Information Sciences 179 (2009) 1278–1297

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

I-Fuzzy equivalence relations and I-fuzzy partitions

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ARTICLE INFO

Article history:
Received 11 August 2008
Received in revised form 18 December 2008
Accepted 25 December 2008

Keywords:
Implicator
Conjunctor
Fuzzy equivalence relation
Fuzzy partition
Semi-copula

ABSTRACT

A T-fuzzy equivalence relation is a fuzzy binary relation on a set $\mathbb{X}$ which is reflexive, symmetric and T-transitive for a t-norm T. Recently, Mesiar et al. [R. Mesiar, B. Reusch, H. Thiele, Fuzzy equivalence relations and fuzzy partitions, J. Multi-Valued Logic Soft Comput. 12 (2006) 167–181] have generalised the t-norm T to any general conjunctor C and investigated the minimal assumptions required on such operations, called *duality fitting conjunctors*, such that the fuzzification of the equivalence relation admits any value from the unit interval and also the one–one correspondence between the fuzzy equivalence relations and fuzzy partitions is preserved. In this work, we conduct a similar study by employing a related form of C-transitivity, viz., I-transitivity, where I is an implicator. We show that although every I-fuzzy equivalence relation can be shown to be a C-fuzzy equivalence relation, there exist C-fuzzy equivalence relations that are not I-fuzzy equivalence relations and hence these concepts are not equivalent. Most importantly, we show that the class of *duality fitting implicators* I is much richer than the residuals of the duality fitting conjunctors in the study of Mesiar et al. We also show that the I-fuzzy partitions have a "constantwise" structure.

1. Introduction

1.1. Equivalence relations and partitions

Let $\mathbb{X}$ be a non-empty set. An equivalence relation on $\mathbb{X}$ is a binary relation on $\mathbb{X}$ that is reflexive, symmetric and transitive. Every equivalence relation gives rise to a partition on the underlying set $\mathbb{X}$. Formally, a partition of a set $\mathbb{X}$ is a collection P of pairwise disjoint subsets of $\mathbb{X}$ whose union is equal to $\mathbb{X}$. Once again, every partition of $\mathbb{X}$ gives rise to an equivalence relation on $\mathbb{X}$.

Let us denote the set of all equivalence relations by $\mathscr{E}(\mathbb{X})$ and the class of all partitions on $\mathbb{X}$ by $\mathscr{P}(\mathbb{X})$. Then, we have the following relations:

- Let $E \in \mathscr{E}(\mathbb{X})$. For every $x \in \mathbb{X}$, let $E_x = \{y \in \mathbb{X} | (x, y) \in E\}$. Then
$$P_E = \{E_x | x \in \mathbb{X}\} \in \mathscr{P}(\mathbb{X}). \tag{1}$$
- Let $P \in \mathscr{P}(\mathbb{X})$. Then
$$E_P = \{(x, y) \in \mathbb{X}^2 | \{x, y\} \in U \text{ for some } U \in P\} \in \mathscr{E}(\mathbb{X}). \tag{2}$$

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doi:10.1016/j.ins.2008.12.027

In fact, the following one–one correspondence between partitions and equivalence relations exists: $E_{P_E} = E$ and $P_{E_P} = P$ for all $E \in \mathscr{E}(\mathbb{X})$ and $P \in \mathscr{P}(\mathbb{X})$, i.e., the relationships (1) and (2) define a bijection between the equivalence relations and partitions.

1.2. Fuzzy equivalence relations and fuzzy partitions

A fuzzification of the above concepts, viz., fuzzy equivalence relation and fuzzy partition, have been dealt with in many works. In fact, the first such definition was proposed by Zadeh himself [60], wherein he proposed the following definition, which have now come to be termed as *similarity relations*.

Definition 1.1. A fuzzy subset E of the Cartesian product $\mathbb{X}^2$ is called a fuzzy equivalence relation on $\mathbb{X}$ if the following properties are satisfied for all $x, y, z \in \mathbb{X}$:

$$\begin{aligned} &E(x,x) = 1 && \text{(Reflexivity)},\\ &E(x,y) = E(y,x) && \text{(Symmetry)},\\ &E(x,z) \geqslant \max_{w \in \mathbb{X}} \min(E(x,w), E(w,z)) && \text{(Transitivity)}. \end{aligned}$$

Of course, the transitivity of E can be, and usually is, written equivalently as follows:

$$\min(E(x,y), E(y,z)) \leqslant E(x,z), \quad x, y, z \in \mathbb{X}. \tag{3}$$

Following Zadeh [60], many works have appeared generalising the above concept of a fuzzy equivalence relation. For instance, many types of reflexivity have been proposed and discussed, see for instance, [7,8,26]. Similarly, other types of transitivity have also been discussed. Bezdek and Harris [2] replaced the operator "min" in (3) by the Łukasiewicz t-norm $T_{\mathbf{LK}}(x,y) = \max(0, x + y - 1)$, while Faurous and Fillard [21] substituted it by the product t-norm $T_{\mathbf{P}}(x,y) = x \cdot y$. Further works have started to consider any general t-norm T instead of "min" (see [17,18,30–35]). Other generalisations have also appeared, see for example, [26,23,28,29].

Analogously, the concept of a fuzzy partition is a generalisation of the parition of a set $\mathbb{X}$. Ruspini [52] was the first to propose one such generalisation. Butnariu [6] proposed another generalisation of the concept of a fuzzy partition based on an equivalent definition of the disjointedness using the Łukasiewicz operators, which was subsequently shown by Mesiar and Rybárik [43] to be equivalent to that of the definition of Ruspini. For works dealing with either different or more general definitions see, for instance, [3–5,16,44,45]. The following is an equivalent form of the above definitions that is more or less established in the literature.

Definition 1.2 (*cf.* [27,29,50,59]). A system P of fuzzy subsets of $\mathbb{X}$ is called a *fuzzy partition* of $\mathbb{X}$ if the following properties are satisfied:

(i) For all $U \in P$ there is *some* $x \in \mathbb{X}$ such that $U(x) = 1$,
(ii) For all $x \in \mathbb{X}$ there is *exactly* one $U \in P$ such that $U(x) = 1$,
(iii) If $U, V \in P$ such that $U(x) = V(y) = 1$ for some $x, y \in \mathbb{X}$, then $U(y) = V(x)$.

1.3. Motivation for this work

Many of the works on fuzzy equivalence relation and fuzzy partition deal either exclusively with fuzzy equivalence relations [7–9,12,14,21,24–26,31,32,36,47,48,56,57] or with fuzzy partitions [3–6,13,16,17,19,11,30,38,39,41,45,52]. The correspondence between fuzzy equivalence relations and fuzzy partitions was discussed, e.g. in [12,22,33,42,44,46,49,50], and a one–one correspondence between them explicitly stated in [53–55]. However, all these generalisations only consider a t-norm to model the fuzzy transitivity in (3) in the concept of fuzzy equivalence relation and the fuzzy disjointedness in the concept of fuzzy partition.

Recently, Mesiar et al. [42] – noticing that the associativity of a t-norm is superfluous in the above context, especially since we never have to aggregate more than two arguments – have substituted a *conjunctor* instead of a t-norm in the above setting and determined the minimal properties required on such operations so that the following properties of fuzzy equivalences and fuzzy partitions are preserved:

Fuzzification: The concepts are such that each value $a \in [0, 1]$ is acceptable as a membership value of a fuzzy equivalence relation (respectively, a member of a fuzzy partition);

Fuzzy extension: Fuzzification of these concepts is such that each equivalence (partition) is also a fuzzy equivalence relation (respectively, fuzzy partition), and vice-versa, each fuzzy equivalence relation which is (respectively, each fuzzy partition with members only) is an equivalence relation (respectively, a partition);

Fitting fuzzy extension: Fuzzy extension of these concepts is such that there is a bijection between the class $\mathscr{F}\mathscr{E}(\mathbb{X})$ of all fuzzy equivalence relations on $\mathbb{X}$ and the class $\mathscr{F}\mathscr{P}(\mathbb{X})$ of all fuzzy partitions on $\mathbb{X}$.

An alternative approach based on implications, which are rather general and not necessarily the residuals of some t-norms, has been considered in [54,55]. In this work, we continue in the spirit of Mesiar et al. [42] with a general implicator, i.e., we investigate the concepts of fuzzy equivalence relation and fuzzy partition by employing a related form of C-transitivity, viz., I-transitivity, where the implicator I is nothing more than a binary operator satisfying the boundary conditions of a implication, as given below.

Definition 1.3. An implicator $I : [0,1]^2 \to [0,1]$ is a mapping whose restriction to the domain {0, 1} coincides with the binary implication, i.e.,

$$I(1,0) = 0; \quad I(0,0) = I(0,1) = I(1,1) = 1.$$

This work differs from [53–55] in that we do not suppose any kind of monotonicity for implicators, while in [53–55] the hybrid monotonicity – decreasing in the first and increasing in the second coordinates – is assumed.

1.4. Organisation of this work

This paper is organised as follows. In Section 2 we introduce the concept of an I-fuzzy equivalence relation and investigate the properties required on a general implicator I so that the 3 properties mentioned in Section 1.3 and considered in the work of Mesiar et al. [42] are satisfied for any I-fuzzy equivalence relation. In Section 3 after introducing the concept of an I-fuzzy partition, similar investigations are carried out. In particular, we show that the conditions required on an implicator I so that a fuzzy relation E on a set $\mathbb{X}$ is an I-fuzzy equivalence relation is different from the conditions required on I for a fuzzy partition P on $\mathbb{X}$ to be an I-fuzzy partition.

In Section 4, we discuss the correspondence between the above concepts of I-fuzzy equivalence relation and I-fuzzy partition and determine the minimal conditions on an implicator to be a *duality fitting implicator*, i.e., an implicator I such that every I-fuzzy equivalence relation is also an I-fuzzy partition and vice-versa. Once again, our studies show that not all of the properties possessed by residuals of left-continuous t-norms are necessary; for example, the exchange principle does not play any role in our study. We determine the exact bounds for the class of duality fitting implicators and propose a rather special subclass of duality fitting implicators which play an important role in the rest of the work. We also show that the I-fuzzy partitions have a "constant-wise" structure.

Sections 5–7 deal with the relationships between the concepts of I-fuzzy equivalence relation and I-fuzzy partition, on the one hand, and the concepts of C-fuzzy equivalence relation and C-fuzzy partition introduced in Mesiar et al. [42], on the other.

In Section 5 we recall the main definitions and relevant results regarding C-fuzzy equivalence relation and C-fuzzy partition. Conjunctors C for whom there exists a one–one correspondence between the concepts of C-fuzzy equivalence relation and C-fuzzy partition are called *duality fitting conjunctors* in [42].

In Section 6, after noting that every I-fuzzy equivalence relation (I-fuzzy partition) is also a C-fuzzy equivalence relation (C-fuzzy partition), we attempt the converse poser, i.e., *is every C-fuzzy equivalence relation (C-fuzzy partition) also an I-fuzzy equivalence relation (I-fuzzy partition)*? Towards this end, we investigate in detail the relationship between duality fitting conjunctors and duality fitting implicators and present many results concerning them. We show that from every duality fitting implicator I we can obtain a duality fitting conjunctor C. From the obtained results we show that there exist C-fuzzy equivalence relations that are not I-fuzzy equivalence relations for any duality fitting implicator I. We also give an example to illustrate the above (see Example 6.18). The classical logic operations of residual and deresiduum play an important role in this section.

In Section 7 we investigate monotonic duality fitting operations. Note that a monotonic duality fitting conjunctor is a commutative semi-copula, while a monotonic duality fitting implicator is a fuzzy implication, which is a minimal generalisation of an implicator on {0, 1} to the unit interval [0, 1]. We give the minimal assumptions required on a conjunctor so that its residual is a fuzzy implication. From the obtained relationships on the bounds of such operators, we show that the class of duality fitting implicators is much richer than the class of residuals obtained from duality fitting conjunctors.

2. *I*-fuzzy equivalence relation

In this section, we define the concept of an I-fuzzy equivalence relation and show that not all implicators give rise to I-fuzzy equivalence relations that accept any value $a \in [0, 1]$. Similarly, not all implicators give rise to *consistent* I-fuzzy equivalence relations (see Definition 2.6). Hence, we investigate the conditions required on an implicator I for it to satisfy the above properties.

Definition 2.1. Let I be an implicator. A fuzzy subset E of the Cartesian product $\mathbb{X}^2$ is called an I-fuzzy equivalence relation on $\mathbb{X}$ if the following properties are satisfied for all $x, y, z \in \mathbb{X}$:

(i) E is reflexive, i.e., $E(x, x) = 1$,
(ii) E is symmetric, i.e., $E(x, y) = E(y, x)$,
(iii) E is I-transitive, i.e., $I(E(x, y), E(y, z)) \geqslant E(x, z)$.

Example 2.2. Not all implicators I give rise to I-fuzzy equivalence relations, i.e., some implicators are such that they allow $E(x, y) \in \{0, 1\}$ only, which is hardly fuzzy! To see this, consider an implicator I such that $I(x, y) = 0$, when $x, y \notin \{0, 1\}$. If $\mathbb{X}$ is non-singleton and $E(x, y) = a \in (0, 1)$, we have

$$I(E(x,y), E(x,x)) = I(a,1) = 0 \ngeq a = E(x,y).$$

Remark 2.3. It should be emphasised that each equivalence relation can be understood as an I-fuzzy equivalence relation with respect to an arbitrary implicator I. Conversely, each I-fuzzy equivalence relation that takes values only from the set $\{0, 1\}$ can be understood as an equivalence relation.

We say an I-fuzzy equivalence relation admits a value $a \in [0, 1]$, if there exists a fuzzy equivalence relation E such that $E(x, y) = a$ for some $x, y \in \mathbb{X}$.

Proposition 2.4. *For a given implicator I the following are equivalent:*

(i) *The concept of I-fuzzy equivalence relation admits any value $a \in [0, 1]$.*
(ii) *I satisfies the following:*
  (a) $I(1, a) \geqslant a$,
  (b) $I(a,1) \geqslant a$,
  (c) $I(a, a) = 1$.

Proof

(i)⇒ (ii): Let there exist a fuzzy equivalence relation E such that $E(x, y) = a \in (0, 1)$. Then
  (a) $I(1,a) = I(E(x,x), E(x,y)) \geqslant E(x,y) = a$,
  (b) $I(a,1) = I(E(x,y), E(x,x)) \geqslant E(x,y) = a$,
  (c) $I(a,a) = I(E(x,y), E(x,y)) \geqslant E(x,x) = E(y,y) = 1$.

(ii)⇒ (i): Let $\mathbb{X}$ be any non-singleton set. Define the fuzzy subset on $\mathbb{X}^2$ as follows: For an $x \neq y \in \mathbb{X}$ let $E(x,y) = E(y,x) = a \in (0,1)$, and $E(u, v) = 1$ for all $(u, v) \neq (x, y) \in \mathbb{X}^2$. Clearly, E is an I-fuzzy equivalence relation. □

In the case of equivalence relations, we know that if there exist $x, y \in \mathbb{X}$ and $x \neq y$ such that $E(x, y) = 1$, i.e., x and y are related to each other, then the equivalence classes of x and y are identical. However, it is possible that an I-fuzzy equivalence relation E w.r.to an implicator I is such that $E(x, y) = 1$ but $E(x, z) \neq E(y, z)$, for some $z \in \mathbb{X}$, as the following example illustrates.

Example 2.5. Consider the following implicator I:

$$I(p,q) = \begin{cases} 0, & \text{if } (p,q) = (1,0), \\ 1, & \text{otherwise}, \end{cases} \qquad p, q \in [0,1].$$

Note that I satisfies the properties in Proposition 2.4, but displays the rather undesirable property mentioned above. For any a, $b \in (0, 1]$ such that $a \neq b$, it can be easily verified that the following is an I-fuzzy equivalence relation on the set $\mathbb{X} = \{x, y, z\}$:

E	x	y	z
x	1	1	a
y	1	1	b
z	a	b	1

Definition 2.6. A fuzzy relation E on a non-empty set $\mathbb{X}$ is said to be *consistent* if $E(x, y) = 1$ for any $x, y \in \mathbb{X}$, then $E_x = E_y$, where $E_x(z) = E(x,z)$, for every $z \in \mathbb{X}$.

Definition 2.7. An implicator I is called an *equivalence relation fitting* implicator if every E that is an I-fuzzy equivalence relation is consistent.

Proposition 2.8. *If an implicator is such that $I(1, a) = a$, for any $a \in [0, 1]$, then I is an equivalence relation fitting implicator.*

Proof. Let E be an I-fuzzy equivalence relation for a given implicator I with the above property. If possible, let for some $x \neq y \in \mathbb{X}$, $E(x, y) = 1$. We will show that for all $z \in \mathbb{X}$, we have $E(x, z) = E(y, z)$. Indeed, we have

$$E(x,z) = I(1, E(x,z)) = I(E(x,y), E(x,z)) \geqslant E(y,z),$$

while

$$E(y,z) = I(1, E(y,z)) = I(E(y,x), E(y,z)) \geqslant E(x,z),$$

from whence we get that $E_x = E_y$, i.e., I is an equivalence relation fitting implicator. □

Remark 2.9. It should be mentioned that the property $I(1, a) = a$ is not necessary for an I to be an equivalence relation fitting implicator. However, the following conditions are necessary whenever I is an equivalence relation fitting implicator:

(i) Whenever $I(1, a) = b > a$, not both $I(a, b) = 1$ and $I(b, a) = 1$,
(ii) Equivalently, if $I(a, b) = I(b, a) = 1$ with $a < b$, then $I(1, a) \neq b$,
(iii) $I(1, a) \neq 1$.

One can easily verify that the I-fuzzy equivalence relation given in Example 2.5 serves as an illustrative counter-example w.r.to the above necessary conditions.

Theorem 2.10. *For an implicator I the following are equivalent:*

(i) *I is an equivalence relation fitting implicator and the concept of I-fuzzy equivalence relation admits any value $a \in [0, 1]$.*
(ii) *I satisfies the following properties for any $a, b \in [0, 1]$:*

$$I(a,b) \geqslant a \wedge b, \quad \text{whenever } a \vee b = 1, \tag{IE1}$$

$$I(a,a) = 1, \tag{IE2}$$

$$\text{If } I(1,a) \geqslant b \quad \text{and} \quad I(1,b) \geqslant a \quad \text{and} \quad I(a,b) = 1 \quad \text{then} \quad a = b. \tag{IE3}$$

Proof

(i)⇒ (ii): If I is such that the concept of I-fuzzy equivalence relation admits any value $a \in [0, 1]$, then from Proposition 2.4 we know that I satisfies all the properties (a)–(c) there. Clearly, (IE1) is only a reformulation of properties (a) and (b), while (IE2) is the property (c). Let I be an equivalence relation fitting implicator such that $I(1, a) \geqslant b$, $I(1, b) \geqslant a$ and $I(a, b) = 1$ for some $a, b \in [0, 1]$. Let us assume to the contrary that $a \neq b$. Let us now consider the fuzzy relation E given in Example 2.5. Then it is immediate that E is an I-fuzzy equivalence relation. However, E is not consistent, contradicting the fact that I is an equivalence relation fitting implicator.

(ii)⇒ (i): From Proposition 2.4 and (IE1), (IE2) we see that the concept of I-fuzzy equivalence relation admits any value $a \in [0, 1]$. On the one hand, if for any $a \in [0, 1]$, $I(1, a) = a$ then from Proposition 2.8 we see that I is an equivalence relation fitting implicator. On the other hand, let E be an I-fuzzy equivalence relation on $\mathbb{X}$ and let $E(x, y) = 1$ for some $x, y \in \mathbb{X}$. For any arbitrary but fixed $z \in \mathbb{X}$, let $a = E(x, z)$ and $b = E(y, z)$. Then,

$$I(a,b) = I(E(x,z), E(y,z)) \geqslant E(x,y) = 1,$$
$$I(1,a) = I(E(y,x), E(x,z)) \geqslant E(y,z) = b,$$
$$I(1,b) = I(E(x,y), E(y,z)) \geqslant E(x,z) = a.$$

From (IE3) we have that $a = E(x, z) = E(y, z) = b$. Since $z \in \mathbb{X}$ was arbitrary, we have that $E_x = E_y$, i.e., E is consistent and I is an equivalence relation fitting implicator. □

3. *I*-Fuzzy partitions

In this section, we define the concept of an I-fuzzy partition and, once again, show that not all implicators give rise to I-fuzzy partitions that accept any value $a \in [0, 1]$. Similarly, not all implicators give rise to *consistent* I-fuzzy partitions (see Definition 3.5). Hence, we investigate the conditions required on an implicator I for it to satisfy the above properties.

Definition 3.1. Let I be an implicator. A system P of fuzzy subsets of $\mathbb{X}$ is called an I-fuzzy partition of $\mathbb{X}$ if the following properties are satisfied, for all $x, y, z \in \mathbb{X}$:

(i) for all $U \in P$ there is some $x \in \mathbb{X}$ such that $U(x) = 1$,
(ii) for all $x \in \mathbb{X}$ there is some $U \in P$ such that $U(x) = 1$,
(iii) if $U(x) = 1$ for some $x \in \mathbb{X}$ then for $y \in \mathbb{X}$ and all $V \in P$ the following inequality holds:

$$I(V(y), U(y)) \geqslant V(x).$$

Remark 3.2

(i) We again emphasise that each partition can be understood as an I-fuzzy partition with respect to any arbitrary implicator I. Conversely, each I-fuzzy partition that takes values only from the set {0, 1} can be understood as a partition.
(ii) Note also that the Definition 3.1 is equivalent to the one given in Definition 1.2.

Once again, we have the following equivalence condition for an implicator I to give rise to an I-fuzzy partition P that admits any value in [0, 1].

Proposition 3.3. *For a given implicator I the following are equivalent:*

(i) *The concept of I-fuzzy partition admits any value $a \in [0, 1]$.*
(ii) *I satisfies the following for any $a \in [0, 1]$:*
  (a) *$I(a, a) = 1$,*
  (b) *there exists $b \in [0, 1]$ such that $I(1, a) \geqslant b$ and $I(1, b) \geqslant a$.*

Proof

(i)⇒ (ii): Let P be an I-fuzzy partition and let $U \in P$ be such that $U(x) = 1$ and $U(y) = a$ for some $x, y \in \mathbb{X}$ and $a \in (0, 1)$. Then

$$I(a,a) = I(U(y),U(y)) \geqslant U(x) = 1.$$

Let $V \in P$ be such that $V(y) = 1$. Then

$$I(1,V(x)) = I(U(x),V(x)) \geqslant U(y) = a,$$

and

$$I(1,a) = I(V(y),U(y)) \geqslant V(x).$$

Now, letting $b = V(x)$ we have the result.

(ii)⇒ (i): Let I satisfy the above properties. Let $\mathbb{X}$ be any non-singleton set. Define the fuzzy subsets on $\mathbb{X}$ as follows:

$$U(y) = \begin{cases} 1, & y = x, \\ a, & y \neq x, \end{cases} \quad V(y) = \begin{cases} b, & y = x, \\ 1, & y \neq x. \end{cases}$$

From the following inequalities, where $y \neq x$,

$$I(V(y),U(y)) = I(1,a) \geqslant b = V(x),$$
$$I(U(x),V(x)) = I(1,b) \geqslant a = U(y),$$

we see that the system $P = \{U, V\}$ is an I-fuzzy partition. □

Note that even if an implicator I is such that the concept of I-fuzzy partition admits any value $a \in [0, 1]$, it is possible that I is such that the concept of I-fuzzy equivalence relation does not admit all values $a \in [0, 1]$. The following example illustrates this.

Example 3.4. Let us define an implicator $I : [0,1]^2 \to [0,1]$ as follows:

$$I(p,q) = \begin{cases} 0, & \text{if } (p,q) = (1,0), \\ 1 - q, & \text{if } p = 1 \text{ and } q \in (0,1), \\ 1, & \text{otherwise}, \end{cases} \quad p,q \in [0,1].$$

From Proposition 3.3 we see that I is such that the concept of I-fuzzy partition admits any value $a \in [0, 1]$. Let $\mathbb{X} = \{x,y\}$. Let us define two fuzzy sets U, V on $\mathbb{X}$ as follows:

$$U(p) = \begin{cases} 1, & p = x, \\ a, & p = y, \end{cases} \qquad V(y) = \begin{cases} b, & p = x, \\ 1, & p = y, \end{cases}$$

where $a, b \in [0, 1]$ are such that $a + b \leqslant 1$. Once again, it is a routine verification that $P = \{U, V\}$ is an I-fuzzy partition. However, from Proposition 2.4 we see that the I-fuzzy equivalence relations on $\mathbb{X}$ do not admit the values $a \in (0.5, 1]$.

Once again, in the case of partitions we know that two members of a parition P on $\mathbb{X}$ are either non-overlapping or identical. However, it is possible that there exist an I-fuzzy partition P w.r.to an implicator I, such that there exist $U, V \in P$ with $U(x) = V(x) = 1$ but $U \neq V$ (see Example 3.7 below).

Definition 3.5. A fuzzy partition P of a non-empty set $\mathbb{X}$ is said to be *consistent*, if P is such that whenever for any elements U, $V \in P$ there exists $x \in \mathbb{X}$ such that $U(x) = V(x) = 1$ then $U = V$.

Definition 3.6. An implicator I is called a *partition fitting implicator* if every fuzzy partition P that is an I-fuzzy partition is consistent.

Example 3.7. Note that not all implicators are partition fitting implicators. For example, let us once again consider the implicator I given in Example 2.5. It can be easily verified that the $P = \{U, V, W\}$ as defined below is an I-fuzzy partition on the set $\mathbb{X} = \{x, y, z\}$, for any a, $b \in [0, 1]$ but $a \neq b$:

$$U(p) = \begin{cases} 1, & p = x \text{ or } y, \\ a, & p = z, \end{cases} \quad V(p) = \begin{cases} a, & p = x, \\ b, & p = y, \\ 1, & p = z, \end{cases} \quad W(p) = \begin{cases} 1, & p = x \text{ or } y, \\ b, & p = z. \end{cases}$$

Note that $U(x) = W(x) = 1$, but that $U(z) \neq W(z)$, i.e., P is not consistent.

Proposition 3.8. *If an implicator I is such that $I(a, b) = 1 \Rightarrow a \leqslant b$, for any a, $b \in [0, 1]$, then I is a partition fitting implicator.*

Proof. Let P be a system of I-fuzzy partition such that for some $U, V \in P$ and some $x \in \mathbb{X}$, $U(x) = V(x) = 1$. Indeed, for any $z \in \mathbb{X}$, we have

$$I(V(y), U(y)) \geqslant V(x) = 1 \Rightarrow V(y) \leqslant U(y),$$

for any $y \in \mathbb{X}$ and similarly, $U(y) \leqslant V(y)$. Hence $U = V$, i.e., I is a partition fitting implicator. □

Remark 3.9

(i) In Proposition 3.8, one could, instead, ask for the following condition: $I(a, b) = 1 \Rightarrow a \geqslant b$, for any a, $b \in [0, 1]$. However, this would be absurd, since $I(0, 1) = 1$ would imply that $0 \geqslant 1$!!!
(ii) Once again, the above property is not necessary for an I to be a partition fitting implicator. However, interestingly, precisely the same conditions given in Remark 2.9 can be shown to be necessary for an I to be a partition fitting implicator.
(iii) Example 3.7 also shows that if $I(a, 0) = 1$ for any $a > 0$, then again I is not a partition fitting implicator.

Theorem 3.10. *For an implicator I the following are equivalent:*

(i) *I is a partition fitting implicator and the concept of I-fuzzy partition admits any value $a \in [0, 1]$.*
(ii) *I satisfies the following properties:*

$$I(a, a) = 1, \quad \text{for any } a \in [0, 1], \tag{IP1}$$
$$\text{For any } a \in [0, 1] \quad \text{there exists } b \in [0, 1] \quad \text{such that } I(1, a) \geqslant b \quad \text{and} \quad I(1, b) \geqslant a, \tag{IP2}$$
$$\text{For any } a, b \in [0, 1], \quad I(a, b) = 1 \Rightarrow a \leqslant b. \tag{IP3}$$

Proof

(i)⇒ (ii): Let I be a partition fitting implicator and let the concept of I-fuzzy partition admit any value $a \in [0, 1]$. The fact that I satisfies (IP1) and (IP2) is immediate from Proposition 3.3. To see that (IP3) is necessary, let us assume that for some a, $b \in [0, 1]$ with $a > b$, I is such that $I(a, b) = 1$. By (IP2), for the above a, b there exist $a', b' \in [0, 1]$ such that

$$I(1, a) \geqslant b' \quad \text{and} \quad I(1, b') \geqslant a, \tag{4}$$
$$I(1, b) \geqslant a' \quad \text{and} \quad I(1, a') \geqslant b. \tag{5}$$

Let $c = \min(a', b')$. Note that it suffices to consider the case $c \in (0, 1]$. Indeed, if $c = 0$ then either $a = 0$ or $b = 0$ from (IP2). $a = 0$ is a contradiction to our assumption, while if $b = 0$ then from Example 3.7 and Remark 3.9(iii) we know that I is not a partition fitting implicator, a contradiction to our hypothesis. With respect to the above c, let us define the function $I_c : [0, 1]^2 \to [0, 1]$ as follows:

$$I_c(p, q) = \begin{cases} 1, & \text{if } p \leqslant q, \\ 1, & \text{if } q = c, \\ I(p, q), & \text{otherwise,} \end{cases} \qquad p, q \in [0, 1].$$

Note that I_c is an implicator such that $I_c(a,b) = 1$ for the above considered $a, b \in [0, 1]$ with $a > b$. Let $\mathbb{X} = \{x, y\}$ and consider the partition $P = \{U, V, W\}$ on $\mathbb{X}$ as follows:

$$U(p) = \begin{cases} 1, & p = x, \\ a, & p = y, \end{cases} \quad V(p) = \begin{cases} c, & p = x, \\ 1, & p = y, \end{cases} \quad W(p) = \begin{cases} 1, & p = x, \\ b, & p = y. \end{cases}$$

A routine calculation shows that P is an I_c-fuzzy partition However, note that $U \neq W$, i.e., P is not consistent.

(ii)⇒ (i): The sufficiency is immediate from Propositions 3.3 and 3.8. □

Remark 3.11. Note that the conditions in Theorems 2.10 and 3.10 are not equivalent. To see this, consider the following implicator:

$$I(p,q) = \begin{cases} 0, & \text{if } p \in (0,1) \quad \text{and} \quad q = 1, \\ 0, & \text{if } p > q \quad \text{and} \quad p \neq 1, \\ q, & \text{if } p = 1, \\ 1, & \text{otherwise}, \end{cases} \qquad p, q \in [0,1].$$

It is easily verifiable that I satisfies all the conditions in Theorem 3.10, viz., IP1, IP2 and IP3, but does not satisfy (IE1), since $I(a,1) = 0 \not\geqslant a$ for any $a \in (0, 1)$. Note, however, that I satisfies (IE2) and (IE3) – for the only applicable value $a = 0.5$ – of Theorem 2.10.

4. Correspondence between *I*-fuzzy equivalence relations and *I*-fuzzy partitions

As noted in Section 1, there exists a one–one correspondence between partitions and equivalence relations given by the relationships (1) and (2). Hence, it is only natural to expect, and in fact enforce, such a correspondence between their I-fuzzy counterparts. However, from Theorems 2.10 and 3.10 and Remark 3.11 it is immediate that there are I-fuzzy partitions without the corresponding counterpart among I-fuzzy equivalence relations (see also Example 3.4).

In the following, we propose relations, similar to Eqs. (1) and (2), in the case of I-fuzzy partitions and I-fuzzy equivalence relations and determine conditions on an implicator I so as to obtain a one–one correspondence between these concepts.

Definition 4.1

(i) Let I be an equivalence fitting implicator and E be an I-fuzzy equivalence relation on $\mathbb{X}$. Let us denote by P_E the set of all fuzzy sets on $\mathbb{X}$, $P_E = \{E_x | x \in \mathbb{X}\}$.
(ii) Let I be a partition fitting implicator and P be an I-fuzzy partition on $\mathbb{X}$. Let us denote by E_P the fuzzy set on $\mathbb{X}^2$ given by $E_P(x,y) = \inf\{U(y) | U \in P, U(x) = 1\}$ for any $x, y \in \mathbb{X}$.

Theorem 4.2. *Let I be an equivalence fitting implicator and E an I-fuzzy equivalence relation on $\mathbb{X}$. Then P_E is always an I-fuzzy partition.*

Proof. From the definition of P_E it is obvious that P_E satisfies both the axioms (i) and (ii) in Definition 3.1. To see that P_E satisfies also the axiom (iii), let $U = E_u$ and $U(x) = 1$ for some $x, u \in \mathbb{X}$. Then, we know that $E_u = E_x$. Now, for any $v, y \in \mathbb{X}$ and letting $V = E_v$ we have

$$I(V(y), U(y) = I(E_v(y), E_u(y)) = I(E(v,y), E(y,u)) \geqslant E(v,u) = E_u(v) = E_x(v) = E_v(x) = V(x). \quad \square$$

Theorem 4.3. *Let I be a partition fitting implicator. The following are equivalent:*

(i) *For any non-empty set $\mathbb{X}$ and any I-fuzzy partition P, E_P is an I-fuzzy equivalence relation on $\mathbb{X}$.*
(ii) *$I(1, a) = a$ for all $a \in [0, 1]$.*

Proof

(i)⇒ (ii): Let $I(1, a) = b > a$, for some $a \in [0, 1]$. Consider the partition $P = \{U, V\}$ on a set $\mathbb{X} = \{x, y\}$ with $U(x) = V(y) = 1$, $U(y) = a$ and $V(x) = b$. It can be easily verified that P is an I-fuzzy partition of $\mathbb{X}$. However, $E_P(x,y) = U(y) = a \neq b = V(x) = E(y,x)$, i.e., E_P is not an I-fuzzy equivalence relation on $\mathbb{X}$.

(ii)⇒ (i): Let $I(1, a) = a$ for all $a \in [0, 1]$. Let P be an I-fuzzy partition on some universe $\mathbb{X}$. Firstly, note that $E_P(x,x) = 1$ for all $x \in \mathbb{X}$, i.e., E_P is reflexive.To see that E_P is symmetric, we have to show that the equality $U(x) = V(y) = 1$ for some U, $V \in P$ and $x, y \in \mathbb{X}$ ensures the equality $U(y) = V(x)$. This follows easily from the following inequalities. Since P is an I-fuzzy partition, with $V(y) = 1$ we have that $I(U(x), V(x)) \geqslant U(y)$ for any $x, y \in \mathbb{X}$ and any $U \in P$.

$$V(x) = I(1, V(x)) = I(U(x), V(x)) \geqslant U(y),$$

and similarly, $U(y) \geqslant V(x)$.To see that E_P is I-transitive, i.e., for any $x, y, z \in \mathbb{X}$ we need to show that

$$I(E_P(x,y), E_P(y,z)) \geqslant E_P(x,z). \tag{6}$$

However, note that from the above we have that if $U(x) = V(z) = 1$ then $V(x) = U(z)$ for any $x, z \in \mathbb{X}$ and hence

$$I(U(y), V(y)) \geqslant V(x) = U(z),$$

for any $y \in \mathbb{X}$, which by the definition of E_P implies (6), i.e., E_P is I-transitive. Thus, E_P is an I-fuzzy equivalence relation. □

4.1. Duality fitting implicators

Theorems 4.2 and 4.3 direct us towards conditions on an implicator I to ensure a one–one correspondence between the concepts of I-fuzzy partitions and I-fuzzy equivalence relations.

Definition 4.4. A duality fitting implicator $I : [0,1]^2 \to [0,1]$ is an implicator that also satisfies the following:

$$I(a,a) = 1, \quad a \in [0,1], \tag{IP}$$

$$I(a,1) \geqslant a, \quad a \in [0,1], \tag{RBC}$$

$$I(1,a) = a, \quad a \in [0,1], \tag{LNP}$$

$$I(a,b) = 1 \Rightarrow a \leqslant b, \quad a,b \in [0,1]. \tag{OP'}$$

The set of all duality fitting implicators will be denoted by $\mathscr{I}$.

Remark 4.5

(i) Note that the above conditions on an implicator I to be a duality fitting implicator is minimal - in the sense of ensuring a one–one correspondence between the concepts of I-fuzzy partition and I-fuzzy equivalence relations as the many examples in Sections 2 and 3 have amply demonstrated- and cannot be weakened. Also the conditions are, in general, mutually independent. However, with the additional assumption of monotonicity this is not the case (see Lemma 7.3).
(ii) The class of all duality fitting implicators is convex, i.e., if I_1, I_2 are any duality fitting operators, then for any $\lambda \in [0, 1]$ we have that $\lambda \cdot I_1 + (1-\lambda) \cdot I_2$ is also a duality fitting implicator.
(iii) Also the class of all duality fitting implicators is closed under lattice operations, i.e., if I_1, I_2 are any duality fitting operators, then their pointwise minimum or maximum, viz.,

$$(I_1 \wedge I_2)(p,q) = \min(I_1(p,q), I_2(p,q)),$$
$$(I_1 \vee I_2)(p,q) = \max(I_1(p,q), I_2(p,q)), \quad p,q \in [0,1],$$

are also duality fitting implicators.
(iv) Finally, we would like to mention that the peculiar labeling of the above properties is so done as to retain consistency with the terms used in literature and will become clear later on in Section 7. In the context of fuzzy implications, (IP) is also known as the *identity principle*, (LNP) is the *left-neutrality property* and (OP') is related to the *ordering property* of a fuzzy implication, while (RBC) is a right-boundary condition.

For a given non-empty set $\mathbb{X}$, let us denote the set of all I-fuzzy equivalence relations by $\mathscr{IFE}(\mathbb{X})$ and the set of all I-fuzzy partitions by $\mathscr{IFP}(\mathbb{X})$. We have the following main result of this section.

Theorem 4.6. *Let $I \in \mathscr{I}$, i.e., I is a duality fitting implicator and $\mathbb{X}$ any non-empty set. Then there exists a bijection between the set of all I-fuzzy equivalence relations and I-fuzzy partitions. Namely, for every $E \in \mathscr{IFE}(\mathbb{X})$, $P_E \in \mathscr{IFP}(\mathbb{X})$, and conversely, for every $P \in \mathscr{IFP}(\mathbb{X})$, $E_P \in \mathscr{IFE}(\mathbb{X})$. Moreover, $E_{P_E} = E$ and $P_{E_P} = P$.*

Proof. Let $I \in \mathscr{I}$ and $\mathbb{X}$ any non-empty set. From Definition 4.4 and Theorems 4.2 and 4.3 we have that for every $E \in \mathscr{IFE}(\mathbb{X})$, $P_E \in \mathscr{IFP}(\mathbb{X})$, and conversely, for every $P \in \mathscr{IFP}(\mathbb{X})$, $E_P \in \mathscr{IFE}(\mathbb{X})$.

Let E be a given I-fuzzy equivalence relation on $\mathbb{X}$. The corresponding I-fuzzy partition is given by $P_E = \{E_x | x \in \mathbb{X}\}$. Let $x, y \in \mathbb{X}$. Then

$$E_{P_E}(x,y) = \inf\{E_z(y) | z \in \mathbb{X}, E_z(x) = 1\} = E(x,y).$$

The last equality follows from Proposition 2.8 and the fact that E being an I-fuzzy equivalence relation for an $I \in \mathscr{I}$ is consistent, i.e., $E_z(x) = E(z,x) = 1$ ensures $E_z = E_x$, and consequently $E_z(y) = E_x(y) = E(x,y)$.

Now, let P be a given I-fuzzy partition on $\mathbb{X}$. The equality $P_{E_P} = P$ is equivalent to the equality $U = E_{P_x}$ for all $U \in P$ and $x \in \mathbb{X}$ such that $U(x) = 1$. Assuming these conditions, let $y \in \mathbb{X}$. Then

$$E_{P_x}(y) = E_P(x,y) = \inf\{V(y) | V \in P, V(x) = 1\} = U(y),$$

where the last claim follows from the fact that $V(x) = 1$ implies that $V = U$, i.e., $V(y) = U(y)$. $\square$

4.2. Bounds on duality fitting implicators

Remark 4.7

(i) It can be easily seen that the class of duality fitting implicators $\mathscr{I}$ is bounded below and above by, but not including, the following implicators:

$$I_*(p,q) = \begin{cases} 1, & \text{if } (p,q) = (0,1) \text{ or } p = q, \\ p \wedge q, & \text{if } p \vee q = 1, \\ 0, & \text{otherwise}, \end{cases} \qquad I^*(p,q) = \begin{cases} 1, & \text{if } p < 1, \\ q, & \text{if } p = 1, \end{cases}$$

i.e., if $I \in \mathscr{I}$ then $I_* \leqslant I < I^*$. I^* is the Weber implication (see [58]), which however, is not a duality fitting implicator since it does not satisfy (OP').

(ii) Let $\mathbb{X}$ be any non-empty set and E a fuzzy set on $\mathbb{X}^2$. Let $J, I \in \mathscr{I}$ such that $J \geqslant I$. If E is I-transitive, then E is also J-transitive. Hence every I-fuzzy equivalence relation is a J-fuzzy equivalence relation, and simultaneously, every I-fuzzy partition is also a J-fuzzy partition.

(iii) Let us denote the set of all I-fuzzy equivalences on a non-empty set $\mathbb{X}$ w.r.to a duality fitting implicator I by $\mathscr{IFE}_I(\mathbb{X})$. If $J, I \in \mathscr{I}$ such that $J \geqslant I$, then we have the following inclusions:

$$\mathscr{IFE}_I(\mathbb{X}) \subseteq \mathscr{IFE}_J(\mathbb{X}) \quad \text{and} \quad \mathscr{IFP}_I(\mathbb{X}) \subseteq \mathscr{IFP}_J(\mathbb{X}).$$

(iv) In particular, every I_*-fuzzy equivalence relation is an I-fuzzy equivalence relation and every I-fuzzy equivalence relation is an I^*-fuzzy equivalence relation, i.e., for any $I \in \mathscr{I}$ we have the following inclusions:

$$\mathscr{IFE}_{I_*}(\mathbb{X}) \subseteq \mathscr{IFE}_I(\mathbb{X}) \subseteq \mathscr{IFE}_{I^*}(\mathbb{X}),$$
$$\mathscr{IFP}_{I_*}(\mathbb{X}) \subseteq \mathscr{IFP}_I(\mathbb{X}) \subseteq \mathscr{IFP}_{I^*}(\mathbb{X}).$$

(v) Let P be any I-fuzzy partition w.r.to an $I \in \mathscr{I}$. If $U \in P$ is such that, for some $x, y \in \mathbb{X}$, $U(x) = U(y) = 1$, then for all $V \in P$ we have that $V(x) = V(y)$. This follows from the fact that I satisfies property (iv) in Definition 4.4. In other words, I-fuzzy partitions have "constant-wise" structure.

(vi) Let $I, J \in \mathscr{I}$ and let $\mathbb{X}$ be non-empty. If E is both an I-fuzzy equivalence relation and J-fuzzy equivalence relation on $\mathbb{X}$, then E is also an $I \wedge J$-fuzzy equivalence relation on $\mathbb{X}$. Of course, E is also an $I \vee J$-fuzzy equivalence relation on $\mathbb{X}$.

The proofs of the following lemmas are quite straightforward.

Lemma 4.8. *An $E \in \mathscr{IFE}_{I_*}(\mathbb{X})$ if and only if E is reflexive, symmetric, consistent and whenever for some $x, z \in \mathbb{X}$ there exists $y \in \mathbb{X}$ such that $E_y(x) \neq E_y(z)$ then $E(x, z) = 0$.*

Lemma 4.9. *A $P \in \mathscr{IFP}_{I_*}(\mathbb{X})$ if and only if P is consistent and whenever there exist U, $V \in P$ and $x, y \in \mathbb{X}$ such that $U(x) = 1$ and $U(y) \neq V(y)$ then $x \notin supp(V)$, where $supp(V) = \{x \in \mathbb{X} | V(x) > 0\}$.*

4.3. A special class of duality fitting implicators

We now introduce a rather special class of duality fitting implicators.

Definition 4.10. Let $\epsilon \in (0, 1]$. Let us define the function $I_{[\epsilon]} : [0,1]^2 \to [0,1]$ as follows:

$$I_{[\epsilon]}(p,q) = \begin{cases} 1, & \text{if } p \leqslant q, \\ q, & \text{if } p = 1, \\ 1 - \epsilon, & \text{otherwise}, \end{cases} \qquad p, q \in [0,1].$$

Remark 4.11. Let us denote by $I_{[E]}$ the above family of functions, i.e., $I_{[E]} = \{I_{[\epsilon]} | \epsilon \in (0,1]\}$. Then the following remarks can be made about $I_{[E]}$.

(i) Clearly, every member of this family is a duality fitting implicator, i.e., $I_{[E]} \subsetneq \mathscr{I}$.

(ii) This family is monotonically increasing with decreasing parameter ϵ and

$$I^* = \lim_{\epsilon \to 0} I_{[\epsilon]} = \sup I_{[E]}.$$

Hence we can write $I^* = I_{[0]}$.

(iii) Moreover, if $I \in \mathscr{I}$, then there exists an $\epsilon \in (0, 1]$ such that $I_{[\epsilon]} \geqslant I$.
(iv) From Remark 4.7(ii) we see that every fuzzy relation E that is I-transitive for an $I \in \mathscr{I}$, is also $I_{[\epsilon]}$-transitive for some $\epsilon \in (0, 1]$. This fact will be used in the sequel.

Remarks 6.5 and 6.13 show further interesting properties of this class of implicators.

5. Conjunctor-based fuzzy equivalences and partitions

As noted in Section 1, in Mesiar et al. [42], the authors have discussed fuzzy equivalence relations based on a general conjunctor C. In this section, after recalling the relevant definitions and results, we investigate the relations between the concepts of C-fuzzy equivalence relations and C-fuzzy partitions with the concepts of I-fuzzy equivalence relations and I-fuzzy partitions, proposed in this work.

Definition 5.1. A conjunctor $C : [0, 1]^2 \to [0, 1]$ is a mapping whose restriction to the domain $\{0, 1\}$ coincides with the binary conjunction, i.e.,

$$C(0,0) = C(1,0) = C(0,1) = 0; C(1,1) = 1.$$

Definition 5.2. ([42], Definition 2.2). Let C be a conjunctor. A fuzzy subset E of the Cartesian product $\mathbb{X}^2$ is called a C-fuzzy equivalence relation on $\mathbb{X}$ if E is reflexive, symmetric and is C-transitive, i.e., for all $x, y, z \in \mathbb{X}$,

$$C(E(x,y), E(y,z)) \leqslant E(x,z).$$

Definition 5.3. ([42], Definition 2.3). Let C be a conjunctor. A system P of fuzzy subsets of $\mathbb{X}$ is called a C-fuzzy partition of $\mathbb{X}$ if the following properties are satisfied, for all $x, y, z \in \mathbb{X}$:

(i) for all $U \in P$ there is some $x \in \mathbb{X}$ such that $U(x) = 1$,
(ii) for all $x \in \mathbb{X}$ there is some $U \in P$ such that $U(x) = 1$,
(iii) if $U(x) = 1$ for some $x \in \mathbb{X}$ then for all $y \in \mathbb{X}$ and all $V \in P$ the following inequality holds:

$$C(V(x), V(y)) \leqslant U(y). \tag{CFP}$$

Definition 5.4. A $C : [0, 1]^2 \to [0, 1]$ is said to satisfy the *right neutrality property*, if

$$C(p, 1) = p, \quad p \in [0, 1]. \tag{RNP}$$

Definition 5.5. ([42], Definition 3.2). A $C : [0, 1]^2 \to [0, 1]$ is called a *duality fitting conjunctor* if it is a commutative conjunctor that satisfies (RNP), or equivalently (LNP), i.e., for all $a \in [0, 1]$ it is $C(a,1) = C(1, a) = a$. The set of all duality fitting conjunctors will be denoted by $\mathscr{C}$.

Theorem 5.6. ([42], Theorem 3.1). *Let C be a duality fitting conjunctor, i.e., $C \in \mathscr{C}$. Then there exists a bijection between the set of all C-fuzzy equivalence relations, denoted as $\mathscr{CFE}(\mathbb{X})$ and C-fuzzy partitions, denoted as $\mathscr{CFP}(\mathbb{X})$. Namely, for each C-fuzzy equivalence relation E, P_E is a C-fuzzy partition, and for each C-fuzzy partition P, E_P is a C-fuzzy equivalence relation. Moreover, $E_{P_E} = E$ and $P_{E_P} = P$.*

Remark 5.7. ([42], pp. 175–177).

(i) The class of duality fitting conjunctors $\mathscr{C}$ is bounded below and above by the following conjunctors:

$$C_*(p,q) = \begin{cases} 0, & \text{if } p, q \in [0,1), \\ \min(p,q), & \text{otherwise}, \end{cases} \qquad C^*(p,q) = \begin{cases} 0, & \text{if } (p,q) = (0,0), \\ \min(p,q), & \text{if } \max(p,q) = 1, \\ 1, & \text{otherwise}. \end{cases}$$

C_* is also known as the drastic t-norm $T_{\mathbf{D}}$ (see [37]).

(ii) Let $\mathbb{X}$ be any non-empty set and E a fuzzy set on $\mathbb{X}^2$. Let $C, C' \in \mathscr{C}$ such that $C \geqslant C'$. If E is C-transitive, then E is also C'-transitive. Hence every C-fuzzy equivalence relation is a C'-fuzzy equivalence relation, and simultaneously, every C-fuzzy partition is also a C'-fuzzy partition.
(iii) Unlike in the case of duality fitting implicators, for duality fitting conjunctors we have the reverse inclusions, i.e., if C is any duality fitting conjunctor such that $C_* \leqslant C \leqslant C^*$, then

$$\mathscr{CFE}_{C_*}(\mathbb{X}) \supseteq \mathscr{CFE}_C(\mathbb{X}) \supseteq \mathscr{CFE}_{C^*}(\mathbb{X}),$$
$$\mathscr{CFP}_{C_*}(\mathbb{X}) \supseteq \mathscr{CFP}_C(\mathbb{X}) \supseteq \mathscr{CFP}_{C^*}(\mathbb{X}).$$

(iv) An $E \in \mathscr{CFE}_{C_*}(\mathbb{X})$ if and only if E is reflexive, symmetric and consistent.
(v) An $E \in \mathscr{CFE}_{C^*}(\mathbb{X})$ if and only if $E \in \mathscr{CFE}_{C_*}(\mathbb{X})$ and there exists $a \in [0, 1]$ such that, for all $x, y \in \mathbb{X}$, $E(x, y) \in [a,1]$. Comparing with Lemma 4.8 it is clear that the concepts of C^*-fuzzy equivalence relation and I_*-fuzzy equivalence relation are different.
(vi) A $P \in \mathscr{CFP}_{C_*}(\mathbb{X})$ if and only if $\{ker(U)|U \in P\}$ is a partition of $\mathbb{X}$ and $U(x) = V(y)$ whenever for any $x, y \in \mathbb{X}$ and $U, V \in P$ it is $U(y) = V(x) = 1$.
(vii) An $P \in \mathscr{CFP}_{C^*}(\mathbb{X})$ if and only if P is a partition or $P = (U, V)$, where $ker(U) = \mathbb{X} \setminus ker(V)$ and $U(x) = V(y) = a \in [0, 1]$ whenever $x \in ker(V), y \in ker(U)$.

6. Comparison of the concepts of *I*- and *C*-fuzzy equivalences and partitions

It is immediately interesting to investigate the relation between the concepts of *I*- and *C*-fuzzy equivalences and partitions. Firstly, we show that every *I*-fuzzy equivalence (and hence every *I*-fuzzy partition) is also a *C*-fuzzy equivalence (*C*-fuzzy partition).

Lemma 6.1. *Let $\mathbb{X}$ be any non-empty set. If I is a duality fitting implicator, then every I-fuzzy equivalence relation (equivalently, I-fuzzy partition) on $\mathbb{X}$ is a C-fuzzy equivalence relation (equivalently, I-fuzzy partition) on $\mathbb{X}$ for some duality fitting conjunctor C.*

Proof. Since $I \in \mathscr{I}$, I is an equivalence relation fitting implicator and hence by Definition 2.7, if $E \in \mathscr{IFE}(\mathbb{X})$ then E is reflexive, symmetric and consistent. Thus, by Remark 5.7 (iv), we see that $E \in \mathscr{CFP}_{C_*}(\mathbb{X})$. In other words, $\mathscr{IFE}(\mathbb{X}) \subset \mathscr{CFE}(\mathbb{X})$. □

In the rest of this section, we attempt to answer the converse question, i.e., *given a set $\mathbb{X}$, is every C-fuzzy equivalence relation* (*equivalently*, *C-fuzzy partition*) *on $\mathbb{X}$ for a duality fitting conjunctor C also an I-fuzzy equivalence relation* (*equivalently*, *an I-fuzzy partition*) *on $\mathbb{X}$*, *with I a duality fitting implicator?* As can be seen the answer to this poser is not straightforward and we resort to some concepts relating conjunctors and implicators from classical logics.

6.1. Relations between conjunctors and implicators

In some classical logics, for instance, Intuitionistic logic, where only a conjunctor is available, an implicator is obtained as its residual. This relation has been generalised to the framework of fuzzy logic with much success and the residuals of many generalisations conjunctors in fuzzy logic, viz., *t*-norms, uninorms, *t*-subnorms, etc, are widely studied and applied (see, for instance, [1,10,40,51]).

In fact, for certain classes of left-continuous conjunctors, a *C* belonging to this class and the residual *I* obtained from *C* are such that the pair (*C*, *I*) satisfies the following *residuation property*:

$$C(p,q) \leqslant r \iff I(p,r) \geqslant q, \quad p,q,r \in [0,1]. \tag{RP}$$

One can immediately note that the *C*-transitivity in Definition 5.2(iii) and the *I*-transitivity in Definition 2.1(iii) can be made to relate to each other if the duality fitting conjunctor and the duality fitting implicator satisfy (RP). In fact, the conjunctors from the above classes are in a one–one correspondence with their residuals. Thus, in the following we investigate the relations between duality fitting conjunctors and duality fitting implicators.

6.2. Residuals of duality fitting conjunctors

Definition 6.2. Let *C* be a function from $[0, 1]^2$ to [0, 1]. A function $I_C : [0,1]^2 \to [0,1]$ defined as follows

$$I_C(p,q) = \sup\{t \in [0,1]|C(p,t) \leqslant q\}, \quad p,q \in [0,1], \tag{7}$$

is called the residual of *C*.

Remark 6.3

(i) In the case *C* and I_C are restricted to $\{0, 1\}^2$, if *C* is a conjunctor, then I_C is an implicator.
(ii) If C, C' are any two functions from $[0, 1]^2$ to [0, 1] such that $C \leqslant C'$, then their residuals have the opposite ordering, i.e., $I_C \geqslant I_{C'}$. Indeed, for any $x, y, t \in [0,1]$ we have that

$$C'(p,t) \leqslant q \text{ implies } C(p,t) \leqslant q,$$
$$\text{i.e., } \{t \in [0,1]|C'(p,t) \leqslant q\} \subset \{t \in [0,1]|C(p,t) \leqslant q\},$$
$$\text{i.e., } \sup\{t \in [0,1]|C'(p,t) \leqslant q\} \leqslant \sup\{t \in [0,1]|C(p,t) \leqslant q\},$$
$$\text{i.e., } I_{C'}(p,q) \leqslant I_C(p,q).$$

We also have the following result regarding a binary operation and its residual.

Theorem 6.4 (cf. [15], Theorem 4.1). *Let C be a function from $[0, 1]^2$ to [0, 1] and let I_C denote its residual. Then the following are valid:*

(i) $I_C(0,0) = 1$ *if and only if C(0, 1) = 0.*
(ii) $I_C(1,1) = 1$ *always.*
(iii) $I_C(1,0) = 0$ *if and only if C(1, x) > 0 for all $x \in (0, 1]$.*

From Theorem 6.4 and the definition of a duality fitting conjunctor (see Definition 5.5) it immediately follows that if $C \in \mathscr{C}$ then its residual I_C is an implicator. However, this residual I_C is not always a duality fitting implicator. For example, we know that if $C \in \mathscr{C}$, then $C_* \leqslant C \leqslant C^*$. The residual of C^* is $I_{C^*} = I_{[1]}$ which is a duality fitting implicator (see Remark 4.11(i)), while that of C_* is $I_{C_*} = I^* = I_{[0]}$, which is not a duality fitting implicator, since $I^*(p,q) = 1$ even if $p > q$.

Remark 6.5. Let us consider the family $I_{[E]}$ of duality fitting implicators.

(i) Recall that I_{C^*} and I_{C_*} are, respectively, the lower and upper bounds of the family $I_{[E]}$.
(ii) Note that the class of duality fitting implicators is much richer than the class of residuals obtained from duality fitting conjunctors, i.e., if C is any duality fitting conjunctor, then we have $I_* < I_{C^*} = I_{[1]} \leqslant I_C < I^* = I_{[0]}$. Of course, it is clear that there exist $I \in \mathscr{I}$ such that $I_* \leqslant I < I_{C^*}$.
(iii) However, it should be emphasized that the residuals obtained from duality fitting conjunctors strictly contains the family $I_{[E]}$ of duality fitting implicators. To see this, consider the Łukasiewicz t-norm $T_{\mathbf{LK}}(p,q) = \max(0, p+q-1)$ which is a monotonic duality fitting conjunctor such that $T_{\mathbf{D}} \leqslant T_{\mathbf{LK}} \leqslant T_{\mathbf{M}}$. However, its residual is the fuzzy implication $I_{\mathbf{LK}}(x,y) = \min(1, 1-x+y)$ and $I_{\mathbf{LK}} \notin I_{[E]}$.

Unfortunately, the properties available for a duality fitting conjunctor are not strong enough to obtain conditions underwhich its residual is a duality fitting implicator. However, if the condition of monotonicity is also assumed, then necessary and sufficient conditions underwhich the residuals of these monotonic duality fitting conjunctors become duality fitting implicators can be obtained. This study will be taken up in Section 7.2.

6.3. Deresiduum of duality fitting implicators

Let us now turn to the converse procedure of obtaining a conjunctor from an implicator. Once again we will be interested only in duality fitting operators.

Definition 6.6. Let $I : [0,1]^2 \to [0,1]$ be any function. We define a mapping $C_I : [0,1]^2 \to [0,1]$ as

$$C_I(p,q) = \inf\{t \in [0,1] | I(p,t) \geqslant q\}, \quad p,q \in [0,1]. \tag{8}$$

C_I is also known as the deresiduum of I (see e.g., [20]).

Remark 6.7

(i) Note that in the case I and C_I are restricted to $\{0, 1\}^2$, if I is an implicator, then C_I is a conjunctor.
(ii) If I, I' are any two functions from $[0, 1]^2$ to [0, 1] such that $I \leqslant I'$, then their deresidua have the opposite ordering, i.e., $C_I \geqslant C_{I'}$. This can be proven analogously to Remark 6.3(ii) using the fact that if $A \subseteq B$ then $\inf A \geqslant \inf B$.

Remark 6.8. Once again, not for every $I \in \mathscr{I}$ we have that its deresiduum $C_I \in \mathscr{C}$. To see this, consider the Łukasiewicz implication $I_{\mathbf{LK}} \in \mathscr{I}$ and whose deresiduum is the Łukasiewicz t-norm $C_{I_{\mathbf{LK}}} = T_{\mathbf{LK}} \in \mathscr{C}$. Whereas, the deresiduum of $I_* \in \mathscr{I}$ is

$$C_{I_*}(p,q) = \begin{cases} p \wedge q, & \text{if } p \vee q = 1, \\ 0, & \text{if } p \wedge q = 0, \quad p,q \in [0,1], \\ p, & \text{otherwise}, \end{cases} \tag{9}$$

which is not commutative and hence $C_{I_*} \notin \mathscr{C}$.

Towards characterizing $I \in \mathscr{I}$ whose deresiduum $C_I \in \mathscr{C}$, we introduce the following important algebraic property.

Definition 6.9. An $I : [0,1]^2 \to [0,1]$ is said to satisfy the *weak exchangeability* property, if

$$I(p,r) \geqslant q \Longleftrightarrow I(q,r) \geqslant p, \quad p,q,r \in [0,1]. \tag{WE}$$

Lemma 6.10. *Let $I : [0,1]^2 \to [0,1]$ be any function and C_I its deresiduum as defined in (8). Then,*

(i) *If I satisfies (LNP) then C_I satisfies (LNP).*
(ii) *If I satisfies (IP) and (OP') then C_I satisfies (RNP).*
(iii) *If I satisfies (WE) then C_I is commutative.*

Proof

(i) That C_I satisfies (LNP) follows from the following equality:

$$C_I(1,q) = \inf\{t \in [0,1] | I(1,t) \geqslant q\} = q, \text{ by (LNP)}.$$

(ii) By (IP) we know that $p \in \mathscr{A} = \{t \in [0,1] | I(p,t) \geqslant 1\}$ for any $p \in [0, 1]$. To see that $p = \inf \mathscr{A}$, if possible, let there exist a $q < p$ such that $I(p, q) = 1$. However, this is a contradiction to the fact that I satisfies (OP'). Thus, we have $C_I(p,1) = \inf\{t \in [0,1] | I(p,t) \geqslant 1\} = p$ for all $p \in [0, 1]$, i.e., C_I satisfies (RNP).
(iii) Let $p, q \in [0, 1]$ be fixed. then

$$C_I(p,q) = \inf\{t \in [0,1] | I(p,t) \geqslant q\},$$
$$C_I(q,p) = \inf\{t \in [0,1] | I(q,t) \geqslant p\}.$$

It is clear now that if I satisfies (WE) then C_I is commutative. □

It can be easily verified that while $I_{\mathbf{LK}}$ satisfies (WE), I_* does not. However, if an $I \in \mathscr{I}$, then it satisfies both (LNP) and (OP') and hence its deresiduum C_I is such that $C_I(1,p) = C_I(p,1) = p$ for all $p \in [0, 1]$. The only property C_I lacks is commutativity.

Definition 6.11. Let $I \in \mathscr{I}$ satisfy (LNP) and (OP') and let C_I be its deresiduum. The *modified deresiduum* of I is defined as

$$\widetilde{C}_I(p,q) = \min(C_I(p,q), C_I(q,p)), \quad p,q \in [0,1]. \tag{10}$$

The following result is now immediate.

Proposition 6.12. *Let $I \in \mathscr{I}$ satisfy (LNP) and (OP') and let C_I be its deresiduum. The modified deresiduum is a duality fitting conjunctor, i.e, $\widetilde{C_I} \in \mathscr{C}$. Moreover, $\widetilde{C_I} \leqslant C_I$.*

Remark 6.13. Let us once again consider the family $I_{[E]}$ of duality fitting implicators.

(i) It can be easily verified that for no $\epsilon \in (0, 1]$ does $I_{[\epsilon]}$ satisfy (WE) and hence the deresidua of $I_{[E]}$ are not duality fitting conjunctors. Of course, their modified deresidua are duality fitting conjunctors. In fact, for any $\epsilon \in (0, 1]$ the deresiduum of $I_{[\epsilon]}$ is given as follows:

$$C_{I_{[\epsilon]}}(p,q) = \begin{cases} p \wedge q, & \text{if } p \vee q = 1, \\ 0, & \text{if } p \wedge q = 0, \\ p, & \text{if } q > 1-\epsilon, \\ 0, & \text{if } q \leqslant 1-\epsilon, \end{cases} \quad p,q \in [0,1]. \tag{11}$$

(ii) Note that $C_{I_{[1]}} = C_{I_*}$ given in (9). Now, for any $I \in \mathscr{I}$ such that $I_* \leqslant I \leqslant I_{[1]}$, we know from Remark 6.7(ii) that $C_{I_{[1]}} \geqslant C_I \geqslant C_{I_*}$, i.e., C_I is exactly as given in (9). Thus the deresiduum of $I_{[1]}$ is the upper bound of the deresidua of the class of all duality fitting implicators $\mathscr{I}$.
(iii) Moreover, the modified deresiduum of $I_{[1]}$ is the minimum operation (also a t-norm on $[0, 1]^2$). Thus no duality fitting conjunctor $C > T_{\mathbf{M}}$ can be obtained as a deresiduum of a duality fitting implicator.
(iv) Note that $C_{I_{[\epsilon]}} = C_*$ only if $\epsilon = 0$ and hence cannot be obtained as a deresiduum of any member of the class of $I_{[E]}$.
(v) Finally, and most interestingly, for any $\epsilon \in (0, 1]$ the residual of $C_{I_{[\epsilon]}}$ given in (11) is precisely $I_{[\epsilon]}$, i.e., $I_{C_{I_{[\epsilon]}}} = I_{[\epsilon]}$.

6.4. Relations between the concepts of I- and C-fuzzy equivalences and partitions

Lemma 6.14. *Let $\mathbb{X}$ be any non-empty set and E be a fuzzy set on $\mathbb{X}^2$.*

(i) *Let $C \in \mathscr{C}$ be such that $C > C_*$ and I_C be its residual. If E is C-transitive then E is also I_C-transitive.*
(ii) *Conversely, if E is I-transitive for any $I \in \mathscr{I}$, then E is also C_I-transitive.*

Proof. Let $\mathbb{X}$ be any non-empty set and E be a fuzzy set on $\mathbb{X}^2$.

(i) Let $C \in \mathscr{C}$ be such that $C > C_*$, I_C be its residual and let E be C-transitive. To show that E is I_C-transitive, we need to show that for any $x,y,z \in \mathbb{X}$ we have that $I_C(E(x,y), E(y,z)) \geqslant E(x,z)$. By the definition of I_C, we have

$$I_C(E(x,y),E(y,z)) = \sup\{t|C(E(x,y),t) \leqslant E(y,z)\}.$$

Since E is C-transitive, we have that $C(E(x,y),E(x,z)) \leqslant E(y,z)$ for any $x,y,z \in \mathbb{X}$, i.e.,

$$E(x,z) \in \mathscr{A} = \{t|C(E(x,y),t) \leqslant E(y,z)\},$$

and hence $E(x,z) \leqslant \sup \mathscr{A}$. Now it is clear that $I_C(E(x,y),E(y,z)) \geqslant E(x,z)$, i.e., E is I_C-transitive.

(ii) The proof in this case can be proven similar to the above, using the fact that infimum of a set is less than or equal to any element of the set. □

Theorem 6.15. *Let $\mathbb{X}$ be any non-empty set and E be a fuzzy set on $\mathbb{X}^2$.*

(i) *Let $C \in \mathscr{C}$ be such that $C > C_*$ and its residual I_C satisfies (OP'). If E is a C-fuzzy equivalence relation on $\mathbb{X}$, then there exists an $I \in \mathscr{I}$ such that E is also an I-fuzzy equivalence relation.*

(ii) *Conversely, if E is an I-fuzzy equivalence relation on $\mathbb{X}$ for any $I \in \mathscr{I}$, then there exists a $C \in \mathscr{C}$ such that E is also a C-fuzzy equivalence relation.*

Proof. Let $\mathbb{X}$ be any non-empty set and E be a fuzzy set on $\mathbb{X}^2$.

(i) Let $C \in \mathscr{C}$ be such that $C > C_*$ and let I_C be its residual. If E is a C-fuzzy equivalence relation on $\mathbb{X}$, then E is C-transitive. From Lemma 6.14(i) we know that E is also I_C-transitive. Now, we have the following two cases. On the one hand, if $I_C \in \mathscr{I}$, then E is an I_C-fuzzy equivalence relation and the result follows. On the other hand, if $I_C \notin \mathscr{I}$, then we know from Remark 4.11(iv) that there exists an $\epsilon > 0$ such that $I_{[\epsilon]} \in \mathscr{I}$ and $I_{[\epsilon]} > I_C$ and E is $I_{[\epsilon]}$-transitive, or equivalently E is an $I_{[\epsilon]}$-fuzzy equivalence relation.

(ii) Let $I \in \mathscr{I}$ and let C_I be its deresiduum. If E is an I-fuzzy equivalence relation on $\mathbb{X}$, then E is I-transitive. From Lemma 6.14(ii) we know that E is also C_I-transitive. Once again, we have two scenarios. On the one hand, if $C_I \in \mathscr{I}$, then E is a C_I-fuzzy equivalence relation and the result follows. On the other hand, if $C_I \notin \mathscr{C}$, then we consider the modified deresiduum $\widetilde{C_I}$ of I (see Definition 6.11). Clearly, $\widetilde{C_I} \in \mathscr{C}$. Since $\widetilde{C_I} \leqslant C_I$, from Remark 5.7(ii) we see that E is also $\widetilde{C_I}$-transitive. Thus, E is also a $\widetilde{C_I}$-fuzzy equivalence relation. Hence the result. □

Remark 6.16. It should be emphasised that, in Theorem 6.15(i), the condition that the residual I_C of the duality fitting conjunctor C does satisfy (OP') is important, since without it there does not exist any $\epsilon > 0$ such that $I_{[\epsilon]} > I_C$.

The final result in this section – see also Example 6.18 – shows that there can exist C-fuzzy equivalence relations that are not I-fuzzy equivalence relations for any $I \in \mathscr{I}$.

Proposition 6.17. *Let $\mathbb{X}$ be any non-empty set and E be a fuzzy set on $\mathbb{X}^2$. If E is such that E is a C_*-fuzzy equivalence relation and is not a C-fuzzy equivalence relation for any $C > C_*$, then there does not exist any $I \in \mathscr{I}$ such that E is an I-fuzzy equivalence relation.*

Proof. Let us assume to the contrary that there exists some $I \in \mathscr{I}$ such that E is an I-fuzzy equivalence relation. We know from Remark 4.11(iv) that there exists an $\epsilon > 0$ such that $I_{[\epsilon]} \in \mathscr{I}$ and $I_{[\epsilon]} \geqslant I$ and E is an $I_{[\epsilon]}$-fuzzy equivalence relation. Consequently, from Theorem 6.15(ii) we have that E is also a $\widetilde{C_{I_{[\epsilon]}}}$-fuzzy equivalence relation for the modified deresiduum of $I_{[\epsilon]}$. However, by the hypothesis, since $\widetilde{C_{I_{[\epsilon]}}} \in \mathscr{C}$, we see that $\widetilde{C_{I_{[\epsilon]}}} \leqslant C_*$. Since C_* is the lower bound of $\mathscr{C}$ we see that $\widetilde{C_{I_{[\epsilon]}}} = C_*$, a contradiction to the fact that $\epsilon > 0$ (see Remark 6.13(iv)). Thus, there does not exist any $I \in \mathscr{I}$ such that E is an I-fuzzy equivalence relation. □

Example 6.18. Let $\mathbb{X} = [0,1]$ and consider the following fuzzy relation E on $\mathbb{X}$:

$$E(x,y) = \begin{cases} 0.25, & \text{if } x = 0.5 \text{ and } y \in [0,1] \setminus \{0, 0.5\}, \\ 1 - \max(x,y), & \text{if } \min(x,y) = 0, \\ 1, & \text{if } x = y, \\ 0, & \text{otherwise}, \end{cases} \qquad x,y \in \mathbb{X} = [0,1].$$

It is easy to verify that E, indeed, is a C_*-fuzzy equivalence relation. However, E is not an I-fuzzy equivalence relation for any duality fitting implicator I, i.e., for any $I \in \mathscr{I}$. On the contrary, if E were to be an I-fuzzy equivalence relation for some duality fitting implicator I, then the following inequality must be valid for any $z \in [0,1] \setminus \{0, 0.5\}$:

$$I(0.5, 0.25) = I(E(0, 0.5), E(0.5, z)) \geqslant E(0,z) = 1 - z.$$

However, this implies that $I(0.5, 0.25) = 1$ and I is not a duality fitting implicator, since I violates (OP').

Remark 6.19. The following remarks highlight many interesting properties about the relation E in Example 6.18.

(i) There exist $C \in \mathscr{C}$ such that $C > C_* = T_{\mathbf{D}}$ and the relation E above is a C-fuzzy equivalence relation. For instance, the following duality fitting conjunctor C' is the largest duality fitting conjunctor such that E is a C-fuzzy equivalence relation:

$$C'(p,q) = \begin{cases} p \wedge q, & \text{if } p \vee q = 1, \\ 1, & \text{if } p = q \text{ and } p \neq 0, \\ 0.25, & \text{if } p \wedge q = 0.5, \\ 0, & \text{otherwise}, \end{cases} \qquad p, q \in [0,1].$$

(ii) Now, by Remark 5.7(ii), E is a C-fuzzy equivalence relation for every $C \in \mathscr{C}$ such that $C_* \leqslant C \leqslant C'$.
(iii) Note that $I_{C'}(0.5, 0.3) = 1$ and hence $I_{C'}$ does not satisfy (OP'). Hence, from Remark 6.3 (ii) we see that for no C such that $C_* \leqslant C \leqslant C'$ the residual I_c satisfies (OP$'$).
(iv) However, in the case we consider only monotonic duality fitting conjunctors, which form the topic of the next section, then E is a C-fuzzy equivalence relation only for $C = C_*$.

7. Relations between monotonic duality fitting operations

In this section, we investigate monotonic duality fitting operations. Note that a monotonic duality fitting conjunctor is a commutative semi-copula, while a monotonic duality fitting implicator is a fuzzy implication, which is a minimal generalisation of an implicator on {0, 1} to the whole unit interval [0, 1]. It is important to emphasize that a fuzzy implication is hybrid monotonic, i.e., it is non-increasing in the first variable and non-decreasing in the second variable.

Firstly, we give the definition of a fuzzy implication and determine the exact class of fuzzy implications that could be employed as duality fitting implicators. Then we introduce semi-copulas and some subclasses of them, mainly w.r.to the different types of continuity they exhibit. Though left-continuity of a conjunctor is a must for it to satisfy (RP) with its residual, we show that, in our context, the class of fuzzy implications that are duality fitting implicators can be obtained as residuals of more general semi-copulas.

7.1. Fuzzy implications and semi-copulas

Definition 7.1. A function $I : [0,1]^2 \to [0,1]$ is called a *fuzzy implication* if it satisfies the following conditions for all $p, p_1, p_2, q, q_1, q_2 \in [0,1]$:

$$\text{if } p_1 \leqslant p_2, \text{ then } I(p_1, q) \geqslant I(p_2, q), \text{ i.e., } I(\cdot, q) \text{ is decreasing}, \tag{I1}$$
$$\text{if } q_1 \leqslant q_2, \text{ then } I(p, q_1) \leqslant I(p, q_2), \text{ i.e., } I(p, \cdot) \text{ is increasing}, \tag{I2}$$
$$I(0,0) = 1, \quad I(1,1) = 1, \quad I(1,0) = 0. \tag{I3}$$

The set of all fuzzy implications will be denoted by $\mathscr{F}\mathscr{I}$.

Definition 7.2. An $I \in \mathscr{F}\mathscr{I}$ is said to satisfy the ordering property, if

$$I(p,q) = 1 \iff p \leqslant q, \quad p, q \in [0,1]. \tag{OP}$$

It can be immediately noted that Definition 7.1 is the minimal generalisation of the concept of an implicator with monotonicity. The following interdependencies, which are important in our context, exist among the above algebraic properties of a fuzzy implication I, the proof of which is straight-forward and hence not presented.

Lemma 7.3. *Let I be an implicator.*
(i) *If I satisfies* (I1)*, then* $I(p,1) = 1$ *for any* $p \in [0, 1]$.
(ii) If I satisfies (OP) *then it also satisfies* (IP).
(iii) *Let I satisfy* (I1). I satisfies (IP) *if and only if I satisfies the following*:

$$p \leqslant q \Rightarrow I(p,q) = 1, \quad p, q \in [0,1]. \tag{12}$$

(iv) *If I satisfies* (I1), (IP) *and* (OP') then I satisfies (OP).

Remark 7.4. From the above result and Definition 4.4, it is clear that the class of monotonic duality fitting implicators are exactly the class of fuzzy implications that satisfy (LNP) and (OP).

Definition 7.5 (*cf.*[37,20]). Consider a mapping $C : [0,1]^2 \to [0,1]$.

(i) C is a *semi-copula* if it is monotonic increasing in both variables and $C(1,p) = C(p,1) = p$ for every $p \in [0, 1]$.
(ii) C is said to be *border continuous* in the first variable if for every increasing sequence $\{p_n\}$ in [0, 1] that converges to 1, we have that, for any $q \in [0, 1]$

$$\lim_{n\to\infty} C(p_n, q) = C(\lim_{n\to\infty} p_n, q) = C(1,q) = q.$$

(iii) C is said to be *left-continuous* in the first variable if for every increasing sequence $\{p_n\}$ in [0, 1] we have that, for any $q \in [0, 1]$

$$\lim_{n\to\infty} C(p_n, q) = C(\lim_{n\to\infty} p_n, q).$$

(iv) C is said to be *right-continuous* in the first variable if for every decreasing sequence $\{p_n\}$ in [0, 1] we have that, for any $q \in [0, 1]$

$$\lim_{n\to\infty} C(p_n, q) = C(\lim_{n\to\infty} p_n, q).$$

(v) Border-, left- and right-continuities in the second variable can be defined analagously. In the case the semi-copula is also commutative, then any of the continuities in one variable immediately implies the same in the other.

7.2. Semi-copulas and their residuals

As seen earlier, from a conjunctor C one can obtain an implicator as its residual. Noting that monotonic duality fitting conjunctors are commutative semi-copulas, in the following, we consider this procedure in the case of semi-copulas. From Remark 7.4 we know that the minimal assumptions on a fuzzy implication $I \in \mathcal{FI}$ for it to be a duality fitting implicator are that I satisfies (LNP) and (OP). Hence, in the following we show the minimal assumptions on a semi-copula C whose residual is a duality fitting fuzzy implication.

Theorem 7.6

(i) *If C is a semi-copula that is border-continuous in the second variable then $I_C \in \mathcal{FI}$ and satisfies (LNP) and (OP).*
(ii) *Conversely, if an $I \in \mathcal{FI}$ satisfies (LNP) and (OP), then C_I is a semi-copula that is border-continuous in the second variable.*

Proof. Firstly, we note the following. Let C be any semi-copula and for any fixed $p \in [0, 1]$, consider the vertical segment $C_p(\cdot) = C(p, \cdot)$. Obviously, C_p is a one-variable function from [0, 1] to [0, p]. Now notice that if C is border continuous in the second variable, then for every $p \in (0, 1)$ there exists a neighborhood $U_p = (\alpha_p, 1]$, where $\alpha_p \in (0,1)$ is dependent on the chosen p, such that C_p is continuous on U_p.

(i) This is straight forward. The hybrid monotonicity of I_C follows from the monotonicity of C, while the boundary conditions of I_C follow from the boundary conditions on C. Note also that I_C satisfies (LNP), since

$$I_C(1,q) = \sup\{t | C(1,t) \leqslant q\} = q, \quad q \in [0,1].$$

Let C be a semi-copula that is border-continuous in the second variable. On the contrary, let I_C not satisfy the ordering property (OP). Since for any C we have that $p \leqslant q \Rightarrow I_C(p,q) = 1$, there exists $p_0, q_0 \in (0,1)$ such that $p_0 > q_0$ and $I_C(p_0, q_0) = 1$. Let $p' = C_{p_0}(\alpha_{p_0}) = C(p_0, \alpha_{p_0}) \leqslant p_0$. Now, we have two cases. If $q_0 > p'$ then there exists $t \in U_{p_0}$ and $t \neq 1$ such that $C_{p_0}(t) = q_0$ contradicting our assumption that $I_C(p_0, q_0) = 1$. On the other hand, if $q_0 \leqslant p'$ then by definition $I_C(p_0, q_0) \leqslant \alpha_{p_0} < 1$. Hence I_C satisfies (OP).

(ii) Let $I \in \mathcal{FI}$ and satisfy (LNP) and (OP). Let C_I be defined as in (8).
- C_I **is monotonic in the first variable**: Let $p_1, p_2, q \in [0,1]$ such that $p_1 \leqslant p_2$. By (I1) if a $t \in [0, 1]$ is such that $I(p_2, t) \geqslant q$ then $I(p_1, t) \geqslant q$, i.e.,

$$\{t \in [0,1] | I(p_1, t) \geqslant q\} \supseteq \{t \in [0,1] | I(p_2, t) \geqslant q\},$$

from which we obtain that $C_I(p_1, q) \leqslant C_I(p_2, q)$ for any $p_1, p_2, q \in [0,1]$.
- C_I **is monotonic in the second variable**: Let $p, q_1, q_2 \in [0,1]$ and $q_1 \leqslant q_2$. Once again from (I2) we have

$$\{t \in [0,1] | I(p,t) \geqslant q_1\} \supseteq \{t \in [0,1] | I(p,t) \geqslant q_2\},$$

from which we obtain that $C_I(p, q_1) \leqslant C_I(p, q_2)$ for any $p, q_1, q_2 \in [0,1]$.
- The following equalities show that C_I indeed is a semi-copula (see also Lemma 6.10):

$$C_I(p,1) = \inf\{t \in [0,1] | I(p,t) \geqslant 1\} = p, \text{ by (OP)},$$
$$C_I(1,p) = \inf\{t \in [0,1] | I(1,t) \geqslant p\} = p, \text{ by (LNP)}.$$

- Let I_C satisfy (OP). On the contrary, if C is not border continuous in the second variable, then there exists an $p_0 \in (0,1)$ such that $\lim_{q\to 1^-} C(p_0, q) = z < p_0$. Now, by definition

  $I_C(p_0, z) = \sup\{t \in [0,1] | C(p_0, t) \leqslant z\} = 1,$

  a contradiction to the fact that I_C satisfies (OP). $\square$

Remark 7.7. We emphasize the following important remarks on Theorem 7.6:

(i) The assumptions on an I to obtain a semi-copula C_I as its deresiduum are minimal.
(ii) The semi-copula C is not assumed to be commutative. However, note that this class of semi-copulas does not subsume the class of monotonic duality fitting conjunctors, since they are not expected to be border-continuous.
(iii) It should be emphasized that, in general, border-continuity of a duality fitting conjunctor is not required for the corresponding residual to satisfy (OP). For example, consider the largest duality fitting conjunctor C^*, which is not border-continuous, but whose residual $I_{[1]}$ satisfies (OP).

Theorem 7.8. ([20], Theorem 3.2 and Proposition 3.3).

(i) *If C is a semi-copula that is left-continuous in both the variables then $I_C \in \mathcal{FI}$, satisfies (LNP), (OP) and I_C is left-continuous in the first variable and right-continuous in the second variable.*
(ii) *Conversely, if an $I \in \mathcal{FI}$ satisfies (LNP), (OP) and is left-continuous in the first variable and right-continuous in the second variable, then C_I is a semi-copula that is left-continuous in both the variables.*

Now, we give a characterisation of monotonic left-continuous duality fitting conjuctors C, i.e., left-continuous commutative semi-copula, such that their residuals I_C are not only duality fitting implicators but also such that every C-fuzzy equivalence relation E on a set $\mathbb{X}$ is also an I_C-fuzzy equivalence relation and vice-versa, i.e., $E \in \mathcal{CFE}_C(\mathbb{X}) \Longleftrightarrow E \in \mathcal{IFE}_{I_C}(\mathbb{X})$.

Theorem 7.9. ([20], Theorem 3.5).

(i) *If C is a left-continuous commutative semi-copula, then $I_C \in \mathcal{FI}$, is left-continuous in the first variable and right-continuous in the second variable and satisfies (LNP), (OP) and (WE).*
(ii) *Conversely, if an $I \in \mathcal{FI}$, is left-continuous in the first variable and right-continuous in the second variable and satisfies (LNP), (OP) and (WE), then C_I is a left-continuous commutative semi-copula.*

7.3. Bounds on monotonic duality fitting operations

We know that the class of duality fitting conjunctors is bounded below by C_* and above by C^*, while the class of duality fitting implicators is bounded below by I_* and above (but not including) by I^*.

As noted in [42, p. 175], monotonic duality fitting conjunctors are binary aggregation operators on [0, 1] with neutral element 1 and annihilator 0, which are symmetric non-decreasing functions between C_*, which is the weakest t-norm known as the drastic t-norm $T_{\mathbf{D}}$, and the strongest t-norm $T_{\mathbf{M}}$, the minimum function. Interestingly, their residuals form the bounds for the monotonic duality fitting implicators, i.e., any $I \in \mathcal{FI}$ that is also a duality fitting implicator is bounded below by the Gödel implication

$$I_{\mathbf{GD}}(p,q) = \begin{cases} 1, & \text{if } p \leqslant q, \\ q, & \text{if } p > q, \end{cases}$$

and bounded above by, but not including, I^* which is the residual of C_*.

8. Concluding remarks

In this work we have investigated fuzzy equivalence relations and fuzzy partitions where the respective transitivities were defined with respect to an implicator I, instead of a conjunctor C as done in Mesiar et al. [42]. We have determined the minimal conditions on an implicator to be a *duality fitting implicator*, i.e., an implicator I such that every I-fuzzy equivalence relation is also a I-fuzzy partition and vice-versa. Our studies show that not all of the properties possessed by residuals of left-continuous t-norms is quite relevant, for example, the exchange principle does not play any role in our study. This can also be seen from the fact that not all residuals of duality fitting conjunctors, i.e., conjunctors C such that every C-fuzzy equivalence relation is also a C-fuzzy partition and vice-versa, are duality fitting implicators. Also there exist duality fitting implicators that cannot be obtained as residuals of duality fitting conjunctors.

Regarding the relationships between the concepts of C-fuzzy equivalence relations and I-fuzzy equivalence relations, we have shown that although, in general, every C-fuzzy equivalence relation can be shown to be an I-fuzzy equivalence relation and vice-versa, there does exist C-fuzzy equivalence relations that are not I-fuzzy equivalence relations for any duality fitting implicator even though C is a duality fitting conjunctor, showing that these concepts are not equivalent. Note also that, due to Lemma 4.8, for a non-singleton $\mathbb{X}$, we see that the smallest class $\mathscr{IFE}_{I_*}(\mathbb{X})$ of all fuzzy equivalence relations on $\mathbb{X}$ based on a duality fitting implicator is a proper subclass of the smallest class $\mathscr{CFE}_{C^*}(\mathbb{X})$ of all fuzzy equivalence relations on $\mathbb{X}$ based on a duality fitting conjunctor (see Remark 5.7(v)).

It should be remarked that, in the Definition 3.1 of an I-fuzzy partition, one could instead use the following equivalent condition in the case of partitions:

(iii′) if $U(x) = 1$ for some $x \in \mathbb{X}$ then for all $y \in \mathbb{X}$ and all $V \in P$ the following inequality holds:

$$I(V(x), U(y)) \geqslant V(y).$$

However, it can be verified, without much tedium, that considering the above inequality does not alter the conditions eventually required on a duality fitting implicator.

Finally, we remark that all the concepts proposed and the results proven in Klawonn and Jacas [33] in the framework of *-fuzzy equivalence relation with * being a conjunction in a GL-monoid, can be, in a natural way, reformulated and proven for the fuzzy equivalence relations based on the residual of *.

Acknowledgement

This work was done when the first author was a visiting researcher at the Department of Mathematics and Descriptive Geometry, Faculty of Civil Engg., Slovak University of Technology, Bratislava. He gratefully acknowledges the grant obtained from SAIA under the National Scholarship Programme (NSP) for the mobility of Teachers and Researchers that made this visit possible. The second author was supported by the Grants VEGA 1/4209/07, APVV-0012-07 and GAČR 402/08/0618.

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