Mapping Principle, ${P_t}^{-1}[X \cap M]$ is dense in X. Hence, without loss of generality, we may assume that $P_t x_i \in M$ for each $i < n$.

Thus $W := \bigcap_{i<n} U(P_t x_i, \varepsilon)$ is a *weak** neighborhood of 0 and $W \in M$, because $P_t x_i \in M$ and $\varepsilon \in M$. By elementarity, there is $a \in A \cap M \cap W$. It follows that $a \in Y_t$, i.e. $P_t^* a = a$. Given $i < n$, we have $a(x_i) = (P_t^* a) x_i = a(P_t x_i)$. Thus $a \in U(x_i, \varepsilon)$. Finally, $a \in A \cap M \cap U$.

Corollary 19 *Let X be a Banach space and let $D, E \subseteq X^*$ be norming spaces induced by projectional skeletons. If $D \cap E$ is total then $D = E$.*

PROOF. Note that $D \cap E$ is a linear space. Assuming it is total, it must be *weak** dense. Thus, given $p \in D$, we have that $p \in \operatorname{cl}_*(D \cap E)$ so $p \in \operatorname{cl}_*(A)$ for some countable

$A \subseteq D \cap E$ (Theorem 18(b)). By Theorem 18(a), $\mathrm{cl}_*(A) \subseteq D \cap E$. This shows that $D \subseteq E$. By symmetry, $D = E$.

Corollary 20 *Let X be a Banach space and let $S \subseteq X^*$ generate projections in X. Further, let D be the smallest linear subspace of X^* containing S and such that $\mathrm{cl}_*(A) \subseteq D$ for every countable set $A \subseteq D$. Then D is induced by a projectional skeleton in X.*

PROOF. By Theorem 15, there exists a projectional skeleton $\mathfrak{s}$ in X such that $S \subseteq D(\mathfrak{s})$. By Theorem 18(a), $D \subseteq D(\mathfrak{s})$. On the other hand, D is *weak** dense, because it is a norming (and hence total) linear space. Hence, given $p \in D(\mathfrak{s})$, we have that $p \in \mathrm{cl}_*(D)$ so, by Theorem 18(b), $p \in \mathrm{cl}_*(A)$ for some countable $A \subseteq D$. Hence $p \in D$. This shows that $D = D(\mathfrak{s})$.

The results of this section show that Banach spaces with a projectional skeleton have very similar properties to Plichko spaces. In fact, we know only one basic example distinguishing those two classes: the space $\mathcal{C}(\omega_2 + 1)$ which, by the result of Kalenda [19], does not have any countably norming Markushevich basis. We shall come back to this example in Section 7.

5 Plichko spaces and projectional skeletons

We prove a preservation theorem for projectional sequences of Plichko spaces. As an application, we show that every Banach space with a commutative projectional skeleton is Plichko. A projectional skeleton $\{P_s\}_{s\in\Gamma}$ is *commutative* if $P_s \circ P_t = P_t \circ P_s$ holds for every $s, t \in \Gamma$.

Proposition 21 *Let X be a Plichko space with a Σ-space $D \subseteq X^*$. Then there exists a commutative projectional skeleton $\{P_s\}_{s\in\Gamma}$ on X such that $D = D(\mathfrak{s})$.*

PROOF. Let $G \subseteq X$ be a linearly dense set witnessing that D is a Σ-space. Fix a big enough regular cardinal θ and let Γ be the family of all countable elementary substructures M of $\langle H(\theta), \in\rangle$ such that $X, D, G \in M$. By Proposition 6 and the proof of Theorem 15, we know that $\mathfrak{s} = \{P_M\}_{M\in\Gamma}$ is a projectional skeleton in X such that $D \subseteq D(\mathfrak{s})$, where P_M is induced by $\langle X, D, M\rangle$, i.e. $\operatorname{im} P_M = X_M$ and $\ker P_M = {}^{\perp}(D \cap M)$. Theorem 18(b) says that $D(\mathfrak{s})$ is *weak** countably tight. On the other hand, D is *weak** countably closed, i.e. $\mathrm{cl}_* A \subseteq D$ whenever $A \subseteq D$ is countable. Thus $D = D(\mathfrak{s})$.

It remains to show that $\mathfrak{s}$ is commutative. Given $M \in \Gamma$, define $r_M = P_M \restriction G$. We claim that

$$(*) \qquad r_M(x) = \begin{cases} x, & \text{if } x \in G \cap M, \\ 0, & \text{if } x \in G \setminus M. \end{cases}$$

Indeed, $G \cap M \subseteq \operatorname{cl}(X \cap M) = \operatorname{im} P_M$. If $x \in G \setminus M$ then for $y \in D \cap M$ we have that $x \notin \operatorname{suppt}(y, G)$, because $\operatorname{suppt}(y, G) \subseteq M$. Hence $y(x) = 0$ for $y \in D \cap M$ and therefore $x \in {}^{\perp}(D \cap M) = \ker P_M$.

Using (*), we see that $r_M \circ r_N = r_{M \cap N}$ for $M, N \in \Gamma$. Since G is linearly dense in X, this shows that $P_M \circ P_N = P_{M \cap N} = P_N \circ P_M$ for every $M, N \in \Gamma$. This completes the proof.

5.1 Preservation theorem

Lemma 22 *Let $\langle X, D\rangle$ be a Plichko pair and let $P \colon X \to X$ be a bounded projection such that $P^* D \subseteq D$. Then $\langle \ker P, D \cap \ker P^* \rangle$ is a Plichko pair.*

PROOF. Let A be a linearly dense subset of X such that $|\operatorname{suppt}(y, A)| \leqslant \aleph_0$ for every $y \in D$. Let $B = \{a - Pa \colon a \in A\}$. Then B is linearly dense in $\ker P = \operatorname{im}(\mathrm{id}_X - P)$. Let $E = D \cap \ker P^*$. Fix $y \in D$ with $\|y\| = 1$. Let $z = y - P^* y$. Then $z \in D$, because P^* preserves D. Further, $z \in \ker P^*$ and $\|z\| \leqslant 1 + \|P\|$. Given $x \in \ker P$, we have $|z(x)| = |y(x) - (P^*y)x| = |y(x) - y(Px)| = |y(x)|$. This shows that E is norming for $\ker P$. Finally, given $y \in E$, we have $y(a - Pa) = y(a) - y(Pa) = y(a) - (P^*y)a = y(a)$, so $\operatorname{suppt}(y, B) = \operatorname{suppt}(y, A)$. Thus, B witnesses that $\langle \ker P, E\rangle$ is a Plichko pair.

Theorem 23 *Let $\{P_\alpha\}_{\alpha<\kappa}$ be a projectional sequence in a Banach space X and let $D \subseteq X^*$ be a norming space such that*

$$D = \bigcup_{\alpha<\kappa} P_\alpha^* D$$

and $\langle P_\alpha X, P_\alpha^* D\rangle$ is a Plichko pair for each $\alpha < \kappa$. Then $\langle X, D\rangle$ is a Plichko pair.

Note that we do not assume that the above projections are uniformly bounded.

PROOF. We construct inductively a family of sets $\{A_\alpha\}_{\alpha<\kappa}$ such that

(i) A_α is a linearly dense subset of $P_\alpha X$;
(ii) $\alpha < \beta \implies A_\alpha \subseteq A_\beta$;
(iii) $\operatorname{suppt}(y, A_\alpha)$ is countable for every $y \in D$;

(iv) $P_\alpha a = 0$ whenever $a \in A_\beta \setminus A_\alpha$.

We start with a linearly dense set $A_0 \subseteq P_0 X$ witnessing that $\langle P_0 X, P_0^* D\rangle$ is a Plichko pair. We must check (iii). Observe that $a = P_0 a$ and consequently $y(a) = (P_0^* y)a$ for every $a \in A_0$ and $y \in X^*$. Hence $\operatorname{suppt}(y, A_0) = \operatorname{suppt}(P_0^* y, A_0)$ is countable and (iii) holds. Now, fix an ordinal $\delta > 0$ and assume $\{A_\alpha\}_{\alpha<\delta}$ has already been defined.

Suppose first that $\delta = \beta + 1$. Let $Y = P_\delta X$ and let $D' = \{y \restriction Y : y \in D\} \subseteq Y^*$. Then $\langle Y, D'\rangle$ is a Plichko pair and $P_\beta \restriction Y$ is a projection whose dual preserves D'. By Lemma 22, there is a linearly dense subset B of $\ker P_\beta \cap Y$ witnessing that $\langle \ker P_\beta \cap Y, E\rangle$ is a Plichko pair, where $E = D' \cap \ker[(P_\beta \restriction Y)^*]$. It follows that $\operatorname{suppt}(y, B)$ is countable whenever $y \in D \cap \ker P_\beta^*$. Define $A_\delta = A_\beta \cup B$. Conditions (i), (ii) and (iv) are obviously satisfied. It remains to check (iii). Fix $y \in D$. By the induction hypothesis, $\operatorname{suppt}(y, A_\beta)$ is countable. Let $z = y - P_\beta^* y$. Then $z \in D \cap \ker P_\beta$, so $\operatorname{suppt}(z, B)$ is countable. Finally, given $b \in B$ we have

$$z(b) = y(b) - (P_\beta^* y)b = y(b) - y(P_\beta b) = y(b),$$

therefore $\operatorname{suppt}(y, B) = \operatorname{suppt}(z, B)$ is countable. This shows (iii), because $\operatorname{suppt}(y, A_\delta)$ $= \operatorname{suppt}(y, A_\beta) \cup \operatorname{suppt}(y, B)$.

Suppose now that δ is a limit ordinal. Define $A_\delta = \bigcup_{\xi<\delta} A_\xi$. Clearly, conditions (i) and (ii) are satisfied. Condition (iv) is obvious, so it remains to check (iii). Fix $y \in D$. There is nothing to prove if δ has a countable cofinality, because then $\operatorname{suppt}(y, A_\delta) = \bigcup_{n\in\omega} \operatorname{suppt}(y, A_{\xi_n})$, where $\{\xi_n\}_{n\in\omega}$ is a cofinal sequence in δ. Assume $\operatorname{cf}\delta > \aleph_0$. Since $A_\delta \subseteq P_\delta X$, we see that $\operatorname{suppt}(y, A_\delta) = \operatorname{suppt}(P_\delta^* y, A_\delta)$. Thus, we may assume that $y = P_\delta^* y$. Further, $P_\delta^* D$ is contained in a Σ-space, therefore it is *weak** countably tight. On the other hand, $\bigcup_{\xi<\delta} P_\xi^* D$ is *weak** dense in $P_\delta^* X^*$ and hence, since δ has uncountable cofinality, there exists $\alpha < \delta$ such that $y \in P_\alpha^* D$. It follows that $P_\alpha^* y = y$. In particular, $\operatorname{suppt}(y, A_\delta) = \operatorname{suppt}(y, A_\alpha)$, because if $a \in A_\delta \setminus A_\alpha$ then $y(a) = (P_\alpha^* y)a = y(P_\alpha a) = 0$, by (iv). By the induction hypothesis, $\operatorname{suppt}(y, A_\delta)$ is countable. This shows (iii).

Finally, set $A = \bigcup_{\alpha<\kappa} A_\alpha$. By (i), A is linearly dense in X. It remains to check that $\operatorname{suppt}(y, A)$ is countable for every $y \in D$. Fix $y \in D$. By the assumption, $y \in P_\alpha^* D$ for some $\alpha < \kappa$. Thus $P_\xi^* y = y$ whenever $\alpha \leqslant \xi < \kappa$ and we conclude like in the limit case of the above construction. This completes the proof.

The above theorem should be compared with the results of S. Gul′ko (see [12,13,14]), where similar preservation was proved for topological spaces which have a continuous injection into a Σ-product.

Corollary 24 *Let X be a Banach space with a projectional sequence $\{P_\alpha\}_{\alpha<\kappa}$ such that $P_\alpha X$ is weakly Lindelöf determined for each $\alpha < \kappa$. Then X is a Plichko space.*

Corollary 25 *Given a Banach space X, the following properties are equivalent.*

(a) X^* generates projections in X.
(b) X is weakly Lindelöf determined.

By a result of Orihuela, Schachermayer and Valdivia [31], the above properties are also equivalent to "$\langle \overline{\mathrm{B}}_{X^*}, weak^* \rangle$ is Corson compact".

It is natural to ask when X (as a subspace of X^{**}) generates projections in X^*. By a result of Fabián and Godefroy [9] this is the case when X is Asplund. Specifically, assuming X is an Asplund space, the authors of [9] construct a projectional generator $\langle X, \Phi \rangle$ in X^*. On the other hand, Orihuela and Valdivia noted in [32, Thm. 3] that the existence of a projectional generator with domain X and with values in X^* implies that X is Asplund. Recall that a Banach space X is *Asplund* if the dual of every separable subspace of X is separable. Assume X generates projections in X^* and fix a separable subspace Y of X. Fix a countable $M \preceq H(\theta)$ such that $X \in M$ and $Y \cap M$ is dense in Y and let $P\colon X^* \to X^*$ be the projection with $\operatorname{im} P = \operatorname{cl}(X^* \cap M)$ and $\ker P = (X \cap M)^{\perp}$. Then $P^* y = y$ for every $y \in Y$, because $Y \subseteq \operatorname{cl}_*(X \cap M)$. Fix $\varphi \in Y^*$ and let $\psi \in X^*$ be an extension of φ. Then $(P\psi)y = \psi(P^* y) = \psi(y) = \varphi(y)$ for every $y \in Y$. Thus, $\varphi = (P\psi) \restriction Y$. It follows that Y^* is separable because so is $\operatorname{im} P$. Summarizing, we have:

Proposition 26 *Given a Banach space X, the following properties are equivalent.*
(a) X is Asplund.
(b) X generates projections in X^*.

5.2 A characterization of Plichko spaces

Theorem 27 *Let X be a Banach space and let $r \geqslant 1$. The following properties are equivalent.*
(a) X has a commutative r-projectional skeleton.
(b) X is an r-Plichko space.

PROOF. Implication (b) $\implies$ (a) is contained in Proposition 21. For the converse implication, we use Theorem 12, Lemma 11 and induction on the density of X. Suppose we have proved that (a) $\implies$ (b) for spaces of density $< \kappa$ and fix a Banach space X of density κ with a commutative r-projectional skeleton $\{P_s\}_{s \in \Gamma}$. By (the proof of) Theorem 12, there exists an r-projectional resolution of the identity $\mathfrak{s} = \{P_\alpha\}_{\alpha < \kappa}$ on X such that for each $\alpha < \kappa$ there is a directed set $S_\alpha \subseteq \Gamma$ with $P_\alpha x = \lim_{s \in S_\alpha} P_s x$ for $x \in X$ (to be formal, we need to assume that $\Gamma \cap \kappa = \emptyset$). Observe that, by continuity, $P_s \circ P_\alpha = P_\alpha \circ P_s$ holds for every $s \in \Gamma$ and $\alpha < \kappa$. Let D be the norming space induced by $\mathfrak{s}$. Fix $y \in D$ and fix $\alpha < \kappa$. Let $s \in \Gamma$ be such that $y = P_s^* y$. Then $P_\alpha^* y = P_\alpha^* P_s^* y = P_s^* P_\alpha^* y \in D$. Hence $P_\alpha^* D \subseteq D$. Now use Theorem 23. In case where

$\mathrm{cf}\,\kappa = \aleph_0$, it may happen that $D \neq \bigcup_{\alpha<\kappa} P_\alpha^* D$, but we may replace D by $\bigcup_{\alpha<\kappa} P_\alpha^* D$, still having an r-norming space. By Theorem 23, $\langle X, D\rangle$ is a Plichko pair, therefore X is r-Plichko.

6 Spaces of continuous functions

In this section we shall discuss a natural class of compact spaces K for which $\mathcal{C}(K)$ has a projectional skeleton.

Let $\mathcal{R}_0$ denote the class of all compacta which have a retractional skeleton. Following [25], a *retractional skeleton* (briefly: *r-skeleton*) in a compact space K is a family of retractions $\{r_s\}_{s\in\Gamma}$, indexed by an up-directed poset Γ, satisfying the following conditions:

(1) $s \leqslant t \implies r_s = r_s \circ r_t = r_t \circ r_s$.
(2) For every $x \in X$, $x = \lim_{s\in\Gamma} r_s(x)$.
(3) $r_s[X]$ is metrizable for each $s \in \Gamma$.
(4) Given $s_0 < s_1 < \dots$ in Γ, $t = \sup_{n\in\omega} s_n$ exists and $r_t(x) = \lim_{n\to\infty} r_{s_n}(x)$ for every $x \in K$.

It has been proved in [25] that Valdivia compacta are precisely those compact spaces which have a commutative r-skeleton. The ordinal $\omega_2 + 1$ is an example of a space in class $\mathcal{R}_0$ which is not Valdivia compact. A retractional skeleton in a space of the form $\kappa + 1$, where κ is an uncountable cardinal, is described in [25, Example 6.4].

It is clear that every r-skeleton induces a 1-projectional skeleton on the space of continuous functions; that is:

Proposition 28 *Let K be a compact space with a retractional skeleton $\{r_s\}_{s\in\Gamma}$. Then $\{r_s^*\}_{s\in\Gamma}$ is a projectional skeleton in $\mathcal{C}(K)$, where r_s^* denotes the transformation adjoint to r_s.*

A simple application of Lemma 11 shows that every space from class $\mathcal{R}_0$ can be decomposed into a continuous inverse sequence of retractions onto smaller spaces in class $\mathcal{R}_0$ (notion dual to a PRI). This shows that $\mathcal{R}_0 \subseteq \mathcal{R}$, where $\mathcal{R}$ is the smallest class of spaces containing all metric compacta and closed under limits of continuous inverse sequences of retractions (see [6,23]). Note that class $\mathcal{R}_0$ restricted to spaces of weight $\leqslant \aleph_1$ coincides with the class of Valdivia compacta ([25, Cor. 4.3]). This is not the case with class $\mathcal{R}$ (see [25, Example 4.6(b)]), therefore $\mathcal{R}_0 \neq \mathcal{R}$.

Let us admit that the converse to Proposition 28 fails, namely there exist compact spaces $K \notin \mathcal{R}$ such that $\mathcal{C}(K)$ is 1-Plichko, see [3]. On the other hand, by Lemma 10, we have:

Proposition 29 *Let $D \subseteq X^*$ be a 1-norming space which generates projections in a Banach space X. Then $\overline{\mathrm{B}}_{X^*}$ endowed with the weak* topology belongs to class $\mathcal{R}_0$.*

The following results are dual to Theorems 15, 17 and 18 respectively.

Theorem 30 *Let K be a compact space and let $D \subseteq K$ be a dense countably closed set. The following properties are equivalent:*

(a) $K \in \mathcal{R}_0$ and D is induced by an r-skeleton in K.

(b) For every sufficiently big cardinal θ, for every countable elementary substructure M of $H(\theta)$ with $K, D \in M$, the quotient $q_K^M \colon K \to K/M$ restricted to $\mathrm{cl}(D \cap M)$ is one-to-one.

PROOF. Assume (a) and fix a countable $M \preceq \langle H(\theta), \in\rangle$ such that $K, D \in M$. By elementarity, there exists $\{r_s\}_{s\in\Gamma} \in M$ which is an r-skeleton in K such that $D = \bigcup_{s\in\Gamma} r_s[K]$. Fix $x, y \in \mathrm{cl}(D\cap M)$, $x \neq y$. Let $t = \sup(\Gamma \cap M)$. By elementarity, $D \cap M \subseteq r_t[K]$. Indeed, if $x \in D \cap M$ then $r_s(x) = x$ for some $s \in \Gamma \cap M$, hence $r_t(x) = x$. Thus, also $\mathrm{cl}(D\cap M) \subseteq r_t[K]$. It follows that $x = r_t(x)$ and $y = r_t(y)$. Let $\{s_n\}_{n\in\omega} \subseteq \Gamma \cap M$ be increasing and such that $t = \sup_{n\in\omega} s_n$. Then $x = \lim_{n\to\infty} r_{s_n}(x)$ and $y = \lim_{n\to\infty} r_{s_n}(y)$. It follows that $r_{s_k}(x) \neq r_{s_k}(y)$ for all but finitely many $k \in \omega$. Fix such k and let $r = r_{s_k}$. Note that $K, r \in M$ and $r[K]$ is second countable. By elementarity and by Proposition 2(c), there exists a countable family $\mathcal{F} \in M$ consisting of continuous real functions on $r[K]$ which separates the points. Choose $f \in \mathcal{F}$ so that $f(r(x)) \neq f(r(y))$. Then $f \circ r \in \mathcal{C}(K) \cap M$, which shows that $x \not\sim_M y$. Thus (a) $\implies$ (b).

The proof of (b) $\implies$ (a) is similar to that of (b) $\implies$ (a) in Theorem 15: the family Γ of all countable $M \preceq \langle H(\theta), \in\rangle$ with $K, D \in M$ is an r-skeleton on K, since each q_K^M can be treated as a retraction of K onto $\mathrm{cl}(K \cap M)$.

Note that countably tight spaces in class $\mathcal{R}_0$ are precisely Corson compacta. Thus, in case K is Corson compact, we have $D = K$ and the above theorem gives Bandlow's characterization [4].

Proposition 31 *Class $\mathcal{R}_0$ is closed under arbitrary products.*

PROOF. Let $\{K_\alpha \colon \alpha \in \kappa\}$ be a family of spaces in $\mathcal{R}_0$ and let $K = \prod_{\alpha\in\kappa} K_\alpha$. For each $\alpha \in \kappa$ choose a dense set $D_\alpha \subseteq K_\alpha$ which is induced by a fixed r-skeleton in K_α. Fix $p \in \prod_{\alpha\in\kappa} D_\alpha$ and let D be the Σ-product of $\{D_\alpha\}_{\alpha\in\kappa}$ based on p, i.e.

$$D = \{x \in \prod_{\alpha\in\kappa} D_\alpha \colon |\operatorname{suppt}_p(x)| \leqslant \aleph_0\},$$

where $\operatorname{suppt}_p(x) := \{\alpha \in \kappa \colon x(\alpha) \neq p(\alpha)\}$. It is clear that D is countably closed and dense in K. We check condition (b) of Theorem 30. Let $\theta > \kappa$ be a regular cardinal such that for every $\alpha < \kappa$ statement (b) of Theorem 30 holds for every countable $M \preceq H(\theta)$ with $K_\alpha, D_\alpha \in M$. Fix a countable $M \preceq H(\theta)$ with $K, D \in M$. Note that M "knows" that K is the product of the family $\{K_\alpha\}_{\alpha\in\kappa}$. Indeed, by elementarity, there is $x \in M \cap K$. Then $\kappa = \operatorname{dom}(x) \in M$ and hence also the function $\alpha \mapsto K_\alpha$ is an element of M. Finally, M satisfies "$x \in K$ iff for every $\alpha \in \kappa$, $x(\alpha) \in K_\alpha$", which means that K is the product of $\{K_\alpha\}_{\alpha\in\kappa}$. In particular, if $\alpha \in \kappa \cap M$ then $K_\alpha \in M$. Similarly, M "knows" that D is the Σ-product of $\{D_\alpha\}_{\alpha\in\kappa}$ based on some $q \in K \cap M$. Assuming $\kappa > \aleph_0$, we have that $q = p$, because the unique constant function in D specifies the base point. If $\kappa = \aleph_0$ then $D = \prod_{\alpha\in\kappa} D_\alpha$ and the point p becomes irrelevant. In any case, we may assume that $p \in M$.

Let $S = \kappa \cap M$. Observe that

$$\operatorname{cl}(D \cap M) \subseteq \{x \in K \colon \operatorname{suppt}_p(x) \subseteq S\}.$$

Fix $x \neq y$ in $\operatorname{cl}(D \cap M)$. Then $x(\alpha) \neq y(\alpha)$ for some α. By the above remark, $\alpha \in M$. Thus $K_\alpha \in M$, being the α-th projection of K. Similarly, $D_\alpha \in M$ and therefore $q^M_{K_\alpha}$ is one-to-one on $\operatorname{cl}(D_\alpha \cap M)$. Finally, if $f \in \mathcal{C}(K_\alpha) \cap M$ separates $x(\alpha)$ from $y(\alpha)$, then $g = f \circ \operatorname{pr}_\alpha \in M$ and $g(x) \neq g(y)$, where pr_α denotes the projection onto the α-th coordinate. This shows that q^M_K is one-to-one on $\operatorname{cl}(D \cap M)$. By Theorem 30, $K \in \mathcal{R}_0$.

Theorem 32 *Assume $\{R_s\}_{s\in\Gamma}$ is an r-skeleton in a compact space K and let $D = \bigcup_{s\in\Gamma} R_s[K]$. Then*

1. *D is dense in K and for every countable set $A \subseteq D$ the closure $\operatorname{cl}_K(A)$ is metrizable and contained in D.*
2. *D is a countably tight Fréchet space.*
3. *D is a normal space and $K = \beta D$.*

We shall say that D is the *dense set induced by* $\{R_s\}_{s\in\Gamma}$.

PROOF. (1) follows from the σ-directedness of Γ: every countable subset of D is contained in some $K_s := R_s[K]$. This also shows that D is Fréchet. The countable tightness of D follows from Theorem 18(b), because $D \subseteq \mathcal{C}(K)^*$ generates projections in $\mathcal{C}(K)$.

It remains to prove that D is a normal space and that $K = \beta D$. Fix disjoint relatively closed sets $A, B \subseteq D$. We claim that $\operatorname{cl}_K(A) \cap \operatorname{cl}_K(B) = \emptyset$. This will also show that $K = \beta D$. Suppose $p \in \operatorname{cl}_K(A) \cap \operatorname{cl}_K(B)$ and fix a countable $M \preceq H(\theta)$ (where θ is sufficiently big) such that $A, B, p, \{R_s\}_{s\in\Gamma} \in M$. Let $\delta = \sup(\Gamma \cap M)$. Then $R_\delta(p) \in D$, so we may assume that $R_\delta(p) \notin A$ (interchanging the roles of A and B, if necessary). Recall that $K_\delta := R_\delta[K]$ is the limit of inverse system $\langle K_s, R^t_s, \Gamma \cap M\rangle$, where $R^t_s = R_s \restriction K_t$ (see [25, Lemma 3.4]). Thus, there are $t \in \Gamma \cap M$ and an open set $V \subseteq K_t$ such that $U := K_\delta \cap (R_t)^{-1}[V]$ is a neighborhood of p in K_δ disjoint from A. On the other hand, $(R_t)^{-1}[V] \cap A \neq \emptyset$. Since K_t is second countable, we may assume that $V \in M$. Thus, by elementarity, there is $a \in M$ such that $a \in (R_t)^{-1}[V] \cap A$. Finally, $a \in K_\delta$, so $a \in U \cap A$, a contradiction.

A Banach space analogue of part (3) of the above result looks as follows.

Proposition 33 *Let D be a norming space induced by a projectional skeleton $\{P_s\}_{s\in\Gamma}$ in a Banach space X. Then for every weak* continuous function $f\colon D \to \mathbb{R}$ there exists $t \in \Gamma$ such that $f = f \circ P_t^* \restriction D$.*

PROOF. Fix $n > 0$ and consider $K_n = n\overline{\mathrm{B}}_{X^*}$. Then $\{P_s^* \restriction K_n\}_{s\in\Gamma}$ is a retractional skeleton in K_n. By [25, Lemma 5.1], there exists $s_n \in \Gamma$ such that $f_n = f \circ P_{s_n}^* \restriction (D \cap K_n)$. We may assume that $s_1 \leqslant s_2 \leqslant \dots$. Let $t = \sup_{n\in\omega} s_n$. Then $f = f \circ P_t^* \restriction D$.

We are now able to determine *weak** compact subsets of spaces induced by projectional skeletons.

Theorem 34 *Assume $D \subseteq X^*$ generates projections in a Banach space X. Then every compact subset of D is Corson.*

PROOF. Let $K \subseteq D$ be compact with respect to the *weak** topology. We use Bandlow's characterization [4], which is a special case of Theorem 30. Fix a big enough cardinal θ and a countable $M \preceq \langle H(\theta), \in \rangle$ and fix $p \neq q$ in $\mathrm{cl}_*(K \cap M)$. Then there is $x \in X$ such that $p(x) \neq q(x)$. Since $\langle X, D \rangle$ has Property Ω (see Proposition 8), there exists $y \in \mathrm{cl}(X \cap M)$ such that $x - y \in {}^\perp(D \cap M)$. In particular, $p(x) = p(y)$ and $q(x) = q(y)$. Now, the continuity of p and q, find $z \in X \cap M$ such that $p(z) \neq q(z)$. Thus the function $\varphi \mapsto \varphi(z)$ is an element of M which separates p and q. This shows that $p \not\sim_M q$. Finally, Bandlow's theorem [4] (or a special case of Theorem 30) shows that K is Corson.

We do not know whether the converse holds, namely whether every norming *weak** Corson compact set generates projections, see Question 5.

A preservation theorem for Valdivia compacta, dual to Theorem 23, looks as follows. We omit its proof, since it can be easily deduced from (the proof of) Theorem 23.

Given a Valdivia compact K, let us call $\langle K, D \rangle$ a *Valdivia pair* if D is dense in K and there is an embedding $j\colon K \to [0,1]^\kappa$ such that $j[D] \subseteq \Sigma(\kappa)$.

Theorem 35 *Let $\{r_\alpha\}_{\alpha<\kappa}$ be a continuous retractive sequence in a compact space K. Let $D \subseteq K$ be a dense set such that for each $\alpha < \kappa$, $\langle r_\alpha[K], r_\alpha[D] \rangle$ is a Valdivia pair and $r_\alpha[D] \subseteq D$ for every $\alpha < \kappa$. Then $\langle K, D \rangle$ is a Valdivia pair.*

The above result leads to another proof of [25, Thm. 6.1], saying that a compact space with a commutative r-skeleton is Valdivia compact. Another corollary is the following.

Corollary 36 *The limit of a continuous retractive inverse sequence of Corson compacta is Valdivia compact.*

7 Final remarks and open problems

As we have already mentioned, the ordinal $\omega_2 + 1$ provides an example of a compact space in class $\mathcal{R}_0$ whose space of continuous functions is not Plichko. An r-skeleton in $\omega_2 + 1$ can be constructed as follows. Denote by Γ the family of all countable closed subsets A of ω_2 such that $0 \in A$ and every isolated point of A is isolated in ω_2. Given $A \in \Gamma$, define $r_A \colon \omega_2 + 1 \to \omega_2 + 1$ by setting $r_A(\alpha) = \max(A \cap [0, \alpha])$. It is straight to check that r_A is a retraction onto A (continuity follows from the assumption concerning isolated points). It is easy to check that $\mathfrak{r} = \{r_A\}_{A\in\Gamma}$ is an r-skeleton. Obviously, this skeleton is not commutative. On the other hand, the dual $\mathcal{C}(\omega_2 + 1)^*$ is 1-Plichko (see [18, Example 4.10(a)] or [17, Example 6.10]).

Example 37 *There exists a* 1*-projectional skeleton* $\mathfrak{s}$ *on* $\ell_1(\omega_2)$ *such that* $D(\mathfrak{s})$ *is not a* Σ*-space, i.e.* $\langle \ell_1(\omega_2), D(\mathfrak{s})\rangle$ *is not a Plichko pair.*

PROOF. We shall use ω_2+1 instead of ω_2 as the coordinate set, because $\omega_2+1 = [0, \omega_2]$ has the maximal element with respect to the natural well order. Let Γ consist of all countable subsets S of ω_2+1 such that $\omega_2 \in S$. Given $S \in \Gamma$, define $f_S \colon \omega_2+1 \to \omega_2+1$ by $f_S(\alpha) = \min(S \cap [\alpha, \omega_2])$. Note that f_S is generally discontinuous with respect to the interval topology on $\omega_2 + 1$. Further, define $Q_S \colon \ell_1(\omega_2 + 1) \to \ell_1(\omega_2 + 1)$ by setting $(Q_S x)(\alpha) = \sum_{\xi\in f_S^{-1}(\alpha)} x(\xi)$. It is clear that Q_S is a well defined linear projection onto $\ell_1(S) \subseteq \ell_1(\omega_2 + 1)$. Further, $\|Q_S\| = 1$. Fix $S \subseteq T$ in Γ. Clearly, $Q_T \circ Q_S = Q_S$, because Q_T is identity on $\ell_1(T) \supseteq \ell_1(S)$. Now observe that $f_S \circ f_T = f_S$. Thus, given $x \in \ell_1(\omega_2+1)$ and $\alpha \in \omega_2+1$, we have that $f_S^{-1}(\alpha) = \bigcup_{\xi\in f_S^{-1}(\alpha)} f_T^{-1}(\xi)$ and consequently

$$(Q_S Q_T x)(\alpha) = \sum_{\xi\in f_S^{-1}(\alpha)} (Q_T x)(\xi) = \sum_{\xi\in f_S^{-1}(\alpha)} \sum_{\eta\in f_T^{-1}(\xi)} x(\eta) = \sum_{\xi\in f_S^{-1}(\alpha)} x(\xi) = (Q_S x)(\alpha).$$

Finally, given $S_0 \subseteq S_1 \subseteq \dots$ in Γ, the set $S_\infty = \bigcup_{n\in\omega} S_n$ is an element of Γ and $\bigcup_{n\in\omega} \operatorname{im}(Q_{S_n})$ is clearly dense in $\operatorname{im}(Q_{S_\infty})$. It follows that $\mathfrak{s} = \{Q_S\}_{S\in\Gamma}$ is a projectional skeleton in $\ell_1(\omega_2 + 1)$.

Now suppose that $\langle \ell_1(\omega_2), D(\mathfrak{s})\rangle$ is a Plichko pair. By Theorem 27, there exists a commutative projectional skeleton $\mathfrak{r} = \{P_t\}_{t\in\Delta}$ such that $D(\mathfrak{r}) = D(\mathfrak{s})$. By Lemma 14, there exists a cofinal subset $\Gamma' \subseteq \Gamma$ such that for every $S \in \Gamma'$ there is $t = t(S) \in \Delta$ with

$Q_S = P_t$. In particular, $Q_S \circ Q_T = Q_T \circ Q_S$ whenever $S, T \in \Gamma'$. We shall derive a contradiction by finding $S, T \in \Gamma'$ such that $Q_S \circ Q_T \neq Q_T \circ Q_S$.

Given $S \in \Gamma'$, define $\varphi(S) = \sup(S \cap \omega_2)$. Construct a chain $\{S_\alpha\}_{\alpha<\omega_1}$ in Γ' so that $\varphi(\alpha) < \varphi(\beta)$ whenever $\alpha < \beta$. This is possible, because Γ' is cofinal in Γ. Let $\delta = \sup_{\alpha<\omega_1} \varphi(\alpha)$. Fix $T \in \Gamma'$ such that $\varphi(T) > \delta$. Then $\sup(T \cap \delta) < \delta$, because δ has cofinality ω_1. Find $\alpha < \omega_1$ such that $\sup(T \cap \delta) < \varphi(S_\alpha) < \delta$. Let $S = S_\alpha$. Fix $\xi \in S_\alpha$ such that $\sup(T \cap \delta) < \xi$. Then $f_T(\xi) > \delta$ and hence $f_S f_T(\xi) = \omega_2$, because $\varphi(S) < \delta$. On the other hand, $f_S(\xi) = \xi < \varphi(T)$ and hence $f_T f_S(\xi) \leqslant \varphi(T) < \omega_2$. It follows that $f_S f_T(\xi) \neq f_T f_S(\xi)$. Considering the characteristic function of $\{\xi\}$ as an element of $\ell_1(\omega_2 + 1)$, we conclude that $Q_S \circ Q_T \neq Q_T \circ Q_S$.

Given a norming space $D \subseteq X^*$, let $\mathcal{T}_D$ denote the topology on X induced by D, i.e. $\mathcal{T}_D = \sigma(X, D)$. It can be shown that $\langle X, \mathcal{T}_D \rangle$ is Lindelöf, whenever D generates projections in X. On the other hand, by the result of Kalenda [16, Thm. 2.3], D is a Σ-space iff $\langle X, \mathcal{T}_D \rangle$ is primarily Lindelöf and $D \cap \overline{\mathrm{B}}_{X^*}$ is *weak** countably compact (the last assumption cannot be dropped, see [17, Example 2.10(ii),(iii)]).

Problem 1 *Assume $D \subseteq X^*$ is a norming space. Find a topological property of $\langle X, \mathcal{T}_D \rangle$ which says when $D = D(\mathfrak{s})$ for some projectional skeleton $\mathfrak{s}$ in X.*

Question 1 *Assume X has a 1-projectional skeleton. Does X have a projectional generator?*

Question 2 *Assume X is a Banach space of density $> \aleph_1$ and $\mathcal{F}$ is a directed family of 1-complemented separable subspaces such that $X = \bigcup \mathcal{F}$ and $\mathrm{cl}(\bigcup_{n\in\omega} F_n) \in \mathcal{F}$ whenever $\{F_n \colon n \in \omega\} \subseteq \mathcal{F}$. Does X necessarily have a projectional skeleton?*

Note that if X has density $\aleph_1$ then the above assumptions imply the existence of a PRI, see [24, Lemma 6.1].

Question 3 *Let X be a Banach space with a projectional skeleton. Does every closed subspace of X have the separable complementation property?*

Note that, by the main result of [24], a closed subspace of a Plichko space of density $\aleph_1$ may not have a projectional skeleton.

The following question has already been asked by Ondřej Kalenda [19].

Question 4 *Is $\mathcal{C}(\omega_2 + 1)$ embeddable into a Plichko space?*

Question 5 *Assume $K \subseteq X^*$ is Corson compact and norming for X. Does K generate projections in X?*

If K is a Corson compact in the dual of a Banach space X and K is norming for X, then X embeds into $\mathcal{C}(K)$. Thus, if $\mathcal{C}(K)$ is WLD then so is X and consequently K generates projections in X. It follows that the above question at least consistently has affirmative answer: it is relatively consistent with the usual axioms of set theory that $\mathcal{C}(K)$ is WLD for every Corson compact K (see e.g. [2, Remark 3.2.3)]).

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