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# A COLLECTOR FOR INFORMATION WITHOUT PROBABILITY IN A FUZZY SETTING

Doretta Vivona and Maria Divari

In the fuzzy setting, we define a collector of fuzzy information without probability, which allows us to consider the reliability of the observers. This problem is transformed in a system of functional equations. We give the general solution of that system for collectors which are compatible with composition law of the kind "inf".

*Keywords:* information measure, system of functional equations

*AMS Subject Classification:* 93E12, 62A10, 62F15

## 1. INTRODUCTION

In the subjective theory of information without probability [9, 10, 11, 12, 15] and in the crisp setting, B. Forte and others [3, 7, 8] have supposed that each group of observers $E$ provides an amount of information $J(A, E)$ from the same event $A$. Moreover they supposed that, for each $E$, the information is compositive (in the sense of [13] with the same law with an additive reliability coefficient $\lambda(E)$.

B. Forte has defined a *collector* as a function $\Phi$:

$$J(A, E_1 \cup E_2) = \Phi\Big(\lambda(E_1), \lambda(E_2), J(A, E_1), J(A, E_2)\Big)$$

for every event $A$ and disjoint groups $E_1,\ E_2$.

Putting $x = \lambda(E_1),\ y = \lambda(E_2),\ u = J(A, E_1),\ v = J(A, E_2)$, Aczél, Forte and Ng in [1, 2] gave the solution in the Shannon case:

$$\Phi(x, y, u, v) = -c \log\left(\frac{x\ e^{-u/c} + y\ e^{-v/c}}{x + y}\right),$$

where $c$ is the constant related to the Shannon information; when the information $J$ is of the kind $\wedge$, Benvenuti, Divari and Pandolfi obtained a more general class of solutions (see [4]).

In a previous paper [16] we have defined collectors of $\wedge$-compositive information without probability for fuzzy sets of events, crisp sets of observers with a reliability coefficient defined in a probabilistic space.

In this paper we shall introduce fuzzy collectors for crisp groups of observers with a fuzzy $\vee$-additive measure of reliability.

Evidently, if we restrict our considerations to crisp sets, the collectors studied in [4] are recovered. One of the main aim of this paper is also to enlight interesting ideas from [4] which are not so known in the wider community.

## 2. PRELIMINARIES

In the setting of *fuzzy sets* [17], we consider the following model:

1) $\Omega$ is an abstract space, $\mathcal{F}$ is an algebra of fuzzy sets such that $(\Omega, \mathcal{F})$ is a fuzzy measurable space, the elements of $\mathcal{F}$ are the *observable events*. Recall that for $A$ and $B \in \mathcal{F}$, whose membership functions are $f_A$ and $f_B$, respectively, it holds: $f_{A\cup B} = f_A \vee f_B, f_{A\cap B} = f_A \wedge f_B, f_{A^c} = 1 - f_A$;

2) $\mathcal{O}$ is another abstract space (space of observers), $\mathcal{E}$ is a $\sigma$-algebra contained in $\mathcal{P}(\mathcal{O})$, whose elements are groups of *observers*;

3) a fuzzy $\vee$-additive measure $\mu$ is defined on the measurable space $(\mathcal{O}, \mathcal{E})$: $\mu(\emptyset) = 0, \mu(\mathcal{O}) = \overline{\mu} \in ]0, +\infty]$, $\mu$ is non-decreasing with respect to the inclusion of the elements of $\mathcal{E}$ and $\mu(E_1 \cup E_2) = \mu(E_1) \vee \mu(E_2)\ \forall\, E_1, E_2 \in \mathcal{E}$; if $E \in \mathcal{E}$, $\mu(E)$ is called *fuzzy reliability coefficient*;

4) an information measure $J$, called *fuzzy information* (see [5, 6]), linked to the group of observers, is a map $J : \mathcal{F} \times \mathcal{E} \to \overline{\mathbb{R}}^+$ such that, fixed $E \in \mathcal{E}, E \neq \emptyset, \neq \mathcal{O}$ for all $A, B \in \mathcal{F}$

4i) $A \subset B \Rightarrow J(A, E) \geq J(B, E)$,

4ii) $J(\emptyset, E) = +\infty,\ J(\Omega, E) = 0$;

5) every information measure $J(\cdot, E)$ is $F_E$*-compositive* i. e. for every $E \in \mathcal{E}, \mathrm{E} \neq \emptyset$ there exists a map $F_E : \Gamma_E \to \overline{\mathbb{R}}^+$, where $\Gamma_E = \{(x, y) \,/\, \exists\, A,\, B \in \mathcal{F}$ with $x = J(A, E),\ y = J(B, E), f_A \wedge f_B = 0\}$ such that

$$J(A \cup B; E) = F_E \Big( J(A, E), J(B, E) \Big) . \tag{1}$$

Evidently $F_E$ is commutative, associative and $F_E(x, +\infty) = x$ , for all $x \in \operatorname{Ran} J(\cdot, E)$.

Throughout this paper we deal with universal composition rule $F = \wedge$,

$$J(A \cup B, E) = F[J(A, E), J(B, E)] = J(A, E) \wedge J(B, E). \tag{2}$$

Note that due the idempotency of the operator $\wedge$ we need not to require the disjontness $f_A \wedge f_B = 0$ in the above equality (2).

We call $\wedge$-*compositive fuzzy information* a fuzzy information $J$ which satisfies (2) for every $E \in \mathcal{E}$.

## 3. COLLECTOR OF $\wedge$-COMPOSITIVE FUZZY INFORMATION

In the previous paper [16] we have defined a collector for crisp sets.

Here, in the fuzzy setting, we give the definition of collector which we shall call *fuzzy collector*.

**Definition 3.1.** A *fuzzy collector* for a given reliability measure $\mu$ is a continuous function $\Psi$

$$\Psi : \Sigma \to \overline{\mathbb{R}}^+$$

where $\Sigma \subset \Big([0,\overline{\mu}]\times\overline{\mathbb{R}}^+\Big)^2$, $\overline{\mu} = \mu(\mathcal{O})$, such that for every pair of two disjoint groups $E_1$ and $E_2$ of observers with reliability coefficients $\mu(E_1)$ and $\mu(E_2)$ it holds

$$J(A, E_1 \cup E_2) = \Psi\Big(\mu(E_1), J(A, E_1), \mu(E_2), J(A, E_2)\Big). \tag{3}$$

## 4. PROPERTIES OF A FUZZY COLLECTOR $\Psi$

In this section we present the properties if a fuzzy collector is expressed by $\Psi$. They are:

$(i)$ (commutativity):

$$\Psi\Big(\mu(E_1), J(A, E_1), \mu(E_2), J(A, E_2)\Big) = \Psi\Big(\mu(E_2), J(A, E_2), \mu(E_1), J(A, E_1)\Big),$$

$\forall\, A \in \mathcal{F}, E_1, E_2 \in \mathcal{E}$, as $J(A, E_1 \cup E_2) = J(A, E_2 \cup E_1)$;

$(ii)$ (associativity):

$$\Psi\Big(\mu(E_1) \vee \mu(E_2), \Psi\Big(\mu(E_1), J(A, E_1), \mu(E_2), J(A, E_2)\Big), \mu(E_3), J(A, E_3)\Big)$$

$$= \Psi\Big(\mu(E_1), J(A, E_1), \mu(E_2) \;\vee\; \mu(E_3), \Psi\Big(\mu(E_2), J(A, E_2), \mu(E_3), J(A, E_3)\Big)\Big),$$

$\forall\, A \in \mathcal{F},\; E_1, E_2, E_3 \in \mathcal{E}$, as $J(A, (E_1 \cup E_2) \cup E_3) = J(A, E_1 \cup (E_2 \cup E_3)$;

$(iii)$ (universal value $J(\emptyset, E) = +\infty$ ):

$$\Psi\Big(\mu(E_1), +\infty, \mu(E_2), +\infty\Big) = +\infty,$$

as $J(\emptyset, E_1 \cup E_2) = +\infty$;

$(iv)$ (universal value $J(\Omega, E) = 0)$):

$$\Psi\Big(\mu(E_1), 0, \mu(E_2), 0\Big) = 0,$$

as $J(\Omega, E_1 \cup E_2) = 0$.

If the information of the group of observers is $\wedge$-compositive in the sense of (2) we can add another property:

$(v)$ (compatibility condition between the $\wedge$-compositivity of $J$ and the collector $\Psi$):

$$\Psi\Big(\mu(E_1),\Big[J(A,E_1)\ \wedge J(B,E_1)\Big],\mu(E_2),\Big[J(A,E_2)\ \wedge J(B,E_2)\Big]\Big)$$
$$= \Psi\Big(\mu(E_1),J(A,E_1),\mu(E_2),J(A,E_2)\Big)\ \wedge\Psi\Big(\mu(E_1),J(B,E_1),J(B,E_2),\mu(E_2),\Big)$$
$$\forall\, A,B\in\mathcal{F},\ \mathrm{E}_1,\mathrm{E}_2\in\mathcal{E}.$$

In fact, from (2) it is $J(A\cup B,E_1\cup E_2)=J(A,E_1\cup E_2)\wedge J(B,E_1\cup E_2)$, and, on the other hand, from (3), we get $J(A\cup B,E_1\cup E_2)=$

$$\Psi\Big(\mu(E_1),J(A\cup B,E_1),\mu(E_2),J(A\cup B,E_2)\Big)$$
$$=\Psi\Big(\mu(E_1),\Big[J(A,E_1)\ \wedge J(B,E_1)\Big],\mu(E_2),\Big[J(A,E_2)\wedge J(B,E_2)\Big]\Big).$$

## 5. SYSTEM OF FUNCTIONAL EQUATIONS

Put $\mu(E_1)=x,\mu(E_2)=y,\mu(E_3)=z$, with $x,y,z\in[0,1]$. The function $\Psi$ given in (3) is defined in the domain $\Sigma^2=([0,\overline{\mu}]\times\overline{\mathbb{R}}^+)^2$. Moreover we set $J(A,E_1)=u,\ J(A,E_2)=v,\ J(B,E_1)=u',\ J(B,E_2)=v',\ J(A,E_3)=w$.

Now we rewrite the conditions $[(i)-(v)]$ in order to obtain a system of functional equations. The equations are:

$$\begin{cases}(i') & \Psi\Big(x,u,y,v\Big)\ =\Psi\Big(y,v,x,u\Big)\\ (ii') & \Psi\Big(x,u,y\vee z,\Psi(y,v,z,w)\Big)\ =\Psi\Big(x\vee y,\Psi(x,y,u,v),z,w\Big)\\ (iii') & \Psi\Big(x,+\infty,y,+\infty\Big)\ =+\infty\\ (iv') & \Psi\Big(x,0,y,0\Big)\ =0\\ (v') & \Psi\Big(x,u\wedge u',y,v\wedge v'\Big)\ =\Psi\Big(x,u,y,v\Big)\ \wedge\Psi\Big(x,u',y,v'\Big).\end{cases}$$

In the setting of crisp sets, an analogous system was studied and solved by Benvenuti–Divari–Pandolfi in [4]. We study the system $[(i')-(v')]$ and we give the general solution step by step.

**Theorem 5.1. Main Theorem.** The function $\Psi\Big(x,u,y,v\Big)$ is solution of the system $[(i')-(v')]$ if and only if

$$\Psi\Big(x,u,y,v\Big)\ =g(x,y,u)\wedge g(y,x,v) \tag{4}$$

where the function $g:[0,\overline{\mu}]^2\times\overline{\mathbb{R}}\to\overline{\mathbb{R}}$ fulfills the following properties:

$(\alpha)$ $g$ is non decreasing with respect to $u$ and continuous,

$(\beta)$ $g(x,y,+\infty)=+\infty$,

$(\gamma)$ $g(x,y,0)\wedge g(y,x.0)=0$,

$(\delta)$ $g[x\vee z,y,g(x,z,u)]=g(x,y\vee z,u)$.

P r o o f. Putting $g(x, y, u) = \Psi(x, u, y, +\infty)$, from $(v')$ for $u' = v$, we have the (4). It is easy to verify that every function $\Psi$ with the form (4) and the properties $[(\alpha) - (\delta)]$ is a solution of the system $[(i') - (v')]$. □

For every function $g(x, y, u)$ which satisfies the properties $[(\alpha) - (\delta)]$, we can prove the following Lemmas.

**Lemma 5.2.** For every function $g(x, y, u)$ which satisfies the properties $[(\alpha) - (\delta)]$, we have $g(x, y, 0) = 0$.

P r o o f. From $(\delta)$, for $u = 0$ it is

$$g(x \vee z, y, g(x, z, 0)) = g(x, y \vee z, 0), \tag{5}$$

and then, changing $x$ with $z$

$$g(z \vee x, y, g(z, x, 0)) = g(z, y \vee x, 0). \tag{6}$$

Because of $(\gamma)$, either $g(x, z, 0) = 0$ or $g(z, x, 0) = 0$, from $(\alpha)$, (4) and (5) we get

$$g(x \vee z, y, 0) = g(x, y \vee z, 0) \wedge g(z, y \vee x, 0), \tag{7}$$

and from (7) for $y = 0$

$$g(x \vee z, 0, 0) = g(x, z, 0) \wedge g(x, z, 0) \tag{8}$$

i. e., due to $(\gamma)$,

$$g(x \vee z, 0, 0) = 0 \;\forall\, x,\; z. \tag{9}$$

Finally, from (8) and (9), for $x = z$, we get

$$g(z, z, 0) = 0. \tag{10}$$

For $x \leq z$

$$g(z, x, 0) = g(x, z, 0) \wedge g(z, x, 0),$$

so we obtain, due to $(\gamma)$,

$$g(z, x, 0) = 0 \;\forall\, x \leq z. \tag{11}$$

Putting in (7) $y = x$ and for $x > z$

$$g(x, x, 0) = g(x, x, 0) \wedge g(z, x, 0),$$

$$g(x \vee z, x, 0) = g(x, x \vee z, 0) \wedge g(z, x \vee x, 0).$$

From $(\delta)$, for $u = 0$

$$g(x \vee z, y, g(x, z, 0)) = g(x, y \vee z, 0).$$

By contradiction we suppose $g(z, x, 0) = \lambda > 0$, i. e. $g(x, y, \lambda) = g(x, y \vee z, 0)$. For $y > z$, we get $g(x, y, \lambda) = g(x, y, 0)$: this is impossible as $g$ is non-decreasing with respect to $u$, then

$$g(z, x, 0) = 0 \;\forall\, x,\; z. \tag{12}$$

□

**Lemma 5.3.** For every function $g$ which enjoys $[(\alpha) - (\delta)]$, we have

$$g(x, 0, u) = u \text{ in } [0, \overline{\mu}] \times \overline{\mathbb{R}}^+. \tag{13}$$

Proof. As, from $(\gamma)$ and $(\delta)$, $g(x, 0, 0) = 0$ and $g(x, 0, +\infty) = +\infty$, for every $v \in \overline{\mathbb{R}}^+$ there exists $u$ such that $g(x, 0, u) = v$.

From $(\gamma)$, for $y = z = 0$, we have $g(x, 0, g(x, 0, v)) = g(x, 0, u)$. □

**Lemma 5.4.** Every function $g$ which satisfies $[(\alpha) - (\delta)]$ has the following representation:

$$g(x, y, u) = h[x \vee y, h^{-1}(x, u)] \ (x, y, u) \in [0, 1]^2 \times \overline{\mathbb{R}}^+ \tag{14}$$

with $h : [0, 1] \times \overline{\mathbb{R}}^+ \to \overline{\mathbb{R}}^+$, continuous, non decreasing with respect to $u$ and $h^{-1}$ its pseudo-inverse [14], defined by $h^{-1}(x, v) = \text{Inf}\{\xi \ / \ h(x, \xi) = v\}$.

Proof. Putting $h(x, u) = g(0, x, u)$, for $(\alpha)$ and $(\beta)$ the function $h$ is continuous, monotone and $h(x, 0) = 0$, $h(x, +\infty) = +\infty$, therefore its pseudo-inverse $h^{-1}$ is defined on $[0, 1] \times \overline{\mathbb{R}}^+$. From $(\delta)$, for $x = 0$ and $u = h^{-1}(z, v)$, we have $g(z, y, g(0, z, h^{-1}(z, v)) = g(0, y \wedge z, h^{-1}(z, v))$, i.e. $g(z, y, g(0, z, h^{-1}(z, v)) = h(y \wedge z, h^{-1}(z, v))$. The thesis follows from $h(z, h^{-1}(z, v)) = v$. □

**Remark.** We observe that continuity of $g$ and condition $(\beta)$ imply that $h(x, u) = g(0, x, u)$ is not (definitely) null or constant (unless $= +\infty$). Indeed, if we hold the following situation: $g(x, y, u) = \dfrac{x \ u}{x \vee y}$ (with $0 \cdot +\infty = 0$), then we couldn't find $h^{-1}$, but clearly $g(0, x, u) = 0$, contrary to $(\beta)$.

This situation corresponds to the following example:

Let $\mathcal{O} = \{1, 2, \ldots, \text{n}\}$ be the set of observers, $\mu(E) = \max E$ and $J(A, E) = \frac{-\log \inf f_A}{\mu(E)}$. So, we have: $g(x, y, u) = \frac{x \ u}{x \vee y}$, $h(x, u) = 0$ and the collector is: $\Psi(x, u, y, v) = \frac{x \ u \wedge y \ v}{x \vee y}$.

**Lemma 5.5.** For every function $g$ which satisfies $[(\alpha) - (\delta)]$, the corresponding function $h$ given by (14) enjoys the following properties:

$$h(0, v) = v \in \overline{\mathbb{R}}^+ \tag{15}$$

and

$$h(x, u) = h(x, v) \Rightarrow h(y, u) = h(y, v) \ \forall \, y > x. \tag{16}$$

Proof. The condition (15) follows from the definition of the function $h$ and from Lemma 5.4. Now, we shall prove the (16): in $(\delta)$ setting $x = 0$ it is $g(z, y, g(0, z, u)) = g(0, y \wedge z, u)$ and for (14) we get $h(z \vee y, h^{-1}(z, h(z, h^{-1}(0, u))) = h(y \vee z, h^{-1}(0, u))$, i.e. $h(z \vee y, h^{-1}(z, h(z, u)) = h(y \vee z, u)$.

If $h(z, v) = h(z, u)$ with $v < u$, from definition of $h^{-1}$, we have $h^{-1}(z, h(z, u)) = \text{Inf}\{\xi \, / \, h(z, \xi) = h(z, u)\} = v' \leq v$ and therefore $h(y \wedge z, v') = h(y \wedge z, u)$.

If $v > u$, for the monotonicity of the function $h$ and the arbitrary of $y$, we obtain the (16). □

**Lemma 5.6.** The expression (14) with the function $h(x,u)$ satisfying the conditions of the Lemmas 5.4 and 5.5 gives the general form of the continuous solutions of the system $[(\alpha)-(\delta)]$.

Proof. We shall, now, verify that every function $g(x,y,u)$ defined by (14)

$$g(x,y,u) = h(x \vee y, h^{-1}(x,u))$$

with $h(x,u)$ satisfying the conditions of the Lemmas 5.4 and 5.5 is solution of the system $[(\alpha)-(\delta)]$. In fact, for the properties of $h$ in Lemma 5.5, the properties $(\alpha)$ and $(\beta)$ are verified. The $(\delta)$ becomes $g(x\vee z, y, g(x,z,u) = g(x, y\vee z, u)$ and then

$$h\left(x\vee z\vee y, h^{-1}(x\vee z, h(x\vee z, h^{-1}(x,u)))\right) = h\left(x\vee z\vee y, h^{-1}(x,u)\right). \tag{17}$$

Putting $h^{-1}(x,u) = v$, the (17) becomes $h(x\vee z\vee y, h^{-1}(x\vee z, h(x\vee z, v))) = h(x\vee z\vee y, v)$. Moreover $h^{-1}(x\vee z, h(x\vee z, v)) = \mathrm{Inf}\{\xi \,/\, h(x\vee z,\xi) = h(x\vee z, v)\} = v' \leq v$, with $h(x\vee z, v') = h(x\vee z, v)$. For the (16), as $x\vee z\vee y \geq x\vee z$ and $h(x\vee y\vee z, v') = h(x\vee y\vee z, v)$, we have the $(\delta)$. □

Summarizing the previous Lemmas, we obtain the following main result:

**Theorem 5.7.** The general solution of the system $[(i')-(v')]$ is the function

$$\Psi(x,y,u,v) = h\left(x\vee y, h^{-1}(x,u)\wedge h^{-1}(y,v)\right)$$

where $h : [0,1]\times\overline{\mathbb{R}}^{+} \to \overline{\mathbb{R}}^{+}$ satisfies the following conditions:

— $h(x,\cdot)$ is non-decreasing, continuous, $h(x,0)=0$, $h(x,+\infty)=+\infty$, $\forall x \in (0,\overline{\mu}]$,

— $h(x,u) = h(x,v) \Rightarrow h(y,u) = h(y,v)$ for every $y > x$.

**Example:** Let $h(x,u) = e^{x}\,u$, this function satisfies the hypotheses of the Theorem above; its pseudo-inverse is $h^{-1}(x,v) = \dfrac{v}{e^{x}}$. Then the function $g$ is

$$g(x,y,u) = h(x\vee y, h^{-1}(x,u)) = e^{x\vee y}\,h^{-1}(x,u) = e^{x\vee y}\,u\,e^{-x} = u\,e^{(x\vee y)-x}.$$

Then the collector $\Psi$ has the following expression:

$$\begin{aligned}\Psi\left(x,y,u,v\right) &= g(x,y,u)\wedge g(y,x,v) \\ &= u\,e^{(x\vee y)-x}\wedge v\;e^{(y\vee x)-y} = e^{x\vee y}\;\left(\frac{u}{e^{x}}\wedge\frac{v}{e^{y}}\right).\end{aligned} \tag{18}$$

Let $J$ be an information measure on crisp sets such that $J(E) = e^{-\lambda(E)}$ with $\lambda$ a fuzzy measure $\vee$-additive and $J(A,E)$ a general information depending on the set $E$ of observers instead of general conditional information.

From (3) and (18), we get

$$J(A, E_1\cup E_2) = \frac{J(A,E_1)J(E_1)\wedge J(A,E_2)J(E_2)}{J(E_1\cup E_2)}.$$

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