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REMARKS ON SPECIAL SYMPLECTIC CONNECTIONS

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Abstract. The relation between special symplectic connection and Weyl connections is shown. We give more explicit construction of the special symplectic connections in some real cases.

Contents

1. Introduction

Special symplectic connection on a symplectic manifold (M, ω) is a torsion-free linear connection preserving ω which is special in the sense of definitions in 1.1. The definition of special symplectic connection is rather wide, however, there is a nice link between special symplectic connections and contact parabolic geometries, which was established in the profound paper [1]. The main result of that paper states that any special symplectic connection on M comes via a symplectic reduction from a specific linear connection on a one-diminension bigger contact manifold $\mathcal{C}$, the homogeneous model of some contact parabolic geometry. All the necessary background on contact parabolic geometries is collected in section 2. The construction and the characterization from [1] is quickly reminded in section 3, culminating in Theorem 3.2.

In the next section we provide an alternative and rather direct approach to special symplectic connections. Firstly we reinterpret the previous results in terms of parabolic geometries so that the specific linear connections on $\mathcal{C}$ are exactly the exact Weyl connections corresponding to specific choices of scales. A choice of scale further defines a bundle projection from $T\mathcal{C}$ to the contact distribution $D \subset T\mathcal{C}$ and this gives rise to a partial contact connection on D. By the very construction, the only ingredient which yields the special symplectic connection on M is just the partial contact connection associated to the choice of scale, Proposition 4.2.

Finally, the direct construction of special symplectic connections works via a pull-back of an ambient symplectic connection on the total space of the canonical scale bundle $\hat{\mathcal{C}} \to \mathcal{C}$. Namely for some real cases we can find a convenient ambient connection on $\hat{\mathcal{C}}$ and then compare the exact Weyl connection and the pull-back connection on $\mathcal{C}$ corresponding to the choice of scale so that they coincide on the contact distribution D, Theorem 4.3. By the previous results, they give rise the same symplectic connection on M after the reduction. This construction applies to the

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projective contact structures, CR structures of hypersurface type, and Lagrangean contact structures, which are dealt in subsections 4.4, 4.5, and 4.6, respectively.

1.1. Special symplectic connections. Given a smooth manifold M with a symplectic structure $\omega \in \Omega^2(M)$, linear connection ∇ on M is said to be *symplectic* if it is torsion free and ω is parallel with respect to ∇. There is a lot of symplectic connections to a given symplectic structure, hence studying this subject, further restrictive conditions appear. Following the article [1], we consider the *special symplectic connections* defined as symplectic connections belonging to some of the following classes:

(i) *Connections of Ricci type.* The curvature tensor of a symplectic connection decomposes under the action of the symplectic group into two irreducible components. One of them corresponds to the Ricci curvature and the other one is the Ricci-flat part. If the curvature tensor consists only of the Ricci curvature part, then the connection is said to be of Ricci type.

(ii) *Bochner–Kähler connections.* Let the symplectic form be the Kähler form of a (pseudo-)Kähler metric and let the connection preserve this (pseudo-)Kähler structure. The curvature tensor decomposes similarly as in the previous case into two parts but this time under the action of the (pseudo-)unitary group. These are called Ricci curvature and Bochner curvature. If the Bochner curvature vanishes, the connection is called Bochner–Kähler.

(iii) *Bochner–bi-Lagrangean connections.* A bi-Lagrangean structure on a symplectic manifold consists of two complementary Lagrangean distributions. If a symplectic connection preserves such structure, i.e. both the Lagrangean distributions are parallel, then again the curvature tensor decomposes into the Ricci and Bochner part. If the Bochner curvature vanishes, we speak about Bochner–bi-Lagrangean connections.

(iv) *Connections with special symplectic holonomies.* We say that a symplectic connection has special symplectic holonomy if its holonomy is contained in a proper absolutely irreducible subgroup of the symplectic group. Special symplectic holonomies are completely classified and studied by various people.

Note that all the previous definitions admit an analogy in complex/holomorphic setting but we are dealing only with the real structures in this paper.

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2. Contact parabolic geometries and Weyl connections

In the this section we provide necessary background from the parabolic geometries and generalized Weyl structures as can be found in [11] and [3].

2.1. Contact parabolic geometries. Semisimple Lie algebra admits a *contact grading* if there is a grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\mathfrak{g}_{-2}$ is one dimensional and the Lie bracket $[\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is non-degenerate. If $\mathfrak{g}$ admits a contact grading, then $\mathfrak{g}$ has to be simple. Any complex simple Lie algebra, except $\mathfrak{sl}(2,\mathbb{C})$, admits a unique contact grading, but the existence is not generally guaranteed in real case. However, the split real form of complex simple Lie algebra and most of non-compact non-complex real Lie algebras admit a contact grading.

Let $\mathfrak{g}$ be a real simple Lie algebra admitting a contact grading, let $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the corresponding parabolic subalgebra, and let $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Let further $\mathfrak{z}(\mathfrak{g}_0)$ be the center of $\mathfrak{g}_0$. Let $E \in \mathfrak{z}(\mathfrak{g}_0)$ be the grading element of $\mathfrak{g}$ and let $\mathfrak{g}_0' \subset \mathfrak{g}_0$

be the orthogonal complement of E with respect to the Killing form on $\mathfrak{g}$. From the invariance of the Killing form and the fact that $[\mathfrak{g}_{-2}, \mathfrak{g}_2] = \langle E \rangle$, the subalgebra $\mathfrak{g}_0' \subset \mathfrak{g}_0$ is equivalently characterized by the fact that $[\mathfrak{g}_0', \mathfrak{g}_2] = 0$. For later use let us denote $\mathfrak{p}' := \mathfrak{g}_0' \oplus \mathfrak{p}_+$.

For a semisimple Lie group G and a parabolic subgroup $P \subset G$, *parabolic geometry* of type (G, P) on a smooth manifold M consists of a principal P-bundle $\mathcal{G} \to M$ and a Cartan connection $\eta \in \Omega^1(\mathcal{G}, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G. If $\mathfrak{g}$ is simple Lie algebra admitting a contact grading and the Lie subalgebra $\mathfrak{p} \subset \mathfrak{g}$ of P corresponds to this grading, then we speak about *contact parabolic geometry.* The contact grading of $\mathfrak{g}$ gives rise to a contact structure on M as follows. Under the usual identification $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ via η, the P-invariant subspace $(\mathfrak{g}_{-1} \oplus \mathfrak{p})/\mathfrak{p} \subset \mathfrak{g}/\mathfrak{p}$, defines a distribution $D \subset TM$, namely

$$D \cong \mathcal{G} \times_P (\mathfrak{g}_{-1} \oplus \mathfrak{p})/\mathfrak{p}. \tag{1}$$

Parabolic geometry is *regular* if the Levi bracket, induced on the graded tangent bundle $\mathrm{gr}(TM) = D \oplus TM/D$ from the Lie bracket of vector fields, coincides with the algebraic bracket determined by the Lie bracket on $\mathfrak{g}_- = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$. For regular contact parabolic geometries, the distribution $D \subset TM$ defined by (1) is a contact distribution.

The contact distribution can be also given as a kernel of the contact form θ on M, that is a form on M satisfying $\theta \wedge (d\theta)^n \neq 0$. The (unique) Reeb vector field $R \in \mathfrak{X}(M)$, defined as $d\theta(R, \_) = 0$ and $\theta(R) = 1$, then leads to the decomposition $TM \cong D \oplus \mathbb{R} \cdot R$ of the tangent bundle.

2.2. Weyl structures. Let $(\mathcal{G} \to M, \eta)$ be a parabolic geometry of type (G, P). Let $\mathfrak{p} \subset \mathfrak{g}$ be the Lie algebras of the Lie groups $P \subset G$ and let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ be the corresponding grading of $\mathfrak{g}$. Let $G_0 \subset P$ be the Lie group with Lie algebra $\mathfrak{g}_0$ and let $P_+ := \exp \mathfrak{p}_+$ so that $P = G_0 \rtimes P_+$. Let further $\mathcal{G}_0 := \mathcal{G}/P_+ \to M$ be the underlying G_0-bundle and let $\pi_0 : \mathcal{G} \to \mathcal{G}_0$ be the canonical projection. A *Weyl structure* for the parabolic geometry $(\mathcal{G} \to M, \eta)$ is a global smooth G_0-equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of the projection π_0, i.e. a reduction of $\mathcal{G} \to M$ to the structure group $G_0 \subset P$.

Denote by η_i the $\mathfrak{g}_i$–component of the Cartan connection $\eta \in \Omega^1(\mathcal{G}, \mathfrak{g})$. For a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$, the pull-back $\sigma^*\eta_0$ defines a principal connection on the principal bundle $\mathcal{G}_0$. It is called the *Weyl connection* of the Weyl stucture σ.

Next, the form $\sigma^*\eta_-$, is the soldering form. It provides the reduction of the bundle $(\mathcal{G} \to M)$ to the structure group G_0 and $\sigma^*\eta_+ =: \mathsf{P}^\sigma$, is the Rho-tensor.

Any Weyl connection induces connections on all bundles associated to $\mathcal{G}_0$, in particular, there is an induced linear connection on TM. By definition, any Weyl connection preserves the underlying G_0-structure on M. On the other hand, there are particularly convenient bundles such that the induced connection from $\sigma^*\eta_0$ is sufficient to determine whole the Weyl structure σ. These are the so called bundles of scales, oriented line bundles over M defined as follows.

2.3. Scales and exact Weyl connections. Let $\mathcal{L} \to M$ be a principal $\mathbb{R}_+$-bundle associated to $\mathcal{G}_0$. This is determined by a group homomorphism $\lambda : G_0 \to \mathbb{R}_+$ whose derivative is denoted by $\lambda' : \mathfrak{g}_0 \to \mathbb{R}$. The Lie algebra $\mathfrak{g}_0$ is reductive, i.e. $\mathfrak{g}_0$ splits into a direct sum of the center $\mathfrak{z}(\mathfrak{g}_0)$ and the semisimple part, hence the only elements that can act non-trivially by λ' are from $\mathfrak{z}(\mathfrak{g}_0)$. Next, the restriction of the Killing form B to $\mathfrak{g}_0$ and further to $\mathfrak{z}(\mathfrak{g}_0)$ is non-degenerate. Altogether, for any representation $\lambda' : \mathfrak{g}_0 \to \mathbb{R}$ there is a unique element $E_\lambda \in \mathfrak{z}(\mathfrak{g}_0)$ such that

$$\lambda'(A) = B(E_\lambda, A) \tag{2}$$

for all $A \in \mathfrak{g}_0$. By Schur's lemma, E_λ acts by a real scalar on any irreducible representation of G_0. An element $E_\lambda \in \mathfrak{z}(\mathfrak{g}_0)$ is called a *scaling element* if it acts by a non-zero real scalar on each G_0-irreducible component of $\mathfrak{p}_+$. (In general, the grading element of $\mathfrak{g}$ is a scaling element.) A *bundle of scales* is a principal $\mathbb{R}_+$-bundle associated to $\mathcal{G}_0$ via a homomorphism $\lambda : G_0 \to \mathbb{R}_+$, whose derivative is given by (2) for some scaling element E_λ. Bundle of scales $\mathcal{L}^\lambda \to M$ corresponding to λ is naturally identified with $\mathcal{G}_0/\ker\lambda$, the orbit space of the action of the normal subgroup $\ker\lambda \subset G_0$ on $\mathcal{G}_0$.

Let $\mathcal{L}^\lambda \to M$ be a fixed bundle of scales and let $\sigma : \mathcal{G}_0 \to \mathcal{G}$ be a Weyl structure of a parabolic geometry $(\mathcal{G} \to M, \eta)$. Then the Weyl connection $\sigma^*\eta_0$ on $\mathcal{G}_0$ induces a principal connection on $\mathcal{L}^\lambda$ and [3, Theorem 3.12] shows that this mapping establishes a bijective correspondence between the set of Weyl structures and the set of principal connections on $\mathcal{L}^\lambda$. Note that the surjectivity part of the statement is rather implicit, however there is a distinguished subclass of Weyl structures which allow more satisfactory interpretation, namely the *exact Weyl strucures* defined as follows. Any bundle of scales is trivial and so it admits global smooth sections, which we usually refer to as *choices of scale*. Any choice of scale gives rise to a flat principal connection on $\mathcal{L}^\lambda$ and the corresponding Weyl structure is then called *exact*.

Furthermore, due to the identification $\mathcal{L}^\lambda = \mathcal{G}_0/\ker\lambda$, the sections of $\mathcal{L}^\lambda \to M$ are in a bijective correspondence with reductions of the principal bundle $\mathcal{G}_0 \to M$ to the structure group $\ker\lambda \subset G_0$. Altogether for any choice of scale, the composition of the two reductions above is a reduction of $\mathcal{G} \to M$ to the structure group $\ker\lambda \subset G_0 \subset P$; let us denote the resulting bundle by $\mathcal{G}'_0$. Hence the corresponding exact Weyl connection has holonomy in $\ker\lambda$ and by general principles from the theory of G-structures, it preserves the geometric quantity corresponding to the choice of scale. In the cases of contact parabolic geometries, the canonical candidate for the bundle of scales is the bundle of positive contact forms. Easily, this is the scale bundle corresponding to a (non-zero scalar multiple of) the grading element $E \in \mathfrak{z}(\mathfrak{g}_0)$ and in the homogeneous case one recovers the cone $\hat{\mathcal{C}} \to \mathcal{C}$, cf. remark 3.1(c). To be more presice, any Weyl connection preserves the underlying G_0-structure so in particular the contact ditribution $D \subset TM$; the exact Weyl connection corresponding to the contact one-form $\theta \in \Omega^1(M)$ preserves moreover θ, i.e. θ is parallel.

3. Characterization of special symplectic connections

In this section the quick review of the construction of the special symplectic connections from the article [1] is described.

3.1. Adjoint orbit and its projectivization. Let $\mathfrak{g}$ be a real simple Lie algebra admitting a contact grading and let $e^2_+ \in \mathfrak{g}$ be a maximal root element, i.e. a generator of $\mathfrak{g}_2$. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Consider the adjoint orbit of e^2_+ and its oriented projectivization:

$$\hat{\mathcal{C}} := \mathrm{Ad}_G(e^2_+) \subset \mathfrak{g},\ \mathcal{C} := \mathcal{P}^o(\hat{\mathcal{C}}) \subset \mathcal{P}^o(\mathfrak{g}). \tag{3}$$

The restriction of the natural projection $p : \mathfrak{g}\backslash\{0\} \to \mathcal{P}^o(\mathfrak{g})$ to $\hat{\mathcal{C}}$ yields the principal $\mathbb{R}_+$-bundle $p : \hat{\mathcal{C}} \to \mathcal{C}$, which we call *the cone*. The right action of $\mathbb{R}_+$ is just the multiplication by positive real scalars. The fundamental vector field of this action is the *Euler vector field* $\hat{E}$ defined as $\hat{E}(x) := x$, for any $x \in \hat{\mathcal{C}} \subset \mathfrak{g}$.

Since $\hat{\mathcal{C}}$ is an adjoint orbit of G in $\mathfrak{g}$, and $\mathfrak{g}$ can be identified with $\mathfrak{g}^*$ via the Killing form, there is a canonical G-invariant symplectic form $\hat{\Omega}$ on $\hat{\mathcal{C}}$. For any

$X, Y \in \mathfrak{g}$ and $\alpha \in \hat{\mathcal{C}} \subset \mathfrak{g}^*$, the value of $\hat{\Omega}$ is given by the formula

$$\hat{\Omega}(\mathrm{ad}^*_X(\alpha), \mathrm{ad}^*_Y(\alpha)) := \alpha([X,Y]),$$

where $\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$ is the infinitesimal coadjoint representation and $\mathrm{ad}^*_X(\alpha) = -\alpha \circ \mathrm{ad}_X$ is viewed as an element of $T_\alpha \hat{\mathcal{C}}$. Under the identification $\mathfrak{g} \cong \mathfrak{g}^*$ the previous formula reads as

$$\hat{\Omega}(\mathrm{ad}_X(a), \mathrm{ad}_Y(a)) = B(a, [X,Y]), \tag{4}$$

for any $X, Y \in \mathfrak{g}$ and $a \in \hat{\mathcal{C}} \subset \mathfrak{g}$, where $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is the Killing form. The Euler vector field and the canonical symplectic form defines a (canonical) G-invariant one-form $\hat{\alpha}$ on $\hat{\mathcal{C}}$ by

$$\hat{\alpha} := \frac{1}{2} \hat{E} \lrcorner \hat{\Omega}. \tag{5}$$

Immediately from definitions it follows that $\mathcal{L}_{\hat{E}} \hat{\Omega} = 2\hat{\Omega}$ and consequently $d\hat{\alpha} = \hat{\Omega}$.

Lemma. *Let $p : \hat{\mathcal{C}} \to \mathcal{C}$ be the cone defined by (3) and let $P' \subset P$ be the connected subgroups in G corresponding to the subalgebras $\mathfrak{p}' \subset \mathfrak{p} \subset \mathfrak{g}$ from 2.1. Then $\hat{\mathcal{C}} \cong G/P'$ and $\mathcal{C} \cong G/P$ so that the contact distribution $D \subset T(G/P)$ is identified with $Tp \cdot \ker \hat{\alpha} \subset T\mathcal{C}$.*

Proof. By definition, the group G acts transitively both on $\hat{\mathcal{C}}$ and $\mathcal{C} = \hat{\mathcal{C}}/\mathbb{R}_+$. Since $[A, e^2_+] = 0$ if and only if $A \in \mathfrak{p}'$ and we assume the Lie subgroup $P' \subset G$ corresponding to $\mathfrak{p}' \subset \mathfrak{g}$ is connected, the stabilizer of e^2_+ is precisely P'. Hence the orbit $\hat{\mathcal{C}}$ is identified with the homogeneous space G/P'. Since $P \supset P'$ is also connected, P/P' is identified with the subgroup $\{\exp tE : t \in \mathbb{R}\} \cong \mathbb{R}_+$ in P. Hence P preserves the ray of positive multiples of e^2_+ so that $\mathcal{C} = \hat{\mathcal{C}}/\mathbb{R}_+$ is identified with G/P.

For the last part of the statement, note that the Euler vector field is generated by (a non-zero multiple of) the grading element $E \in \mathfrak{z}(\mathfrak{g}_0)$. The canonical one-form $\hat{\alpha}$ on $\hat{\mathcal{C}}$ is G-invariant, so it is determined by its value in the origin $o \in G/P'$, i.e. $e^2_+ \in \hat{\mathcal{C}}$, which is a P'-invariant one-form ϕ on $\mathfrak{g}/\mathfrak{p}'$. By (4) and (5), ϕ is explicitly given as $\phi(X) = B(e^2_+, [E, X])$, possibly up to a non-zero scalar multiple. The formula is obviously independent of the representative of X in $\mathfrak{g}/\mathfrak{p}'$ and the kernel of ϕ is just $(\mathfrak{g}_{-1} \oplus \mathfrak{p})/\mathfrak{p}'$. The tangent map of the projection $p : \hat{\mathcal{C}} \to \mathcal{C}$ corresponds to the natural projection $\mathfrak{g}/\mathfrak{p}' \to \mathfrak{g}/\mathfrak{p}$, hence $Tp \cdot \ker \hat{\alpha} \subset T\mathcal{C}$ corresponds to $(\mathfrak{g}_{-1} \oplus \mathfrak{p})/\mathfrak{p} \subset \mathfrak{g}/\mathfrak{p}$ which defines the contact distribution $D \subset T(G/P)$ in (1). □

Remarks. (a) Note that in contrast to the definition of the cone in [1] we do not assume the center of G is trivial. Hence the two approaches differ by a (usually finite) covering. Because of the very local character of all the constructions that follow, this causes no problem and we will not mention the difference below.

(b) The homogeneous space $\hat{\mathcal{C}} \cong G/P'$ is an example of a *G-symplectic homogeneous space*, i.e. a homogeneous space of a Lie group G with G-invariant symplectic structure. According to [7, Corollary to Theorem 5.5.1], for G being semisimple, any G-symplectic homogeneous space is isomorphic to a covering of an orbit in $\mathfrak{g}$, which is thought with the (restriction of the) canonical symplectic form. Moreover the covering map and hence the orbit are unique. This characterization will be useful below.

(c) According to [1, Prop. 3.2], the bundle $\hat{\mathcal{C}} \to \mathcal{C}$ can be identified with the bundle of positive contact forms on $\mathcal{C}$ so that $\hat{\Omega} = d\hat{\alpha}$ corresponds to the restriction of the canonical symplectic form on the cotangent bundle $T^*\mathcal{C}$. In detail, a section $s : \mathcal{C} \to \hat{\mathcal{C}}$ yields the contact one-form $\theta_s := s^*\hat{\alpha}$ and, by the naturality of the exterior differential, $d\theta_s = s^*\hat{\Omega}$.

3.2. General construction. Let a be an element of a real simple Lie algebra $\mathfrak{g}$ admitting a contact grading. With the notation as before, let ξ_a be the fundamental vector field of the left action of G on $\mathcal{C} \cong G/P$ corresponding to $a \in \mathfrak{g}$. Let us denote by $\mathcal{C}_a$ the (open) subset in $\mathcal{C}$ where ξ_a is transverse to the contact distribution $D \subset T\mathcal{C}$ and oriented in accordance with a fixed orientation of $T\mathcal{C}/D$. The vector field ξ_a gives rise to a unique contact one-form θ_a on $\mathcal{C}_a$ such that ξ_a is its Reeb field. In other words, $\theta_a \in \Omega^1(\mathcal{C}_a)$ is uniquely determined by the conditions

$$\ker \theta_a = D \text{ and } \theta_a(\xi_a) = 1. \tag{6}$$

Since ξ_a is a contact symmetry, i.e. $\mathcal{L}_{\xi_a} D \subset D$, it easily follows that $\mathcal{L}_{\xi_a}\theta_a = 0$ and consequently $\xi_a \lrcorner\, d\theta_a = 0$. Let $T_a \subset G$ denote the one-parameter subgroup corresponding to the fixed element $a \in \mathfrak{g}$. We say that an open subset $U \subset \mathcal{C}_a$ is *regular* if the local quotient $M_U := T_a \setminus U$ is a manifold. Since $\xi_a \lrcorner\, d\theta_a = 0$ and $d\theta_a$ has maximal rank, it descends to a symplectic form ω_a on M_U, for any regular $U \subset \mathcal{C}_a$.

Next, let $\pi : G \to G/P \cong \mathcal{C}$ be the canonical P-principal bundle and consider its restriction to $\mathcal{C}_a$. If $\mathcal{C}_a$ in non-empty, then [1, Theorem 3.4] describes explicitly a subset Γ_a in $\pi^{-1}(\mathcal{C}_a) \subset G$, which forms a G_0'-principal bundle over $\mathcal{C}_a$ where G_0' denotes the connected subgroup in G corresponding to the Lie subalgebra $\mathfrak{g}_0' \subset \mathfrak{g}$. For a regular open subset $U \subset \mathcal{C}_a$, denote $\Gamma_U := \pi^{-1}(U) \subset \Gamma_a$ and $B_U := T_a \setminus \Gamma_U$. Then $B_U \to M_U$ is a G_0'-principal bundle and [1, Theorem 3.5] shows that the restriction of the $(\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0')$-component of the Maurer–Cartan form $\mu \in \Omega^1(G, \mathfrak{g})$ to Γ_U descends to a $(\mathfrak{g}_{-1} \oplus \mathfrak{g}_0')$-valued coframe on B_U. Altogether, the bundle $B_U \to M_U$ is interpreted as a classical G_0'-structure and the $\mathfrak{g}_0'$-part of the coframe above induces a linear connection on M_U. It turns out this connection is special symplectic connection with respect to the symplectic form ω_a.

Surprisingly, [1, Theorem B] proves that any special symplectic connection can be at least locally obtained by the previous construction. With an assumption on $\dim \mathfrak{g} \geq 14$, which is equivalent to $\dim M_U \geq 4$, we reformulate the main result of [1] as follows.

Theorem ([1]). *Let $\mathfrak{g}$ be a simple Lie algebra of dimension ≥ 14 admitting a contact grading. With the same notation as above, let $a \in \mathfrak{g}$ be such that $\mathcal{C}_a \subset \mathcal{C}$ is non-empty and let $U \subset \mathcal{C}_a$ be regular. Then*
(a) the local quotient M_U carries a special symplectic connection,
(b) locally, connections from (a) exhaust all special symplectic connection.

An instance of the correspondence between the various classes of special symplectic connections and contact gradings of simple Lie algebras is as follows. For $\dim M_U = 2n$, special symplectic connections of type (i), (ii) and (iii), according to the definitions in 1.1, corresponds to the contact grading of simple Lie algebras $\mathfrak{sp}(2n+2, \mathbb{R})$, $\mathfrak{su}(p+1, q+1)$ with $p + q = n$, and $\mathfrak{sl}(n+2, \mathbb{R})$, respectively. The corresponding contact parabolic structure on $\mathcal{C} \cong G/P$ is the projective contact structure, CR structure of hypersurface type, and Lagrangean contact structure, respectively. Details on each of these structures are treated in the next section in details.

4. Alternative realization of special symplectic connections

Below we describe contact parabolic structures corresponding to special symplectic connections of type (i), (ii) and (iii) as mentioned above. The aim of this section is, for each of the listed cases, to provide the characterization of Theorem 3.2, and so the realization of special symplectic connections, in more explicit and satisfactory way. For this purpose we interpret the model cone $p : \hat{\mathcal{C}} \cong G/P' \to G/P \cong \mathcal{C}$

in each particular case and look for a natural ambient connection $\hat{\nabla}$ on $\hat{\mathcal{C}}$ which is good enough to give rise the easier interpretation. We start with a reinterpretation of the construction from 3.2 in terms of Weyl structures and conenctions.

4.1. Partial contact connections. In order to formulate the next results we need the notion of partial contact connections. For a general distribution $D \subset TM$ on a smooth manifold M, a *partial linear connection* on M is an operator $\Gamma(D) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying the usual conditions for linear connections. In other words, we modify the notion of linear connection on TM just by the requirement to differentiate only in the directions lying in D. If a partial linear connection preserves D, then restricting also the second argument to D yields an operator of the type $\Gamma(D) \times \Gamma(D) \to \Gamma(D)$; in the case the distribution $D \subset TM$ is contact, we speak about the *partial contact connection*.

Given a contact distribution $D \subset TM$ and a classical linear connection ∇ on M, any choice of a contact one-form induces a partial contact connection ∇^Das follows. Let $\theta \in \Omega^1(M)$ be a contact one-form with the contact subbundle D and let R_θ be the corresponding Reeb vector field. This induces a decomposition $TM \cong D \oplus \mathbb{R}$ and in particular a bundle projection $\pi_\theta : TM \to D$, namely the projection in the direction of $\langle R_\theta \rangle \subset TM$. Now for any $X, Y \in \Gamma(D)$, the formula

$$\nabla^D_X Y := \pi_\theta(\nabla_X Y) \tag{7}$$

defines a partial contact connection and we say that ∇^D is induced from ∇ by θ.

The contact torsion of the linear connection is the projection of the classical torsion to the contact distribution D given by the decomposition of the tangent bundle given by the contact form, see above.

4.2. General construction revisited. In the construction of special symplectic connection in 3.2, we started with a choice of an element $a \in \mathfrak{g}$ which in particular induced a contact one-form θ_a on $\mathcal{C}_a$. Then we described the G'_0-principal bundle $\Gamma_a \subset \pi^{-1}(\mathcal{C}_a)$ which is actually a reduction of the P-principal bundle $\pi^{-1}(\mathcal{C}_a) \to \mathcal{C}_a$ to the structure group G'_0. In terms of subsection 2.3, the couple $(\pi^{-1}(\mathcal{C}_a) \to \mathcal{C}_a, \mu)$ forms a flat parabolic geometry of type (G, P) and the contact form θ_a represents a choice of scale. The reduction above is just the exact Weyl structure corresponding to θ_a so that Γ_a is the image of $\mathcal{G}'_0$ and the restriction of the $\mathfrak{g}'_0$-part of the Maurer–Cartan form μ to Γ_a is the exact Weyl connection.

Further restriction to a regular subset $U \subset \mathcal{C}_a$ and the factorization by T_a finally yielded a special symplectic connection on $M_U = T_a \setminus U$. In the current setting together with the definitions in 4.1, it is obvious that the resulting connection on M_U is fully determined by the partial contact connection induced by θ_a from the exact Weyl connection on $U \subset \mathcal{C}_a$ corresponding to θ_a. Since any Weyl connection preserves the contact distribution D, the induced partial contact connection is just the restriction to the directions in D. Altogether, we can recapitulate the results in 3.2 as follows.

Proposition. *Let $a \in \mathfrak{g}$ be so that $\mathcal{C}_a \subset \mathcal{C}$ is non-empty and let $U \subset \mathcal{C}_a$ be regular. Let θ_a be the contact one-form on $U \subset \mathcal{C}_a$ determined by $a \in \mathfrak{g}$ as in* (6).

Then the special symplectic connection on M_U constructed in 3.2 is fully determined by the partial contact connection induced from the exact Weyl connection corresponding to θ_a

4.3. Pull-back connections. Let $p : \hat{\mathcal{C}} \to \mathcal{C}$ be the cone as in 3.1. Any smooth section $s : \mathcal{C} \to \hat{\mathcal{C}}$ determines a principal connection on $\hat{\mathcal{C}}$; the corresponding horizontal lift of vector fields is denoted as $X \mapsto X^{hor}$. An ambient linear connection $\hat{\nabla}$ on $\hat{\mathcal{C}}$ defines a linear connection ∇^s on $\mathcal{C}$ by the formula

$$\nabla^s_X Y := Tp(\hat{\nabla}_{X^{hor}} Y^{hor}). \tag{8}$$

We call ∇^s the *pull-back connection* corresponding to s. On the other hand, for any section s, which we call a choice of scale by 2.3, let $\theta_s = s^*\hat{\alpha}$ be the contact form and let $\bar{\nabla}^s$ be the corresponding exact Weyl connection on $\mathcal{C}$. In the rest of this section, we are looking for an ambient connection $\hat{\nabla}$ on $\hat{\mathcal{C}}$ so that both ∇^s and $\bar{\nabla}^s$ induce the same partial contact connection on $D \subset T\mathcal{C}$. For this reason it turns out that $\hat{\nabla}$ has to be symplectic, i.e. $\hat{\nabla}\hat{\Omega} = 0$.

The following statement provides together with Theorem 3.2 and Proposition 4.2 the desired simple realization of special symplectic connections of type (i), (ii), and (iii) according to the list in 1.1. The point is that in all these cases the ambient connection $\hat{\nabla}$ is very natural and easy to describe.

Theorem. *Let $\hat{\mathcal{C}} \to \mathcal{C}$ be the model cone for $\mathfrak{g} = \mathfrak{sp}(2n+2, \mathbb{R})$, $\mathfrak{su}(p+1, q+1)$ or $\mathfrak{sl}(n+2, \mathbb{R})$. Then there is an ambient symplectic connection $\hat{\nabla}$ on the total space of $\hat{\mathcal{C}}$ so that, for any section $s : \mathcal{C} \to \hat{\mathcal{C}}$, the induced partial contact connections of the exact Weyl connection and the pull-back connection corresponding to s coincide.*

Although the definition of the cone $\hat{\mathcal{C}} \to \mathcal{C}$ is pretty general, its convenient interpretation necessary to find a natural candidate for $\hat{\nabla}$ is no more universal. In order to prove the Theorem, we deal in following three subsections with each case individually. It follows that the reasonable interpretation of the cone in any discussed case is more or less standard and the candidate for an ambient connection $\hat{\nabla}$ is almost canonical. Therefore in the proofs of subsequent Propositions we focus only in the justification of the choices.

Note that a natural guess for $\hat{\nabla}$ to be a G-invariant symplectic connection on $\hat{\mathcal{C}} = G/P'$ does help only for contact projective structures, i.e. the structures corresponding to the contact grading of $\mathfrak{g} = \mathfrak{sp}(2n+2, \mathbb{R})$. This is due to the following statement, which is an immediate corollary of [10, Theorem 3]: For a connected real simple Lie group G with Lie algebra $\mathfrak{g}$, the nilpotent adjoint orbit $\mathcal{C} = \mathrm{Ad}_G(e_+^2)$ admits a G-invariant linear connection if and only if $\mathfrak{g} \cong \mathfrak{sp}(m, \mathbb{R})$.

For a reader's convenience we assume the dimension of $\mathcal{C} = G/P$ to be always $m = 2n + 1$. Consequently, $\dim \hat{\mathcal{C}} = 2n + 2$ and we further continue the convention that all important objects on $\hat{\mathcal{C}}$ are denoted with the hat.

4.4. Contact projective structures. Contact projective structures correspond to the contact grading of the Lie algebra $\mathfrak{g} = \mathfrak{sp}(2n+2, \mathbb{R})$, the only real form of $\mathfrak{sp}(2n+2, \mathbb{C})$ admitting the contact grading. These structures are studied in [6] in whole generality: *contact projective structure* on a contact manifold (M, D) is defined as a contact path geometry such that the paths are among geodesics of a linear connection on M; the paths are then called *contact geodesics*. In analogy to classical projective structures, a contact projective structure is given by a class of linear connections $[\nabla]$ on TM having the same contact torsion and the same non-parametrized geodesics such that the following property is satisfied: if a geodesic is tangent to D in one point then it remains tangent to D everywhere.

The model contact projective structure is observed on the projectivization of symplectic vector space $(\mathbb{R}^{2n+2}, \hat{\Omega})$ with $\hat{\Omega}$ being a standard symplectic form. Let G be the group of linear automorphisms of $\mathbb{R}^{2n+2}$ preserving $\hat{\Omega}$, i.e. $G := Sp(2n+2, \mathbb{R})$. In order to represent conveniently the contact grading of the corresponding Lie algebra, let $\hat{\Omega}$ be given by the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbb{J} & 0 \\ -1 & 0 & 0 \end{pmatrix}$, with respect to the standard basis of $\mathbb{R}^{2n+2}$, where $\mathbb{J} = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$ and $\mathbb{I}_n$ is the identity matrix of rank n. For $\mathbb{J}^t = -\mathbb{J}$, the Lie algebra $\mathfrak{g} = \mathfrak{sp}(2n+2, \mathbb{R})$ is represented by block matrices of the form

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & Z & z \\ X & A & \mathbb{J}Z^t \\ x & -X^t\mathbb{J} & -a \end{pmatrix} : A \in \mathfrak{sp}(2n, \mathbb{R}) \right\},$$

where the non-specified entries are arbitrary, i.e. $x, a, z \in \mathbb{R}$, $X \in \mathbb{R}^{2n}$ and $Z \in \mathbb{R}^{2n*}$, and the fact $A \in \mathfrak{sp}(2n, \mathbb{R})$ means that $A^t\mathbb{J} + \mathbb{J}A = 0$. Particular subspaces of the contact grading $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of $\mathfrak{g}$ correspond to the diagonal pieces of the block decomposition above. For instance, $\mathfrak{g}_{-2}$ is represented by $x \in \mathbb{R}$, $\mathfrak{g}_{-1}$ by $X \in \mathbb{R}^{2n}$, etc. In particular, $\mathfrak{g}_0$ is represented by the pairs $(a, A) \in \mathbb{R} \times \mathfrak{sp}(2n, \mathbb{R})$ so that $\mathfrak{sp}(2n, \mathbb{R})$ is the semisimple part $\mathfrak{g}_0^{ss}$ and the center $\mathfrak{z}(\mathfrak{g}_0)$ is generated by the grading element E corresponding to the pair $(1, 0)$. Following the general setup in 2.1, $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\mathfrak{p}' = \mathfrak{g}_0^{ss} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, and P, P' are the corresponding connected Lie subgroups in G. Schematically, the parabolic subgroup $P \subset G$ is given as

$$P = \left\{ \begin{pmatrix} r & * & * \\ 0 & * & * \\ 0 & 0 & r^{-1} \end{pmatrix} : r \in \mathbb{R}_+ \right\}$$

and $P' \subset P$ corresponds to $r = 1$. Easily, G acts transitively on $\mathbb{R}^{2n+2} \setminus \{0\}$, P' is the stabilizer of the first vector of the standard basis, and P is the stabilizer of the corresponding ray. Hence $\hat{\mathcal{C}} \cong G/P'$ is identified with $\mathbb{R}^{2n+2} \setminus \{0\}$ and its oriented projectivization $\mathcal{C} \cong G/P$ is further identified with the sphere $S^{2n+1} \subset \mathbb{R}^{2n+2}$. Altogether, we have interpreted the model cone for contact projective structures as

$$\hat{\mathcal{C}} \cong \mathbb{R}^{2n+2} \setminus \{0\} \to S^{2n+1} \cong \mathcal{C}.$$

The canonical symplectic form on $\hat{\mathcal{C}}$ corresponds to the standard symplectic form on $\mathbb{R}^{2n+2}$ which is G-invariant by definition. As a particular interpretation of the general definition in 3.1, the contact distribution $D \subset TS^{2n+1}$ is given by $D_v = v^\perp \cap T_vS^{2n+1}$, where $v \in S^{2n+1}$ and $v^\perp = \{x \in \mathbb{R}^{2n+2} : \hat{\Omega}(v, x) = 0\}$.

Next, let $\hat{\nabla}$ be the canonical flat connection on $\mathbb{R}^{2n+2}$. Then the connections on S^{2n+1} defined by (8) form projectively equivalent connections having the great circles as common non-parametrized geodesics. Any great circle is the intersection of S^{2n+1} with a plane passing through 0. If the plane is isotropic with respect to $\hat{\Omega}$, we end up with contact geodesics. Note that no connection in the class preserves the contact distribution, since it is obviously torsion-free, however the induced partial contact connection coincides with the restriction of an exact Weyl connection to D:

Proposition. *Let $\hat{\mathcal{C}} \to \mathcal{C}$ be the model cone for $\mathfrak{g} = \mathfrak{sp}(2n+2, \mathbb{R})$. Then $\mathcal{C} \cong S^{2n+1}$, $\hat{\mathcal{C}} \cong \mathbb{R}^{2n+2} \setminus \{0\}$, $\hat{\Omega}$ corresponds to the standard symplectic form on $\mathbb{R}^{2n+2}$, and the ambient symplectic connection $\hat{\nabla}$ from Theorem 4.3 is the canonical flat connection on $\mathbb{R}^{2n+2}$.*

Proof. Since $\hat{\mathcal{C}} \cong G/P'$, the tangent bundle $T\hat{\mathcal{C}}$ is identified with the associated bundle $G \times_{P'} (\mathfrak{g}/\mathfrak{p}')$ via the Maurer–Cartan form μ on G; the action of P' on $\mathfrak{g}/\mathfrak{p}'$ is induced from the adjoint representation. On the other hand, $\hat{\mathcal{C}} \cong \mathbb{R}^{2n+2} \setminus \{0\}$, so $\mathfrak{g}/\mathfrak{p}' \cong \mathbb{R}^{2n+2}$ as vector spaces. $\mathbb{R}^{2n+2}$ is the standard representation of G and an essential observation for the next development is that its restriction to $P' \subset G$ is isomorphic to the representation of P' on $\mathfrak{g}/\mathfrak{p}'$. Explicitly, the isomorphism $\mathbb{R}^{2n+2} \to \mathfrak{g}/\mathfrak{p}'$ is given by

$$\begin{pmatrix} a \\ X \\ x \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & 0 \\ X & 0 & 0 \\ x & -X^t\mathbb{J} & -a \end{pmatrix} + \mathfrak{p}'. \tag{9}$$

Altogether, $T\hat{\mathcal{C}} \cong G \times_{P'} \mathbb{R}^{2n+2} \cong (G \times_{P'} G) \times_G \mathbb{R}^{2n+2}$, where the (homogeneous) principal bundle $G \times_{P'} G \to \hat{\mathcal{C}}$ represents the symplectic frame bundle of $\hat{\mathcal{C}}$. The

Maurer–Cartan form μ on G extends to a G-invariant principal connection on $G\times_{P'} G$. The later connection induces connections on all associated bundles, in particular, this gives rise to an invariant symplectic connection on $T\hat{\mathcal{C}}$. Such a connection is unique, hence this is the canonical flat connection $\hat{\nabla}$ on $\mathbb{R}^{2n+2}$.

Due to this interpretation of $\hat{\nabla}$, we are going to describe the covariant derivative with respect to $\hat{\nabla}$ in an alternative way which will provide a comparison of the pull-back and exact Weyl connections. For this purpose, let $\hat{X},\hat{Y}\in\mathfrak{X}(\hat{\mathcal{C}})$ be vector fields on $\hat{\mathcal{C}}$ [1]. As in general, the subalgebra $\mathfrak{g}_-\oplus\langle E\rangle\subset\mathfrak{g}$ is isomorphic to $\mathfrak{g}/\mathfrak{p}'\cong\mathbb{R}^{2n+2}$ as a vector space. The action of P' on $\mathfrak{g}/\mathfrak{p}'\cong\mathbb{R}^{2n+2}$ turns $\mathfrak{g}_-\oplus\langle E\rangle$ into a P'-module. Further, the splitting $\mathfrak{g}=(\mathfrak{g}_-\oplus\langle E\rangle)\oplus\mathfrak{p}'$ and the Maurer–Cartan form $\mu\in\hat{\Omega}^1(G,\mathfrak{g})$ gives rise to a general connection on the principal bundle $G\to\hat{\mathcal{C}}$. The horizontal lift $\mu^{-1}(\hat{X})\in\mathfrak{X}(G)$ of $\hat{X}\in\mathfrak{X}(\hat{\mathcal{C}})$ is defined by the equation $\mu(\mu^{-1}(\hat{X})(g))=\hat{X}(g)$ for any $g\in G$, where $\hat{X}$ on the right hand stands for the frame form with values in $\mathfrak{g}_-\oplus\langle E\rangle\subset\mathfrak{g}$. The (frame form of the) covariant derivative of $\hat{Y}$ in the direction of $\hat{X}$ turns out to be expressed as

$$\hat{\nabla}_{\hat{X}}\hat{Y}=T\hat{Y}\cdot\mu^{-1}(\hat{X})+\lambda'(\hat{X})(\hat{Y}), \tag{10}$$

where $\hat{Y}$ is viewed as the frame form with values in $\mathbb{R}^{2n+2}$ and λ' denotes the standard representation $\mathfrak{g}\to\mathfrak{gl}(\mathbb{R}^{2n+2})$; see [11] for details.

From now on, let $s:\mathcal{C}\to\hat{\mathcal{C}}$ be a fixed section of the model cone, i.e. a choice of scale. This provides identifications $T\mathcal{C}\cong\mathcal{G}_0'\times_{G_0'}\mathfrak{g}_-$ and $T\hat{\mathcal{C}}\cong\mathcal{G}_0'\times_{G_0'}(\mathfrak{g}_-\oplus\langle E\rangle)$, where $\mathcal{G}_0'$ is the principal G_0'-bundle as in 2.3. In the definition of the pull-back connection, $X^{hor}\in\mathfrak{X}(\hat{\mathcal{C}})$ denotes the horizontal lift of vector field $X\in\mathfrak{X}(\hat{\mathcal{C}})$ with respect to the principal connection on $\hat{\mathcal{C}}$ determined by s. According to the identifications above, the corresponding frame forms are related as $X^{hor}(u_0)=(0,X(u_0))^t\in\mathbb{R}^{2n+2}\cong\langle E\rangle\oplus\mathfrak{g}_-$, for any $u_0\in\mathcal{G}_0'$. Altogether, for $X,Y\in\mathfrak{X}(\hat{\mathcal{C}})$, the description of $X^{hor},Y^{hor}\in\mathfrak{X}(\hat{\mathcal{C}})$ and the formula (10) yield

$$\hat{\nabla}_{X^{hor}}Y^{hor}=\begin{pmatrix}0\\ TY\cdot\mu^{-1}(X)\end{pmatrix}+\lambda'(X)\begin{pmatrix}0\\ Y\end{pmatrix}. \tag{11}$$

The tangent map of the projection $p:\hat{\mathcal{C}}\to\mathcal{C}$ corresponds to the projection $\pi:\mathfrak{g}_-\oplus\langle E\rangle\to\mathfrak{g}_-$ in the direction of $\langle E\rangle$, hence the result of the covariant derivative ∇^s_XY with respect to the pull-back connection defined by (8) corresponds to the $\mathfrak{g}_-$ part of (11).

On the other hand, the exact Weyl connection corresponding to the choice of scale s is given by the formula

$$\bar{\nabla}^s_XY=TY\cdot\mu^{-1}(X)-\mathrm{ad}(\mathsf{P}^s(X))(Y), \tag{12}$$

see [11, 2.16]. Altogether, the desired comparison of the pull-back connection and the exact Weyl connection determined by s is given by

$$\nabla^s_XY-\bar{\nabla}^s_XY=\pi\left(\lambda'(X+\mathsf{P}^s(X))\begin{pmatrix}0\\ Y\end{pmatrix}\right), \tag{13}$$

where π denotes the projection $\mathfrak{g}_-\oplus\langle E\rangle\to\mathfrak{g}_-$ as before and $(0,\mathrm{ad}(\mathsf{P}^s(X))(Y))^t=\lambda'(\mathsf{P}^s(X))(0,Y)^t$ is easily satisfied according to the identification in (9). In particular, expressing the standard action on the right hand side of (13) for $X,Y\in\Gamma(D)$, the difference tensor turns out to be of the form

$$\nabla^s_XY-\bar{\nabla}^s_XY=-d\theta_s(X,Y)R_s \tag{14}$$

[1] Be aware we will often denote by the same symbols the corresponding frame forms from $C^\infty(G,\mathfrak{g}/\mathfrak{p}')^{P'}$. The meaning of the symbols will be clear from context.

where θ_s and R_s is the contact form and the Reeb vector field, respectively, corresponding to the scale $s : \mathcal{C} \to \hat{\mathcal{C}}$. This shows that the induced partial contact connections of the pull-back connection and the exact Weyl connection determined by s coincide. □

4.5. CR structures of hypersurface type. These structures correspond to the contact grading of the Lie algebra $\mathfrak{g} = \mathfrak{su}(p+1, q+1)$, a real form of $\mathfrak{sl}(n+2, \mathbb{C})$, where $p + q = n$ once for all. In fact the correct full name of the general geometric structure of this type is *non-degenerate partially integrable almost CR structure of hypersurface type*. This structure on a smooth manifold M is given by a contact distribution $D \subset TM$ with a complex structure $J : D \to D$ so that the Levi bracket $\mathcal{L} : D \wedge D \to TM/D$ is compatible with the complex structure, i.e. $\mathcal{L}(J-, J-) = \mathcal{L}(-, -)$ for any $-, - \in \Gamma(D)$. A choice of contact form provides an identification of T_xM/D_x, over any $x \in M$, with $\mathbb{R}$ and the later condition on Levi bracket says that $\mathcal{L}(-, J-)$ is a non-degenerate symmetric bilinear form on D, that is a pseudometric. Hence $\mathcal{L}(-, J-) + i\mathcal{L}(-, -)$ is a Hermitean form on D whose signature (p, q) is the *signature* of the CR structure.

The classical examples of CR structures of the above type are induced on nondegenerate real hypersurfaces in $\mathbb{C}^{n+1}$. In general, for a real submanifold $M \subset \mathbb{C}^{n+1}$, the CR structure on M is induced from the ambient complex space $\mathbb{C}^{n+1}$ so that the distribution D is the maximal complex subbundle in TM, and the complex structure J is the restriction to D of the multiplication by i. The model CR structures of hypersurface type are induced on the so called *hyperquadrics.*A typical hyperquadric of signature (p, q) is described as a graph

$$\mathcal{Q} := \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \Im(w) = h(z, z)\}, \tag{15}$$

or as

$$\mathcal{S} := \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : h(z, z) + |w|^2 = 1\}, \tag{16}$$

where h is a Hermitean form of signature (p, q). It turns out that the induced CR structures on $\mathcal{Q}$ and $\mathcal{S}$ are equivalent and the equivalence is established by the restriction of the biholomorphism $(z, w) \mapsto \left(\frac{z}{w-i}, \frac{1-iw}{w-i}\right)$. Note that this identification is almost global (only the point $(0, i) \in \mathcal{S}$ is mapped to infinity) and projective. In particular, $\mathcal{Q}$ and $\mathcal{S}$ are different affine realizations of a projective hyperquadric in $\mathbb{CP}^{n+1}$ which is identified with the homogeneous space G/P as follows.

Let G be the group of complex linear automorphisms of $\mathbb{C}^{n+2}$ preserving a Hermitean form H of signature $(p + 1, q + 1)$, i.e. $G := SU(p + 1, q + 1)$. Let the Hermitean form H be given by the matrix $\begin{pmatrix} 0 & 0 & -\frac{i}{2} \\ 0 & \mathbb{I} & 0 \\ \frac{i}{2} & 0 & 0 \end{pmatrix}$, with respect to the standard basis $(e_0, e_1, \ldots, e_n, e_{n+1})$, where $\mathbb{I} = \begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & -\mathbb{I}_q \end{pmatrix}$ represents the Hermitean form h of signature (p, q) on $\langle e_1, \ldots, e_n \rangle \subset \mathbb{C}^{n+2}$. According to this choice, the Lie algebra $\mathfrak{g} = \mathfrak{su}(p + 1, q + 1)$ is represented by matrices of the following form with blocks of sizes 1, n, and 1

$$\mathfrak{g} = \left\{ \begin{pmatrix} c & 2iZ & v \\ X & A & \mathbb{I}\bar{Z}^t \\ u & -2i\bar{X}^t\mathbb{I} & -\bar{c} \end{pmatrix} : u, v \in \mathbb{R}, A \in \mathfrak{u}(p, q), \operatorname{tr}(A) + 2i\Im(c) = 0 \right\},$$

where the non-specified entries are arbitrary, i.e. $X \in \mathbb{C}^n, Z \in \mathbb{C}^{n*}$, and $c \in \mathbb{C}$. (Note that $A \in \mathfrak{u}(p, q)$ means $\bar{A}^t\mathbb{I} + \mathbb{I}A = 0$, so in particular $\operatorname{tr}(A)$ is purely imaginary complex number.) The contact grading of $\mathfrak{g}$ is read along the diagonals as in

4.4. In particular, $\mathfrak{g}_0$ is represented by the pairs $(c, A) \in \mathbb{C} \times \mathfrak{u}(p,q)$ with the constrain $\operatorname{tr}(A) + 2i\Im(c) = 0$. The center $\mathfrak{z}(\mathfrak{g}_0)$ is two-dimensional, where the grading element E corresponds to the pair $(1,0)$, and the semisimple part $\mathfrak{g}_0^{ss}$ is isomorphic to $\mathfrak{su}(p,q)$. The subalgebra $\mathfrak{g}_0' \cong \mathfrak{u}(p,q)$ corresponds to the pairs of the form $(-\frac{1}{2}\operatorname{tr}(A), A)$. Subalgebras $\mathfrak{p}' \subset \mathfrak{p} \subset \mathfrak{g}$ are defined as in 2.1, $P' \subset P$ are the corresponding connected subgroups in G. The parabolic subgroup $P \subset G$ is schematically indicated as

$$P = \left\{ \begin{pmatrix} re^{i\theta} & * & * \\ 0 & * & * \\ 0 & 0 & \frac{1}{r}e^{i\theta} \end{pmatrix} : r \in \mathbb{R}_+ \right\}$$

and $P' \subset P$ corresponds to $r = 1$.

Let $\mathcal{N}$ be the set of non-zero null-vectors in $\mathbb{C}^{n+2}$ with respect to the Hermitean form H. Clearly, G preserves (and acts transitively on) $\mathcal{N}$. If $Q \subset G$ denotes the stabilizer of the first vector of the standard basis then $\mathcal{N}$ is identified with the homogeneous space G/Q. Obviously $Q \subset P' \subset P$ corresponds to $r = 1$ and $\theta = 0$ according to the description of P above. Since $P'/Q \cong U(1)$, the group of complex numbers of unit length, the homogeneous space G/P' is identified with $\mathcal{N}/U(1)$. Next $P \supset P'$ is the stabilizer of the complex line generated by the first vector of the standard basis, so the homogeneous space G/P is identified with $\mathcal{N}/\mathbb{C}^*$, the complex projectivization of $\mathcal{N}$. Altogether a natural interpretation of the model cone in this case is

$$\hat{\mathcal{C}} \cong \mathcal{N}/U(1) \to \mathcal{N}/\mathbb{C}^* \cong \mathcal{C}.$$

A direct substitution shows that the hyperquadric $\mathcal{Q}$ from (15) is the intersection of $\mathcal{N}$ with the complex hyperplane $z_0 = 1$. According to the new basis $(e_0 + ie_{n+1}, e_1, \dots, e_n, e_0 - ie_{n+1})$ of $\mathbb{C}^{n+2}$, the Hermitean metric H is in the diagonal form and the hyperquadric $\mathcal{S}$ from (16) is the intersection of $\mathcal{N}$ with the complex hyperplane $z_0' = 1$ (where the dash refers to coordinates with respect to the new basis). This recovers the identification above, in particular, both $\mathcal{Q}$ and $\mathcal{S}$ are identified with $\mathcal{N}/\mathbb{C}^* \cong \mathcal{C}$.

From now on, let $\mathcal{C}$ be the hyperquadric $\mathcal{S}$ in the hyperplane $z_0' = 1$ which we naturally identify with $\mathbb{C}^{n+1}$. This hyperplane without the origin is further identified with $\mathcal{N}/U(1) \cong \hat{\mathcal{C}}$ under the map $(z', w') \mapsto (h(z', z') + |w'|^2, z', w')$. Denote by $\hat{h}$ the induced Hermitean metric (of signature $(p+1, q)$) on this hyperplane and let $\hat{\Omega}$ be its imaginary part. Obviously, both $\hat{h}$ and $\hat{\Omega}$ are G-invariant, so the latter corresponds to the canonical symplectic form by general principles we mentioned in remark 3.1(b). Altogether, the defining equation (16) for $\mathcal{S} \subset \mathbb{C}^{n+1}$ reads as

$$\mathcal{S} = \{ z \in \mathbb{C}^{n+1} : \hat{h}(z, z) = 1 \} \tag{17}$$

and the most satisfactory interpretation of the model cone is

$$\hat{\mathcal{C}} \cong \mathbb{C}^{n+1} \setminus \{0\} \to \mathcal{S} \cong \mathcal{C}.$$

Proposition. *Let $\hat{\mathcal{C}} \to \mathcal{C}$ be the model cone for $\mathfrak{g} = \mathfrak{su}(p+1, q+1)$. Then $\hat{\mathcal{C}} \cong \mathbb{C}^{n+1} \setminus \{0\}$ and $\mathcal{C} \cong \mathcal{S}$, the hyperquadric in $\mathbb{C}^{n+1} \setminus \{0\}$ given by* (17)*, where $\hat{h}$ is a Hermitean metric of signature $(p+1, q)$. Further, $\hat{\Omega}$ corresponds to the imaginary part of $\hat{h}$ and the ambient symplectic connection $\hat{\nabla}$ from Theorem 4.3 is the canonical flat connection on $\mathbb{C}^{n+1}$.*

Proof. The connection $\hat{\nabla}$ is obviously symplectic, i.e. $\hat{\nabla}$ is torsion-free and $\hat{\nabla}\hat{\Omega} = 0$. By definition, $\hat{\Omega}$ is the imaginary part of the Hermitean metric $\hat{h}$ on $\mathbb{C}^{n+1}$. Its real part $\hat{g}$ is then expressed in terms of $\hat{\Omega}$ and the standard complex structure on $\mathbb{C}^{n+1}$ as $\hat{g} = \hat{\Omega}(-, i-)$. This is a real pseudo-metric on $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ of signature $(2p+2, 2q)$ and $\hat{\nabla}$ can be seen as the Levi-Civita connection of $\hat{g}$.

As in general, let $\hat{\alpha} := \hat{E} \lrcorner \hat{\Omega}$. Let $s : \mathcal{S} \to \mathbb{C}^{n+1} \setminus \{0\}$ be a section of the cone and let $\theta := s^* \hat{\alpha}$ be the corresponding contact one-form on $\mathcal{S}$. Then $g := d\theta(-, i-)$ is a non-degenerate symmetric bilinear form on the contact distribution D which has to be preserved by the Weyl connection $\bar{\nabla}^s$. Next, since we deal with the homogeneous model, the contact torsion of $\bar{\nabla}^s$ vanishes. In fact, the corresponding partial contact connection on D is uniquely determined by the fact that (i) it leaves g to be parallel and (ii) its contact torsion vanishes.

In order to prove the statement, it suffices to show that (i) and (ii) is satisfied also by the partial contact connection induced by the pull-back connection ∇^s corresponding to s. However, since $\hat{\nabla}$ is trosion-free, the pull-back connection ∇^s is torsion-free as well, hence the condition (ii) is satisfied trivially. The condition (i) follows as follows: For $X, Y, Z \in \Gamma(D)$, expand

$$(\nabla^s_X g)(Y, Z) = X \cdot d\theta(Y, iZ) - d\theta(\nabla^s_X Y, iZ) - d\theta(Y, i\nabla^s_X Z).$$

Since $\theta = s^* \hat{\alpha}$ and $d\hat{\alpha} = \hat{\Omega}$, by naturality of exterior differential we have $d\theta = s^* \hat{\Omega}$. Next easily, $T_x s \cdot X = X^{hor}|_{s(x)}$ and, by the definition of the pull-back connection in 4.3, $T_x s \cdot (\nabla^s_X Y)|_x = (\hat{\nabla}_{X^{hor}} Y^{hor} + f\hat{E})|_{s(x)}$, for any $x \in \mathcal{S}$ and some function f on $\mathcal{S}$. The previous expression is then rewritten as

$$\begin{aligned} X \cdot \hat{\Omega}(Ts \cdot Y, Ts \cdot iZ) - \hat{\Omega}(Ts \cdot \nabla^s_X Y, Ts \cdot iZ) - \hat{\Omega}(Ts \cdot Y, Ts \cdot i\nabla^s_X Z) = \\ = X^{hor} \cdot \hat{\Omega}(Y^{hor}, iZ^{hor}) - \hat{\Omega}(\hat{\nabla}_{X^{hor}} Y^{hor}, iZ^{hor}) - \hat{\Omega}(Y^{hor}, i\hat{\nabla}_{X^{hor}} Z^{hor}). \end{aligned}$$

But the latter expression is just $(\hat{\nabla}_{X^{hor}} \hat{g})(Y^{hor}, Z^{hor})$ which vanishes trivially by definitions. □

4.6. Lagrangean contact structures. Lagrangean contact structures correspond to the contact grading of $\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{R})$, another real form of $\mathfrak{sl}(n+2, \mathbb{R})$. *Lagrangean contact strucure* on a smooth manifold M consists of the contact distribution $D \subset TM$ and a fixed decomposition $D = L \oplus R$ so that the subbundles L and R are *Lagrangean*, i.e. isotropic with respect to the Levi bracket $\mathcal{L} : D \wedge D \to TM/D$. These structures was profoundly studied in [12] where we refer for a lot of details. The model Lagrangean contact structure appears on the projectivization of the cotangent bundle of real projective space; let us present the algebraic background first.

The contact grading of $\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{R})$ is read diagonally as in 4.4 and 4.5 from the following block decomposition

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & Z_1 & z \\ X_1 & B & Z_2 \\ x & X_2 & c \end{pmatrix} : a + \operatorname{tr}(B) + c = 0 \right\},$$

where as usual the non-specified entries are arbitrary, i.e. $x, a, c, z \in \mathbb{R}$, $X_1, Z_2 \in \mathbb{R}^n$, $X_2, Z_1 \in \mathbb{R}^{n*}$, and $B \in \mathfrak{gl}(n, \mathbb{R})$. The subalgebra $\mathfrak{g}_0$ is represented by the triples $(a, B, c) \in \mathbb{R} \times \mathfrak{gl}(n, \mathbb{R}) \times \mathbb{R}$ so that $a + \operatorname{tr}(B) + c = 0$. The center $\mathfrak{z}(\mathfrak{g}_0)$ is two-dimensional and the grading element E corresponds to $(1, 0, -1)$. The semisimple part $\mathfrak{g}_0^{ss}$ is isomorphic to $\mathfrak{sl}(n, \mathbb{R})$ and the subalgebra $\mathfrak{g}_0' \cong \mathfrak{gl}(n, \mathbb{R})$ is represented by all triples of the form $\left(-\frac{1}{2}\operatorname{tr}(B), B, -\frac{1}{2}\operatorname{tr}(B)\right)$. The subspace $\mathfrak{g}_{-1}$ defining the contact distribution is split as $\mathfrak{g}_{-1} = \mathfrak{g}^L_{-1} \oplus \mathfrak{g}^R_{-1}$, where $\mathfrak{g}^L_{-1}$ is represented by $X_1 \in \mathbb{R}^n$ and $\mathfrak{g}^R_{-1}$ by $X_2 \in \mathbb{R}^{n*}$, so that this splitting is invariant under the adjoint action of $\mathfrak{g}_0$. Furthermore, the subspaces $\mathfrak{g}^L_{-1}$ and $\mathfrak{g}^R_{-1}$ are isotropic w.r. to the bracket $[\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$. Similarly, $\mathfrak{g}_1$ splits as $\mathfrak{g}^L_1 \oplus \mathfrak{g}^R_1$. The subalgebras $\mathfrak{p}' \subset \mathfrak{p} \subset \mathfrak{g}$ are given as before. Let G be the group $SL(n+2, \mathbb{R})$. The connected parabolic subgroup $P \subset G$ corresponding to $\mathfrak{p} \subset \mathfrak{g}$ is schematically indicated as

$$P = \left\{ \begin{pmatrix} pq & * & * \\ 0 & * & * \\ 0 & 0 & \frac{p}{q} \end{pmatrix} : p, q \in \mathbb{R}_+ \right\}$$

and $P' \subset P$ corresponds to $q = 1$.

The homogeneous space G/P is naturally identified with the set of flags of half-lines in hyperplanes in $\mathbb{R}^{n+2}$. Indeed, the standard action of G on $\mathbb{R}^{n+2}$ descends to a transitive action both on rays and hyperplanes in $\mathbb{R}^{n+2}$, so G acts transitively on the set of flags of above type. The subgroup P is the stabilizer of the flag $\ell \subset \rho$ where ℓ and ρ is the ray and the hyperplane generated by the first and the first $n+1$ vectors from the standard basis, respectively. Obviously, $P = \tilde{P} \cap \bar{P}$ where $\tilde{P}$ is the stabilizer of ℓ and $\bar{P}$ stabilizes ρ. Note that both $\tilde{P}$ and $\bar{P}$ are also parabolic.

We claim that $G/P \cong \mathcal{P}^o(T^*S^{n+1})$ which is the oriented projectivization of the cotangent bundle of projective sphere, the oriented projectivization of $\mathbb{R}^{n+2}$. This can be clarified as follows: The projective sphere $S^{n+1} \cong \mathcal{P}^o(\mathbb{R}^{n+2})$ is identified with $G/\tilde{P}$, where $\tilde{P} \subset G$ is the stabilizer of the ray ℓ as above. Let $\tilde{\mathfrak{p}} \subset \mathfrak{g}$ be the Lie algebra of $\tilde{P}$ and let $\mathfrak{g} = \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$ be the corresponding grading of $\mathfrak{g}$. As usual, $(\mathfrak{g}/\tilde{\mathfrak{p}})^* \cong \tilde{\mathfrak{g}}_{-1}^* \cong \tilde{\mathfrak{g}}_1$, hence $T^*S^{n+1} \cong T^*(G/\tilde{P})$ is identified with $G \times_{\tilde{P}} \tilde{\mathfrak{g}}_1$ via the Maurer–Cartan form on G. Now, the adjoint action of $\tilde{P}$ on $\tilde{\mathfrak{g}}_1$ is transitive and an easy direct calculation shows that the stabilizer of a convenient element of $\tilde{\mathfrak{g}}_1$ is precisely $P' \subset P \subset \tilde{P}$; the subgroup $P \subset \tilde{P}$ is the stabilizer of the corresponding ray. Altogether, $\tilde{\mathfrak{g}}_1 \cong \tilde{P}/P'$ and $\mathcal{P}^o(\tilde{\mathfrak{g}}_1) \cong \tilde{P}/P$, so $T^*S^{n+1} \cong G/P'$ and $\mathcal{P}^o(T^*S^{n+1}) \cong G/P$. Hence the interpretation of the model cone for Lagrangean contact structures is

$$\hat{\mathcal{C}} \cong T^*S^{n+1} \to \mathcal{P}^o(T^*S^{n+1}) \cong \mathcal{C}$$

so that the canonical G-invariant symplectic form on $\hat{\mathcal{C}}$ corresponds to the canonical symplectic form on the cotangent bundle T^*S^{n+1}...

Now we are going to expose a general construction following [12]; it turns out this will be useful to find a candidate for the ambient connection $\hat{\nabla}$ on $\hat{\mathcal{C}} \cong T^*S^{n+1}$. Let M be a manifold with linear torsion-free connection ∇ and let $H \subset TT^*M$ be the corresponding horizontal distributions on the cotangent bundle over M. Together with the vertical subbundle V of the projection $p : T^*M \to M$ we have got an almost product structure on T^*M. Let $\hat{\alpha}$ be the canonical one-form and $\hat{\Omega} = d\hat{\alpha}$ the canonical symplectic form on T^*M. By definition of $\hat{\Omega}$, the subbundle V is isotropic w.r. to $\hat{\Omega}$. The complementary subbundle H determined by the connection ∇ is isotropic if and only if ∇ is torsion-free. After the projectivization, the decomposition $V \oplus H = TT^*M$ yields a Lagrangean contact structure on $\mathcal{P}(T^*M)$. Moreover, the almost product structure on T^*M and so the Lagrangean contact structure on $\mathcal{P}(T^*M)$ are independent on the choice of connection from the projectively equivalent class $[\nabla]$. Altogether, starting with a projective structure on a smooth manifold M, this gives rise to a Lagrangean contact structure on the projectivized cotangent bundle of M. Note that in terms of parabolic geometries, this construction is an instance of the so called correspondence space construction [2] which is formally powered by the inclusion $P \subset \tilde{P}$ of parabolic subgroups in G. As a particular implementation of a general principle, locally flat projective structure on M gives rise to a locally flat Lagrangean contact structure on $\mathcal{P}(T^*M)$. This is actually observed elementarily in the previous paragraph provided we consider oriented projectivization instead of the usual one.

Proposition. *Let $\hat{\mathcal{C}} \to \mathcal{C}$ be the model cone for $\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{R})$. Then $\hat{\mathcal{C}} \cong T^*S^{n+1}$, $\mathcal{C} \cong \mathcal{P}^o(T^*S^{n+1})$, and $\hat{\Omega}$ corresponds to the canonical symplectic form on cotangent* *bundle. Let further* $J : TT^*S^{n+1} \to TT^*S^{n+1}$ *be the almost product structure given by the projective structure on* S^{n+1} *as above. Then the bilinear form* $\hat{g} := \hat{\Omega}(-, J-)$ *on* T^*S^{n+1} *is symmetric and non-degenerate and the ambient symplectic connection* $\hat{\nabla}$ *from Theorem 4.3 is the Levi-Civita connection of* $\hat{g}$*.*

Proof. Let $S^{n+1} \subset \mathbb{R}^{n+2}$ be the standard projective sphere. The projective structure $[\nabla]$ is induced from the canonical flat connection in $\mathbb{R}^{n+2}$, in particular, any connection in the class is torsion-free. As before, this ensures that both subbundles V and H from the corresponding decomposition of TT^*S^{n+1} are isotropic with respect to the canonical symplectic form $\hat{\Omega}$. The decomposition $V \oplus H = TT^*S^{n+1}$ determines the product structure J so that V and H is the eigenspace of J corresponding to the eigenvalue 1 and -1, respectively. Since both $\hat{\Omega}$ and J are non-degenerate, the same holds true also for $\hat{g} := \hat{\Omega}(-, J-)$. Since both V and H are isotropic with respect to $\hat{\Omega}$, the bilinear form $\hat{g}$ turns out to be symmetric, hence it is a pseudo-metric on T^*S^{n+1}.

The rest of the proof is completely parallel to that in 4.5 up to the interchange between the almost complex and almost product structure on $\hat{\mathcal{C}}$ and $D \subset T\mathcal{C}$, respectively. □

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