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A NEW ROBUST STABILITY MARGIN

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The aim of this paper is to derive a new robust stability margin. Known sufficient conditions for robust stability stated in gap-metric sense contain inherent conservativeness in the formulation of the various steps. In this paper conservativeness in one of the steps is removed, resulting in a new robustness margin. The key issue is that more information of the specific controller is taken into consideration. The resulting robustness margin is less conservative than the margin in directed gap-metric sense and is as easy to compute. The advantage of this robustness margin will be illustrated by an example.

1. INTRODUCTION

When a perfect model of the real plant is available, it will be non-linear and of extremely high order. In engineering practice the plant will be described by a high order linear model. This nominal model is an approximation of the real plant. The discrepancy between the nominal model and the plant is then approximated by an uncertainty plant model. The plant uncertainty models are allowed to contain a different number of unstable poles. This leads in a natural way to a coprime factor approach of model descriptions.

In the next step a controller will be designed in such a way that it robustly stabilizes the nominal model with a prespecified performance, methods to design such controller are for example given in [2, 4, 8].

The feedback loop will be called robustly stable if the closed loop transfer function $T(P,C)$ remains stable for all plant variations described by the uncertainty plant model.

In some recent papers [2, 6] a sufficient condition for robust stability of a closed loop system under plant perturbations have been stated in the gap-metric. In the gap-metric robustness the nominal plant is factorized in normalized coprime factors. The difference between a perturbed plant and the nominal plant is described by perturbations on the normalized coprime factors of the nominal plant. Robustness of the closed loop for a class of perturbed plants is guaranteed if the norm of the coprime perturbations is small enough. The maximum allowable norm of the perturbations is determined by the infinity norm of the closed loop, hence only crude information about the nominal plant and controller is taken into consideration.

The main idea behind the new and less conservative robustness margin to be considered in this paper is to take more information about the nominal closed loop system into account. One can think about this information as refinement of the infinity norm to frequency dependent maximum singular values, and the directionality of the feedback loops in multivariable systems.

In order to take the closed loop characteristics into account, a normalized coprime factorization of the nominal plant is modified by a normalized coprime factorization the nominal controller.

The difference between a perturbed plant and the nominal plant is now described by perturbations on the modified coprime factors of the nominal plant which includes detailed information about the controller.

It will be shown that this new robustness margin allows a larger class of coprime factor plant perturbations than allowed in the gap-metric.

The layout of this paper is as follows: after some preliminaries in Section 2 stability of a nominal closed loop system is discussed in Section 3. Then the new robustness margin will be derived in Section 4. Relations with other robustness margins, such as the gap-metric, will be discussed in Section 5. The whole procedure will be illustrated by an example in Section 6 followed by conclusions in Section 7.

2. PRELIMINARIES

In this note we adopt the ring theoretic setting of [3, 9] to study stable multivariable linear systems by considering them as transfer function matrices having all entries belonging to the ring $\mathcal{H}$. Here we will identify the ring $\mathcal{H}$ with $\mathbb{R}H_\infty$, the space of stable real rational finite dimensional linear time–invariant continuous–time systems. We consider the class of possibly non–proper and/or unstable multivariable systems as transfer function matrices whose entries are elements of the quotient field $\mathcal{F}$ of $\mathcal{H}$ ($\mathcal{F} := \{a/b \mid a \in \mathcal{H},\ b \in \mathcal{H}\backslash 0\}$). The set of multiplicative units of $\mathcal{H}$ is defined as: $\mathcal{J} := \{h \in \mathcal{H} \mid h^{-1} \in \mathcal{H}\}$. In the sequel systems $P \in \mathcal{F}^{m\times n}$ are denoted as $P \in \mathcal{F}$.

Definition 2.1. [9] A plant $P \in \mathcal{F}$ has a right (left) fractional representation if there exist $N, M(\tilde{N}, \tilde{M}) \in \mathcal{H}$ such that $P = NM^{-1}$ $(= \tilde{M}^{-1}\tilde{N})$.

The pair M, $N(\tilde{M}, \tilde{N})$ is a right (left) coprime fractional representation of P (*rcf* or *lcf*) if it is a right (left) fractional representation of P and there exists $U, V(\tilde{U}, \tilde{V}) \in \mathcal{H}$ such that: $UN + VM = I$ $(\tilde{N}\tilde{U} + \tilde{M}\tilde{V} = I)$

The pair $M, N(\tilde{M}, \tilde{N})$ is called a normalized right (left) coprime fractional representation of P (*nrcf* or *nlcf*) if it is a coprime fractional representation of P and: $M^*M + N^*N = I$ $(\tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I)$ with $M^* = M^T(-s)$.

3. CLOSED LOOP STABILITY

In this paper we will study the closed loop configuration according to Figure 1, where we assume that a stabilizing controller C has been designed for the nominal plant P.

Fig. 1. Closed loop structure.

The closed loop transfer function $T(P,C)$, mapping the external inputs (e_1, e_2) onto the outputs (u, y) is given by:

$$T(P,C) = \begin{bmatrix} I \\ P \end{bmatrix} (I + CP)^{-1} \, [\, I \; C \,] \tag{1}$$

For bounded (e_1, e_2), stability of the closed loop, i.e. the controller C internally stabilizes the plant P, is guaranteed if and only if $T(P,C) \in \mathcal{H}$.

Theorem 3.1. Let $P \in \mathcal{F}$ be given as $P = N\,M^{-1}$ with (N, M) a *rcf* of P and let the controller $C \in \mathcal{F}$ be given as $C = \tilde{X}^{-1}\tilde{Y}$ with $(\tilde{X}, \tilde{Y})$ a *lcf* of C. Then stability of the closed loop is equivalent to:

$$\Lambda = [\, \tilde{X} \; \tilde{Y} \,] \begin{bmatrix} M \\ N \end{bmatrix} \in \mathcal{J} \tag{2}$$

Proof. Inserting a coprime representation of P, C in (1), then $T(P,C)$ can be written as:

$$T(P,C) = \begin{bmatrix} M \\ N \end{bmatrix} \left(\tilde{X}M + \tilde{Y}N \right)^{-1} [\, \tilde{X} \; \tilde{Y} \,]$$

Now pre-multiplying $T(P,C)$ by the Bezout factors of (N, M) and post-multiplying by the Bezout factors of $(\tilde{X}, \tilde{Y})$ proves the theorem. □

For robust stability it is essential that the closed loop transfer function remains stable for plants P_Δ "close to" P which form a feedback system with the controller C. Usually the controller C is designed with knowledge of P only.

4. MAIN RESULT

Theorem 4.1 Assume a controller stabilizes the nominal plant according to Theorem 3.1. Let a perturbed plant P_Δ be given by:

$$\begin{aligned} P_\Delta &= (N_\Delta Q)(M_\Delta Q)^{-1} \quad Q \in \mathcal{H} \\ &= (N - \Delta_N)(M - \Delta_M) \quad \Delta_N, \Delta_M \in \mathcal{H} \end{aligned}$$

with (N_Δ, M_Δ) a *nrcf* of P_Δ, then a sufficient condition for P_Δ to be stabilized is given by

$$\left\| \begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix} \Lambda^{-1} \right\|_\infty < 1.$$

Proof. $T(P_\Delta, C) \in \mathcal{H}$ iff $\Lambda_\Delta^{-1} \in \mathcal{J}$ with $\Lambda_\Delta = [\,\tilde{X}\ \tilde{Y}\,] \begin{bmatrix} M_\Delta Q \\ N_\Delta Q \end{bmatrix} \in \mathcal{J}$. Inserting the definition of the coprime factor perturbations of Theorem 4.1 we have:

$$\begin{aligned} \Lambda_\Delta &= \Lambda - [\,\tilde{X}\ \tilde{Y}\,] \begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix} \\ &= \left(I - [\,\tilde{X}\ \tilde{Y}\,] \begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix} \Lambda^{-1} \right) \Lambda. \end{aligned}$$

Next we use the fact that $\Lambda \in \mathcal{J}$ by definition, the normalized left coprimeness of $[\,\tilde{X}\ \tilde{Y}\,]$ and the small gain theorem to obtain the following sufficient condition for $\Lambda_\Delta \in \mathcal{J}$:

$$\left\| \begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix} \Lambda^{-1} \right\|_\infty < 1$$

which proves the theorem. □

The coprime factor perturbations can be written as the difference between the *nrcf* of the nominal plant P and an arbitrary *rcf* of the perturbed plant P_Δ:

$$\begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} - \begin{bmatrix} M_\Delta \\ N_\Delta \end{bmatrix} Q \tag{3}$$

This means that there is freedom left in Q to choose a specific *rcf* of P_Δ. In view of the robustness measure a Q such that $\left\| \begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix} \Lambda^{-1} \right\|_\infty$ is as small as possible can be determined. Define the smallest value as $\delta_\Lambda(P, P_\Delta)$, then the determination leads to the following minimization:

$$\delta_\Lambda(P, P_\Delta) = \inf_{Q \in \mathcal{H}} \left\| \left(\begin{bmatrix} M \\ N \end{bmatrix} - \begin{bmatrix} M_\Delta \\ N_\Delta \end{bmatrix} Q \right) \Lambda^{-1} \right\|_\infty . \tag{4}$$

Although the optimization is over $Q \in \mathcal{H}$, when $\delta_\Lambda(P, P_\Delta) < 1$ then $Q \in \mathcal{J}$ by virtue of $\Lambda_\Delta \in \mathcal{J}$.

5. RELATION TO OTHER CRITERIA

In this section the relation with other robustness criteria such as the gap-metric and a point-wise criterion is discussed. It will be shown that the derived robustness margin is less conservative then the gap based margin. Finally it will be shown that the whole procedure also provides a less conservative margin when the plant is input-output weighted.

The gap [5] between P and P_Δ can be defined by:

$$\delta(P, P_\Delta) = \inf_{Q \in \mathcal{H}} \left\| \left(\begin{bmatrix} M \\ N \end{bmatrix} - \begin{bmatrix} M_\Delta \\ N_\Delta \end{bmatrix} Q \right) \right\|_\infty \tag{5}$$

where P and P_Δ are factorized according to Theorem 4.1. Next, define:

$$\begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} - \begin{bmatrix} M_\Delta \\ N_\Delta \end{bmatrix} \hat{Q}$$

with $\hat{Q}$ minimizing (5). Using the above defined perturbations $\begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix}$ we have the following sufficient conditions for robust stability:

The closed loop system is stable for all P_Δ, if:

$$\bar{\sigma}\left(\begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix} \Lambda^{-1} \right) < 1, \text{ for all } s \in j\mathbb{R}. \tag{6}$$

Notice that (6) is more restrictive than Theorem 4.1 since $\hat{Q}$ is minimizing (5) instead of (4). A more conservative margin is obtained when the above is terms are split and the equality $\bar{\sigma}(\Lambda^{-1}) = \bar{\sigma}(T(P, C))$ is used:

$$\bar{\sigma}\left(\begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix} \right) \bar{\sigma}\left(T(P, C)\right) < 1, \text{ for all } s \in j\mathbb{R}. \tag{7}$$

A one step more conservative margin is obtained when the infinity norm is used:

$$\left\| \begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix} \right\|_\infty \|T(P, C)\|_\infty < 1. \tag{8}$$

Notice that (8) is the gap-metric robustness [6].

The improvement of the new robustness margin can be interpreted as follows. Starting from the gap-metric robustness (8), less conservativeness is obtained by the refinement of the infinity-norm towards maximum singular values, as in (7). Further refinement is obtained in when directionality of Λ is taken into account, as in (6).

It is easy to see that all margins are the same in the case that $\Lambda = \alpha I$, where $\alpha \in \mathbb{R}$. It can be shown [1] that this corresponds with a particular control design.

[7] proposed a weighted gap-metric by scaling the input- and output spaces according to $P_w = W_o P W_i$, with $W_o, W_i \in \mathcal{J}$. Since P_w is then represented by its *nrcf*, again we obtain a less conservative robustness margin.

6. EXAMPLE

In this section the application of the presented robustness margin will be illustrated using an example. For simplicity only SISO systems are considered, which implies that the improvement of the new robustness margin by taking into account directionality of the feedback loops can not be demonstrated.

Using the control design method described in [2] a controller C of order 2 has been designed on the nominal plant model P of order 4 such that $\|T(P,C)\|_\infty$ is minimized. In Figure 2 the amplitude part of the frequency responses of the nominal plant model P, controller model C and three perturbed plant models P_Δ^i are shown.

Fig. 2. Amplitude part frequency response P (—), P_Δ^1 (- -), P_Δ^2 (. . .), P_Δ^3 (-.-), C (– –).

If the robustness is measured in the gap-metric, the closed loop system $T(P_\Delta, C)$ remains stable provided

$$\delta(P, P_\Delta) \leq \|T(P,C)\|_\infty^{-1}.$$

The gap between the nominal plant and perturbed plants is:

$$\begin{aligned} \delta(P, P_\Delta^1) &= 0.53 \\ \delta(P, P_\Delta^2) &= 0.25 \\ \delta(P, P_\Delta^3) &= 0.69 \end{aligned}$$

The nominal plant,controller pair imply a robustness margin of: $\|T(P,C)\|_\infty^{-1} = 0.14$. It is obvious that the plant perturbations do not satisfy the robustness margin, therefore stability of the perturbed feedback system can not be guaranteed.

Improvement by taking into account the frequency dependency of the feedback system is illustrated in Figure 3, where $\Delta_i = \begin{bmatrix} \Delta_M^i \\ \Delta_N^i \end{bmatrix}$. Plants P_Δ^i are stabilized in the point-wise margin if $\bar{\sigma}(\Delta^i) < \sigma(\Lambda)$, while they are also stabilized in the gap if $\bar{\sigma}(\Delta^i) < \|T(P,C)\|_\infty^{-1}$.

Fig. 3. Frequency response $\sigma(\Lambda)$ (—), $\bar{\sigma}(\Delta_1)$ (- -), $\bar{\sigma}(\Delta_2)$ (...), $\bar{\sigma}(\Delta_3)$ (-.-), $\|T(P,C)\|_\infty^{-1}$ (– –).

The refinement towards the new robustness margin can be seen as follows: The frequency where the largest difference between P and P_Δ lies is not taken into account in $\delta(P, P_\Delta)$, it is in $\delta_\Lambda(P, P_\Delta)$. Thereby the area of allowable $P'_\Delta s$ is extended towards the solid curve in Figure 3.

It can be seen in Figure 3 that the perturbed plants do not satisfy the gap-robustness of (8). P_Δ^2 satisfies the point-wise margin of (7).

When the stability robustness is measured in the new robustness margin (Theorem 4.1), the perturbed closed loop $T(P_\Delta, C_\Delta)$ remains stable provided:

$$\delta_\Lambda(P, P_\Delta) < 1.$$

The lambda-margins between the nominal plant and the perturbed plants controller are:

$$\begin{aligned}\delta_\Lambda(P, P_\Delta^1) &= 0.9\\ \delta_\Lambda(P, P_\Delta^2) &= 0.3\\ \delta_\Lambda(P, P_\Delta^2) &= 0.86\end{aligned}$$

It can be seen easily seen that robust stability of the perturbed feedback system is guaranteed by the new robustness margin.

7. CONCLUSIONS

The derivation of a new robust stability margin has been presented. It has been shown that this margin is less conservative than similar robustness margins stated in the gap-metric. The improvement of the new margin lies in the fact that frequency dependency of the feedback system and directionality of the feedback loops are taken into account. The application of this robustness margin has been illustrated by an example.

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