KYBERNETIKA — VOLUME *30* (1994), NUMBER 3, PAGES 245 – 262

ON SOME ESTIMATION VARIANCES IN SPATIAL STATISTICS

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Several estimators have been developed to estimate the length intensity of fibre processes (see for example Ohser [5], Vedel-Jensen and Kieu [8]). Among them, estimators based on sections of the sample with random planes are popular because of their easy use. Recently, Beneš et al [1] studied an estimator based on projections of the fibre process on hyperplanes. In the present paper the first and second order properties of these estimators will be recalled in the first part. The second part will contain the convergence of the estimator based on serial sections to the estimator based on projections. Two examples will be presented at the end of the paper.

1. ESTIMATORS UNDER STUDY

Let $(R, \mathcal{B}, \nu)^d$ be the d-dimensional Euclidean space with Borel σ-algebra and Lebesgue measure ν. The index d is often omitted in the text. Let $(M, \mathcal{M})$ be the measurable space of one-dimensional subspaces in R^d, which is interpreted here as a hemisphere of axial orientations.

Let P be a probability measure on $\mathcal{M}$. Its Buffon transform $\mathcal{F}_P$ is the function on M:

$$\mathcal{F}_P(l) = \int_M |\cos \sphericalangle(l, m)| P(\mathrm{d}m)$$

where $\sphericalangle(l, m)$ denotes the angle between l and m.

For two probability measures P, Q on $\mathcal{M}$, the Buffon constant is

$$\mathcal{F}_{PQ} = \int_M \mathcal{F}_P(l) Q(\mathrm{d}l) = \mathcal{F}_{QP}$$

$\mathcal{F}_P(l)$ can be interpreted as the mean projection length of a unit segment in R^d of orientation l onto a random line with orientation distribution P.

Let Φ be a stationary random fibre process in R^d, see Stoyan et al [7] for a proper definition. Recall that fibres are images of continuously differentiable curves. For $B \in \mathcal{B}^d$, the total fibre length in B, denoted $\Phi(B)$, is locally finite.

A weighted fibre process Ψ is derived from Φ by joining to each point x of Φ its tangent orientation $m(x)$. There exists $L \in R^+$ and a probability measure P on $\mathcal{M}$

such that the intensity measure Λ of Ψ can be written (cf. [7]):

$$\Lambda(B \times D) = E[\Psi(B \times D)] = L\nu(B)P(D) \qquad B \in \mathcal{B}, D \in \mathcal{M}$$

L is the length intensity of Φ and P its rose of directions. We assume in the following that the density ρ of P exists.

Let B, C be measurable bounded sets of R^d, $\nu(B) > 0$, then (cf. [7]):

$$E(\Phi(B)) = L\nu(B)$$

$$E(\Phi(B)\Phi(C)) = \int\int 1_B(x)\, 1_C(x+h)\mathcal{K}(\mathrm{d}h)\mathrm{d}x$$

$$\mathrm{var}(\Phi(B)) = L^2 \int_{R^d} g_B(x)\,(p(x)-1)\mathrm{d}x$$

where $g_B(x) = \nu(B \cap B_{-x})$, $B_{-x} = \{y - x;\, y \in B\}$, $\mathcal{K}$ is the reduced second moment measure and $p(x)$ is the pair correlation function of Φ. It holds $\mathcal{K}(\mathrm{d}x) = p(x)\mathrm{d}x$, throughout the paper it is assumed that the pair correlation functions studied exist and are continuous in $R^2 - \{0\}$ (excluding the origin of coordinates).

Then an unbiased estimator of L is

$$L_1 = \frac{\Phi(B)}{\nu(B)}$$

with variance

$$\mathrm{var}(L_1) = \frac{L^2}{\nu(B)^2} \int_{R^d} g_B(x)\,(p(x)-1)\mathrm{d}x.$$

Let $(H_i)_{1 \le i \le n}$ be n $(d-1)$-dimensional hyperplanes with normal orientations (l_i), $A_i = \nu_{d-1}(H_i \cap B)$ where ν_{d-1} is the Lebesgue measure in R^{d-1}, and denote $N_i = \nu_0(\Phi \cap H_i \cap B)$ the number of intersection points between Φ and H_i lying in B. Then (Kanatani [2]) $E(N_i) = A_i \mathcal{F}_P(l_i) L$ and an unbiased estimator of L is:

$$L_2 = \frac{1}{n} \sum_{i=1}^{n} \frac{N_i}{A_i \mathcal{F}_P(l_i)}$$

with variance:

$$\mathrm{var}(L_2) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\mathrm{cov}(N_i, N_j)}{A_i A_j \mathcal{F}_P(l_i) \mathcal{F}_P(l_j)}.$$

Let $l \in M$ be a fixed orientation. A random measure Φ_l can be defined where $\Phi_l(C)$ is the sum of the orthogonal projection lengths of all fibres of Φ in C onto l for every $C \in \mathcal{B}$ (cf. [1]) whose intensity is $L_l = L\mathcal{F}_P(l)$. More generally, let Q be a probability measure on $\mathcal{M}$, a random measure Φ_Q is defined as, for $C \in \mathcal{B}$:

$$\Phi_Q(C) = \int_M \Phi_l(C) Q(\mathrm{d}l).$$

Its intensity L_Q is $L\mathcal{F}_{PQ}$ and its pair correlation function $p_Q(x)$ can be expressed in terms of characteristics of Φ (Beneš et al [1]):

$$p_Q(x) = \frac{I_Q(x)}{\mathcal{F}^2_{PQ}} p(x)$$

where $I_Q(x) = \int_M \int_M \mathcal{F}_Q(m_1)\mathcal{F}_Q(m_2)W_x(\mathrm{d}(m_1, m_2))$, W_x is the two point distribution function of Φ (Schwandtke [6]), i. e. the joint distribution of fibre tangent orientations m_1, m_2 in points x_1, x_2 such that $x = x_1 - x_2$ under the condition that these points belong to fibres. Then one gets

$$\mathrm{cov}(\Phi_Q(A), \Phi_Q(B)) = L^2_Q \int_{R^d} g_{A,B}(x)\,(p_Q(x) - 1)\mathrm{d}x, \qquad (1)$$

where $g_{A,B}(x) = \nu(A \cap B_{-x})$ and an unbiased estimator of L is

$$L_3 = \frac{\Phi_Q(B)}{\nu(B)\mathcal{F}_{PQ}},$$

whose variance is

$$\mathrm{var}(L_3) = \frac{L^2}{\nu(B)^2} \int_{R^d} g_B(x)\,(p_Q(x) - 1)\mathrm{d}x.$$

Explicit formulae are given in Section 3 for two fibre processes, namely the Poisson boolean segment process and the Poisson line process.

2. CONVERGENCE OF THE SERIAL SECTION ESTIMATOR TO THE PROJECTION ESTIMATOR IN R^2

Let us denote u the vertical axis, $x = (r, \theta)$ the polar coordinates in R^2, $-\pi \le \theta < \pi$, θ being the colatitude with respect to u, $V_{a,y} = [0, X_y] \times [y, y + a]$ the rectangle with edge length a parallel to u. For fixed $y \in R$, X_y is a real constant (see Figure 1). Let $\Psi_a(y) = \frac{1}{a}\Phi_u(V_{a,y})$ be the total projected fibre length in $V_{a,y}$ divided by a and $N_y = \nu_0(\Phi \cap V_{0,y})$ be the number of intersections of Φ with the basis of $V_{a,y}$. In fact due to stationarity assumptions, the distribution laws of these quantities depend of y through X_y only.

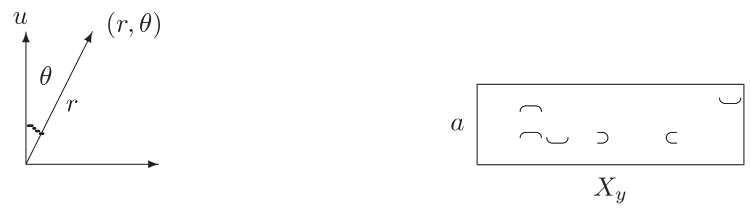

Fig. 1. Polar coordinates with respect to u and rectangle $V_{a,y}$ used is $\Psi_a(y)$ definition.

Lemma 1. Let $f(r,\theta)=\frac{c(\theta)}{r}$ be a continuous function on $R^2-\{0\}$ with

- $c(\theta)\sim K_+(\pi/2-\theta)^{\alpha_+}$ when $\theta\to\pi/2$ for some real constants K_+,α_+; $\alpha_+>0$
- $c(\theta)\sim K_-(\theta+\pi/2)^{\alpha_-}$ when $\theta\to-\pi/2$ for some real constants K_-,α_-; $\alpha_->0$

and let $b\to 0$, $a\to 0$ verifying $0<a\le b^2$ then,

$$\int_{x\in R^2}\nu(V_{b,y}\cap(V_{a,y})_{-x})\,f(x)\mathrm{d}x = baX_y\int_{-\pi/2}^{\pi/2}\frac{c(\theta)}{\cos(\theta)}\mathrm{d}\theta + bao(1) \tag{2}$$

and

$$\int_{x\in R^2}\nu(V_{b,y}\cap(V_{b,y})_{-x})\,f(x)\mathrm{d}x = b^2X_y\int_{-\pi/2}^{\pi/2}\frac{c(\theta)}{\cos(\theta)}\mathrm{d}\theta + b^2o(1).$$

If there exist m_1 and $m_2\in R^+$ such that $0<m_1\le X_y\le m_2$, these convergences are uniform in y.

Proof. The first integral can be written as:

$$\int_{x\in R^2}\nu(V_{a,y}\cap(V_{b,y})_{-x})\,f(x)\mathrm{d}x = J_1+J_2,$$

where

$$J_1=\int_{-\pi}^{0}\int_{r\in R}\nu(V_{a,y}\cap(V_{b,y})_{-x})c(\theta)\mathrm{d}r\mathrm{d}\theta,$$
$$J_2=\int_{0}^{\pi}\int_{r\in R}\nu(V_{a,y}\cap(V_{b,y})_{-x})c(\theta)\mathrm{d}r\mathrm{d}\theta.$$

Let us denote

$$\begin{aligned}
\theta_l &= \arctan(X_y/b) &&= \pi/2-b/X_y+o(b)\\
\theta_l' &= \arctan(X_y/(b-a)) &&= \pi/2-(b-a)/X_y+o(b-a)\\
\theta_l'' &= \arctan(X_y/a) &&= \pi/2-a/X_y+o(a).
\end{aligned}$$

These limits hold uniformly in Y_y if $0<m_1\le X_y\le m_2$.

Then e. g.

$$\begin{aligned}
J_2 \quad = \quad & \int_0^{\theta_l'} \int_0^{\frac{b-a}{\cos(\theta)}} a(X_y - r\sin(\theta))c(\theta)\mathrm{d}r\mathrm{d}\theta \\
& + \int_{\theta_l'}^{\pi/2} \int_0^{\frac{X_y}{\sin(\theta)}} a(X_y - r\sin(\theta))c(\theta)\mathrm{d}r\mathrm{d}\theta \\
& + \int_0^{\theta_l} \int_{\frac{b-a}{\cos(\theta)}}^{\frac{b}{\cos(\theta)}} (b - r\cos(\theta))(X_y - r\sin(\theta))c(\theta)\mathrm{d}r\mathrm{d}\theta \\
& + \int_{\theta_l}^{\theta_l'} \int_{\frac{b-a}{\cos(\theta)}}^{\frac{X_y}{\sin(\theta)}} (b - r\cos(\theta))(X_y - r\sin(\theta))c(\theta)\mathrm{d}r\mathrm{d}\theta \\
& + \int_{\pi/2}^{\pi-\theta_l''} \int_0^{\frac{X_y}{\sin(\theta)}} r|\cos(\theta)|(X_y - r\sin(\theta))c(\theta)\mathrm{d}r\mathrm{d}\theta \\
& + \int_{\pi-\theta_l''}^{\pi} \int_0^{\frac{a}{|\cos(\theta)|}} r|\cos(\theta)|(X_y - r\sin(\theta))c(\theta)\mathrm{d}r\mathrm{d}\theta.
\end{aligned}$$

Denote

$$\begin{array}{ll}
I_1(a,b) = \int_a^b \frac{c(\theta)}{\cos(\theta)}\mathrm{d}\theta & I_2(a,b) = \int_a^b \frac{c(\theta)}{\sin(\theta)}\mathrm{d}\theta \\
I_3(a,b) = \int_a^b \frac{\sin(\theta)c(\theta)}{\cos^2(\theta)}\mathrm{d}\theta & I_4(a,b) = \int_a^b \frac{\cos(\theta)c(\theta)}{\sin^2(\theta)}\mathrm{d}\theta,
\end{array}$$

then integrating we obtain

$$\begin{aligned}
\frac{1}{ab}J_2 \quad = \quad & X_y I_1(0,\theta_l^{'}) - \frac{a}{2b}X_y I_1(\theta_l^{''},\theta_l^{'}) + \frac{b}{2a}X_y I_1(\theta_l,\theta_l^{'}) + \\
& + \frac{X_y^2}{2b}I_2(\theta_l^{'},\pi/2) + \frac{X_y^2}{2a}I_2(\theta_l,\theta_l^{'}) + \frac{a^2-3b^2}{6b}I_3(0,\theta_l^{'}) + \\
& + \frac{b^2}{6a}I_3(\theta_l,\theta_l^{'}) - \frac{X_y^3}{6ab}I_4(\theta_l,\theta_l^{'}) + \frac{X_y^3}{6ab}I_4(\theta_l^{''},\frac{\pi}{2}),
\end{aligned}$$

similarly for J_1. Now evaluating the limits using the assumptions (2) follows. Uniform convergence is ensured by uniform convergence of θ_l, θ_l', θ_l''. □

Lemma 2. Let $h(r,\theta)$ be a continuous function on R^2, and let $b \to 0$ and $a \to 0$ satisfying $0 < a \le b^2$ then,

$$\begin{aligned}
\int_{x\in R^2} \Bigg(& \frac{1}{b^2}\nu(V_{b,y} \cap (V_{b,y})_{-x}) - \frac{2}{ba}\nu(V_{b,y} \cap (V_{a,y})_{-x}) \\
& + \frac{1}{a^2}\nu(V_{a,y} \cap (V_{a,y})_{-x}) \Bigg) h(x)\mathrm{d}x = o(1)
\end{aligned} \tag{3}$$

If there exist m_1 and $m_2 \in R^+$ such that $0 < m_1 \le X_y \le m_2$ this convergence is uniform in y.

Proof. Let $x = (x_1, x_2) \in R \times R$,

$$\nu(V_{b,y} \cap (V_{b,y})_{-x}) = \sup(0, (X_y - |x_1|)) \sup(0, (b - |x_2|))$$

and

$$\nu(V_{b,y} \cap (V_{a,y})_{-x}) = \sup(0, (X_y - |x_1|)) g_{b,a}(x_2)$$

with

$$g_{b,a}(x_2) = \begin{cases} 0 & \text{if} \quad x_2 \leq -a \\ a - |x_2| & \text{if} \quad -a \leq x_2 \leq 0 \\ a & \text{if} \quad 0 \leq x_2 \leq b - a \\ b - x_2 & \text{if} \quad b - a \leq x_2 \leq b \\ 0 & \text{if} \quad x_2 \geq b. \end{cases}$$

Let h_1 be a continuous function on R^+, then

$$\frac{1}{b^2} \int_{x \in \mathbb{R}^2} \nu(V_{b,y} \cap (V_{b,y})_{-x})\, h(x_1) \mathrm{d}x = \int_{x_1 = -X_y}^{X_y} (X_y - |x_1|)\, h(x_1) \mathrm{d}x_1$$

and

$$\begin{aligned} & \frac{1}{ba} \int_{x \in R^2} \nu(V_{b,y} \cap (V_{a,y})_{-x})\, h(x_1) \mathrm{d}x \\ = \; & \frac{1}{ba} \int_{x_1 = -X_y}^{X_y} (X_y - |x_1|)\, h(x_1) \mathrm{d}x_1 \int_{x_2 \in \mathbb{R}} g_{b,a}(x_2) \mathrm{d}x_2 \\ = \; & \frac{1}{ba} \int_{x_1 = -X_y}^{X_y} (X_y - |x_1|)\, h(x_1) \mathrm{d}x_1 \left(\int_{-a}^{0} (a - |x_2|) dx_2 + \int_0^{b-a} a \mathrm{d}x_2 \right. \\ & \left. + \int_{b-a}^{b} (a - |b - a - x_2|) \mathrm{d}x_2 \right) \\ = \; & \int_{x_1 = -X_y}^{X_y} (X_y - |x_1|)\, h(x_1) \mathrm{d}x_1, \end{aligned}$$

so that

$$\begin{aligned} \int_{x \in R^2} & \left(\frac{1}{b^2} \nu(V_{b,y} \cap (V_{b,y})_{-x}) - \frac{2}{ba} \nu(V_{b,y} \cap (V_{a,y})_{-x}) \right. \\ & \left. + \frac{1}{a^2} \nu(V_{a,y} \cap (V_{a,y})_{-x}) \right) h(x_1) \mathrm{d}x = 0 \end{aligned}$$

and

$$\int_{x \in R^2} \left| \frac{1}{b^2} \nu(V_{b,y} \cap (V_{b,y})_{-x}) - \frac{2}{ba} \nu(V_{b,y} \cap (V_{a,y})_{-x}) + \frac{1}{a^2} \nu(V_{a,y} \cap (V_{a,y})_{-x}) \right| \mathrm{d}x \leq 4 X_y^2.$$

Let us consider R^2 with cartesian coordinates. As a function on R^2, h is a continuous function on every compact of R^2. Then, it is uniformly continuous on

$D(0, X_y)$ (the disc of center 0 and radius X_y) so that $h(x_1, x_2) = h(x_1, 0) + o(1)$ uniformly in x_1 for every $x_1 \in [0, X_y]$ and $|x_2| \leq b$.

Finally

$$\begin{aligned}
& \left|\int_{x\in R^2} \left(\frac{1}{b^2}\nu(V_{b,y}\cap(V_{b,y})_{-x}) - \frac{2}{ba}\nu(V_{b,y}\cap(V_{a,y})_{-x}) \right.\right. \\
& \qquad \left.\left. +\frac{1}{a^2}\nu(V_{a,y}\cap(V_{a,y})_{-x})\right) h(x)\mathrm{d}x\right| \\
\leq\ & \left|\int_{x\in R^2} \left(\frac{1}{b^2}\nu(V_{b,y}\cap(V_{b,y})_{-x}) - \frac{2}{ba}\nu(V_{b,y}\cap(V_{a,y})_{-x}) \right.\right. \\
& \qquad \left.\left. +\frac{1}{a^2}\nu(V_{a,y}\cap(V_{a,y})_{-x})\right) h(x_1,0)\mathrm{d}x\right| + \\
+\ & \int_{x\in R^2} \left|\frac{1}{b^2}\nu(V_{b,y}\cap(V_{b,y})_{-x}) - \frac{2}{ba}\nu(V_{b,y}\cap(V_{a,y})_{-x}) \right. \\
& \qquad \left. +\frac{1}{a^2}\nu(V_{a,y}\cap(V_{a,y})_{-x})\right| \mathrm{d}x\, o(1) \\
\leq\ & 4X_y^2 o(1)
\end{aligned}$$

uniformly in Y_y due to the uniform continuity of h. □

Theorem 1. If the pair correlation function $p_u(x)$ on $R^2 - \{0\}$ of the projection measure Φ_u can be written as

$$p_u(r,\theta) - 1 = \frac{c(\theta)}{r} + h(\theta, r) \tag{4}$$

where functions c and h satisfy conditions of Lemma 1 and 2, respectively, then,

$$\Psi_a(y) \to N_y \text{ in quadratic mean for } a \to 0. \tag{5}$$

Under the last condition of Lemma 2, this convergence is uniform in y.

P r o o f. a) Let us suppose $0 < e \leq b^2$, then from equation (1),

$$\begin{aligned}
\mathrm{cov}(\Phi_u(V_{b,y}), \Phi_u(V_{e,y})) &= L^2\mathcal{F}_R^2(u)\int_{x\in R^2} \nu(V_{e,y}\cap((V_{b,y})_{-x})(p_u(x)-1)\mathrm{d}x \\
&= L^2\mathcal{F}_R^2(u)\int_{x\in R^2} \nu(V_{e,y}\cap(V_{b,y})_{-x})\frac{c(\theta)}{r}\mathrm{d}x \\
&+ L^2\mathcal{F}_R^2(u)\int_{x\in R^2} \nu(V_{e,y}\cap(V_{b,y})_{-x})\, h(x)\mathrm{d}x
\end{aligned}$$

$c(\theta)$ fulfills conditions of Lemma 1 and so

$$\begin{aligned}
A(b,e) &= L^2\mathcal{F}_R^2(u)\int_{x\in R^2} \nu(V_{e,y}\cap(V_{b,y})_{-x})\frac{c(\theta)}{r}\mathrm{d}x = \\
&= L^2\mathcal{F}_R^2(u) be X_y \int_{-\pi/2}^{\pi/2} \frac{c(\theta)}{\cos(\theta)}\mathrm{d}\theta + be\, o(1)
\end{aligned}$$

and

$$\begin{aligned} A(b,b) &= L^2\mathcal{F}_R^2(u)\int_{x\in R^2}\nu(V_{b,y}\cap(V_{b,y})_{-x})\frac{c(\theta)}{r}\mathrm{d}x = \\ &= L^2\mathcal{F}_R^2(u)b^2X_y\int_{-\pi/2}^{\pi/2}\frac{c(\theta)}{\cos(\theta)}\mathrm{d}\theta + b^2o(1). \end{aligned}$$

The function $h(x)$ verifies conditions of Lemma 2, using the definition of $\Psi_a(y)$, one gets

$$\begin{aligned} &\operatorname{var}(\Psi_b(y)-\Psi_e(y)) = \frac{1}{b^2}\operatorname{var}(\Phi_u(V_{b,y})) - 2\frac{1}{be}\operatorname{cov}(\Phi_u(V_{b,y}),\Phi_u(V_{e,y})) + \frac{1}{e^2}\operatorname{var}(\Phi_u(V_{e,y})) \\ &= \frac{1}{b^2}A(b,b) - 2\frac{1}{be}A(b,e) + \frac{1}{e^2}A(e,e) \\ &+\int_{x\in R^2}\left(\frac{1}{b^2}\nu(V_{b,y}\cap(V_{b,y})_{-x}) - \frac{2}{be}\nu(V_{b,y}\cap(V_{e,y})_{-x}) + \frac{1}{e^2}\nu(V_{e,y}\cap(V_{e,y})_{-x})\right)h(\theta,r)\mathrm{d}x \\ &= o(1), \end{aligned}$$

the convergence being uniform in Y_y. $\Psi_b(y)$ and $\Psi_e(y)$ having equal means $X_yL\mathcal{F}_R(u)$,

$$\|\Psi_b(y)-\Psi_e(y)\| = o(1)$$

uniformly in Y_y, $\|U\| = EU^2$ denoting the quadratic norm of a random variable U.

b) Let a, b be two positive reals such that $0 < a \le b$

$$\|\Psi_b(y)-\Psi_a(y)\| \le \|\Psi_a(y)-\Psi_{a^2}(y)\| + \|\Psi_b(y)-\Psi_{a^2}(y)\| \le 2o(1)$$

and the series $(\Psi_a(y))$ is Cauchy for all y.

Moreover, the fibres being smooth and locally finite, $\Psi_a(y)\to N_y$ almost surely, and then (Neveu [4]) $\Psi_a(y)\to N_y$ in quadratic mean, and

$$\|\Psi_a(y)-N_y\| = o(1) \tag{6}$$

uniformly in Y_y. □

Lemma 3. Let $f(x) = g(\theta)$ be a continuous function on $R^2-\{0\}$ with

- $\lim_{\theta\to\pi/2} g(\theta)$ exists (denoted $g(\pi/2)$ in the following)
- $\lim_{\theta\to-\pi/2} g(\theta)$ exists (denoted $g(-\pi/2)$ in the following)

and let $b\to 0$, $a\to 0$ verifying $0 < a \le b^2$ then,

$$\int_{x\in R^2}\nu(V_{a,y}\cap(V_{b,y})_{-x})\,f(x)\mathrm{d}x = \frac{ba}{2}X_y^2\,(g(\pi/2)+g(-\pi/2)) + ba\,o(1) \tag{7}$$

and

$$\int_{x\in R^2}\nu(V_{b,y}\cap(V_{b,y})_{-x})\,f(x)\mathrm{d}x = \frac{b^2}{2}X_y^2\,(g(\pi/2)+g(-\pi/2)) + b^2o(1) \tag{8}$$

If there exist m_1 and $m_2 \in R^+$ such that $0 < m_1 \leq X_y \leq m_2$, these convergences are uniform in Y_y.

Proof. It is similar to Lemma 1: Let $f(x) = g(\theta)$, the first integral can be written as:

$$\int_{x\in R^2} \nu(V_{a,y} \cap (V_{b,y})_{-x})\, f(x)\mathrm{d}x = J_1 + J_2,$$

where

$$J_1 = \int_{-\pi}^{0}\int_{r\in R} \nu(V_{a,y} \cap (V_{b,y})_{-x}) g(\theta) r\mathrm{d}r\mathrm{d}\theta,$$
$$J_2 = \int_{0}^{\pi}\int_{r\in R} \nu(V_{a,y} \cap (V_{b,y})_{-x}) g(\theta) r\mathrm{d}r\mathrm{d}\theta.$$

Let us denote $\theta_l = \arctan(X_y/b)$, $\theta_l' = \arctan(X_y/(b-a))$, $\theta_l'' = \arctan(X_y/a)$, then e. g.

$$\begin{aligned} J_2 \quad = \quad & \int_0^{\theta_l'}\int_0^{\frac{b-a}{\cos(\theta)}} a(X_y - r\sin(\theta))g(\theta)r\mathrm{d}r\mathrm{d}\theta \\ & + \int_{\theta_l'}^{\pi/2}\int_0^{\frac{X_y}{\sin(\theta)}} a(X_y - r\sin(\theta))g(\theta)r\mathrm{d}r\mathrm{d}\theta \\ & + \int_0^{\theta_l}\int_{\frac{b-a}{\cos(\theta)}}^{\frac{b}{\cos(\theta)}} (b - r\cos(\theta))(X_y - r\sin(\theta))g(\theta)r\mathrm{d}r\mathrm{d}\theta \\ & + \int_{\theta_l}^{\theta_l'}\int_{\frac{b-a}{\cos(\theta)}}^{\frac{X_y}{\sin(\theta)}} (b - r\cos(\theta))(X_y - r\sin(\theta))g(\theta)r\mathrm{d}r\mathrm{d}\theta \\ & + \int_{\pi/2}^{\pi-\theta_l''}\int_0^{\frac{X_y}{\sin(\theta)}} r|\cos(\theta)|(X_y - r\sin(\theta))g(\theta)r\mathrm{d}r\mathrm{d}\theta \\ & + \int_{\pi-\theta_l''}^{\pi}\int_0^{\frac{a}{|\cos(\theta)|}} r|\cos(\theta)|(X_y - r\sin(\theta))g(\theta)r\mathrm{d}r\mathrm{d}\theta. \end{aligned}$$

Denote

$$I_1(a,b) = \int_a^b \frac{g(\theta)}{\cos^2(\theta)}\mathrm{d}\theta \qquad I_2(a,b) = \int_a^b \frac{g(\theta)\sin(\theta)}{\cos^3(\theta)}\mathrm{d}\theta$$
$$I_3(a,b) = \int_a^b \frac{g(\theta)}{\sin^2(\theta)}\mathrm{d}\theta \qquad I_4(a,b) = \int_a^b \frac{g(\theta)\cos(\theta)}{\sin^3(\theta)}\mathrm{d}\theta,$$

then we obtain

$$\begin{aligned} \frac{1}{ab}J_2 \quad = \quad & -\frac{b^2X_y}{6a}I_1(0,\theta_l) - \frac{(b-a)^3X_y}{6ab}I_1(0,\theta_l') + \frac{a^2}{3b}X_yI_1(\pi-\theta_l'',\pi) \\ & -\frac{b^3}{12a}I_2(0,\theta_l) + \frac{(b-a)^4}{12ab}I_2(0,\theta_l') + \frac{a^3}{4b}I_2(\pi-\theta_l'',\pi) \\ & -\frac{X_y^3}{6b}I_3(\theta_l',\pi/2) + \frac{X_y^3}{6a}I_3(\theta_l,\theta_l') - \frac{X_y^4}{12ab}I_4(\theta_l,\theta_l') + \frac{X_y^4}{12ab}I_4(\pi/2,\pi-\theta_l'') \end{aligned}$$

similarly for J_1. Now evaluating the limits using the assumptions (7) follows and (8) for $a = b$. Uniform convergence is ensured by uniform convergence of θ_l, θ_l', θ_l''. □

Corollary 1. Under the assumptions of Theorem 1, suppose $X_y = X_z = X$,

$$\mathrm{cov}(N_y, N_z) = L^2\mathcal{F}_P^2(u)\int_{-X}^{X}(X-|t|)(p_u(t, y-z)-1)\mathrm{d}t \text{ if } y \neq z \tag{9}$$

If moreover $h(\theta, r) = g(\theta)$ verifies:

- $\lim_{\theta\to\pi/2} g(\theta)$ exists (denoted $g(\pi/2)$ in the following)
- $\lim_{\theta\to-\pi/2} g(\theta)$ exists (denoted $g(-\pi/2)$ in the following)

$$\mathrm{var}(N_y) = L^2\mathcal{F}_P^2(u)(X_y\int_{-\pi/2}^{\pi/2}\frac{c(\theta)}{\cos(\theta)}\mathrm{d}\theta + \frac{X_y^2}{2}(g(\pi/2)+g(-\pi/2))) \tag{10}$$

Proof. The first equality is:

$$\mathrm{cov}(N_y, N_z) - \mathrm{cov}(\Psi_a(y), \Psi_b(z)) = \mathrm{cov}(N_y - \Psi_a(y), N_z) + \mathrm{cov}(\Psi_a(y), N_z - \Psi_b(z))$$

Let $\varepsilon > 0$, A such that $\mathrm{var}(N_y - \Psi_a(y)) \leq \frac{\epsilon^2}{4\mathrm{var}(N_y)}$ for $0 < a < A$ then, for $0 < a < A$, $0 < b < A$,

$$|\mathrm{cov}(N_y, N_z) - \mathrm{cov}(\Psi_a(y), \Psi_b(y))| \leq \varepsilon$$

and $\mathrm{cov}(N_y, N_z) = \lim_{a,b\to 0}\mathrm{cov}(\Psi_a(y), \Psi_b(z))$.

In particular,

$$\begin{aligned}
\mathrm{cov}(N_y, N_z) &= \lim_{a\to 0}\mathrm{cov}(\Psi_a(y), \Psi_a(z)) \\
&= L^2\mathcal{F}_P^2(u)\lim_{a\to 0}\int_{x\in R^2}\frac{1}{a^2}\nu(V_{a,y}\cap(V_{a,z})_{-x}))(p_u(x)-1)\mathrm{d}x \\
&= L^2\mathcal{F}_P^2(u)\lim_{a\to 0}\int_{(t,v)\in R^2}\frac{1}{a^2}1_{[-X,X]}(t)1_{[y-z-a,y-z+a]}(v) \\
&\qquad (X-|t|)(a-|y-z-v|)(p_u(t, y-z+v)-1)\mathrm{d}t\mathrm{d}v
\end{aligned}$$

and the result follows from the continuity of $p_u(x)$ for $x \neq 0$.

The second equality is issued from $\mathrm{var}(N_y) = \lim_{b\to 0}\frac{1}{b^2}\mathrm{var}(\Phi_u(V_{b,y}))$ by applying Theorem 1 with $h(x) = g(\theta)$.

Moreover, as in Theorem 1, one obtains for $b > 0$, $b \to 0$,

$$\begin{aligned}
\mathrm{var}(\Phi_u(V_{b,y})) &= L^2\mathcal{F}_P^2(u)\int_{x\in R^2}\nu(V_{b,y}\cap((V_{b,y})_{-x})(p_u(x)-1)\mathrm{d}x \\
&= L^2\mathcal{F}_P^2(u)\int_{x\in R^2}\nu(V_{b,y}\cap(V_{b,y})_{-x})\frac{c(\theta)}{r}\mathrm{d}x \\
&+ L^2\mathcal{F}_P^2(u)\int_{x\in R^2}\nu(V_{b,y}\cap(V_{b,y})_{-x})g(\theta)\mathrm{d}x.
\end{aligned}$$

Applying Lemma 1 and Lemma 3, one gets

$$\begin{aligned} & L^2\mathcal{F}_P^2(u)\int_{x\in R^2}\nu(V_{b,y}\cap(V_{b,y})_{-x})(\frac{c(\theta)}{r}+g(\theta))\mathrm{d}x \\ = & L^2\mathcal{F}_P^2(u)b^2X_y\int_{-\pi/2}^{\pi/2}\frac{c(\theta)}{\cos(\theta)}\mathrm{d}\theta+b^2\frac{X_y^2}{2}(g(\pi/2)+g(-\pi/2))+b^2o(1)\end{aligned}$$

and

$$\begin{aligned}\mathrm{var}(N_y) &= \lim_{b\to 0}\frac{1}{b^2}\mathrm{var}(\Phi_u(V_{b,y})) \\ &= L^2\mathcal{F}_P^2(u)\left(X_y\int_{-\pi/2}^{\pi/2}\frac{c(\theta)}{\cos(\theta)}\mathrm{d}\theta+\frac{X_y^2}{2}(g(\pi/2)+g(-\pi/2))\right)+o(1).\end{aligned}$$

□

Let H_0 be a hyperplane with normal orientation u, $(H_{ai})_{i\in\mathbb{Z}}=(H_0+iau)_{i\in\mathbb{Z}}$ a series of parallel hyperplanes and denote

$$L_a(B)=a\sum_{i\in\mathbb{Z}}\nu((H_0+iau)\cap B\cap\Phi)=\sum_{i\in\mathbb{Z}}aN_{ia}(B), \tag{11}$$

where $N_{ia}(B)=\nu((H_0+iau)\cap B\cap\Phi)$ is the number of intersection points between Φ and H_{ai} inside B.

Lemma 4. Under the assumptions of Theorem 1 suppose that B is a compact convex set such that there exists a positive constant b for which either $\nu(B\cap H_{ai})>b>0$ or $\nu(B\cap H_{ai})=0$ for all a,i, then for $a\to 0$,

$$L_a(B)\to\Phi_u(B) \text{ in quadratic mean.}$$

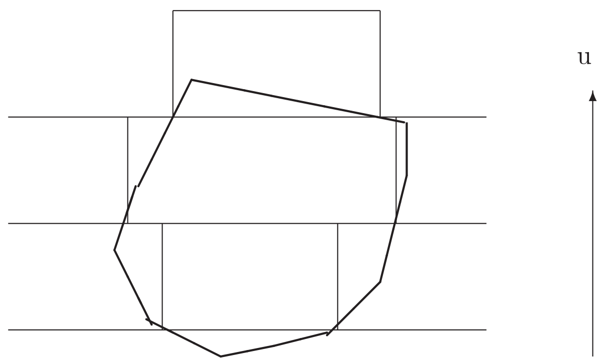

Fig. 2. B compact convex set (in thick lines), serial sections with distance a between consecutive lines (horizontal lines) and B_a union of the rectangles built using the intersection of each line with B as basis and common height a.

P r o o f. a) Let $V_{a,i}$ be the rectangle of basis $B \cap H_{ai}$ and height a, $B_a = \cup_i V_{a,i}$, and

$$\Xi_a(B) = \sum_{i\in\mathbb{Z}} \Phi_u(V_{a,i}) = \sum_{i\in K_a} \Phi_u(V_{a,i})$$

where $K_a = [\inf(i; H_{ia} \cap B \neq \emptyset), \sup(i; H_{ia} \cap B \neq \emptyset)]$.

Then,

$$E(\Xi_a(B)) = E(\Phi_u(\bigcup_{i\in K_a} V_{a,i})) \to E(\Phi_u(B)).$$

Moreover,

$$\begin{aligned} &\operatorname{var}(\Xi_a(B) - \Phi_u(B)) = \operatorname{var}(\Phi_u(B\backslash B_a) + \Phi_u(B_a\backslash B)) \\ \leq\ &\operatorname{var}(\Phi_u(B\backslash B_a)) + \operatorname{var}(\Phi_u(B_a\backslash B)) + 2\sqrt{\operatorname{var}(\Phi_u(B\backslash B_a))\operatorname{var}(\Phi_u(B_a\backslash B))}, \end{aligned}$$

these variances being equal to

$$\operatorname{var}(\Phi_u(B_a\backslash B)) = L^2\mathcal{F}_R^2(u)\int_{x\in R^2} g_{B_a\backslash B}(x)(p_u(x)-1)\mathrm{d}x$$

and

$$\operatorname{var}(\Phi_u(B\backslash B_a)) = L^2\mathcal{F}_R^2(u)\int_{x\in R^2} g_{B\backslash B_a}(x)(p_u(x)-1)\mathrm{d}x.$$

The two functions $g_{B_a\backslash B}(x)$ and $g_{B\backslash B_a}(x)$

- tend to 0 when a tends to 0,
- are dominated by $g_C(x)$ where C is the dilation of B by the disc $D(0,1)$ as soon as $a \leq 1/2$,
- and $L^2\mathcal{F}_R^2(u)\int_{x\in R^2} g_C(x)(p_u(x)-1)\mathrm{d}x = \operatorname{var}(\Phi_u(C))$ is finite,

so that $\operatorname{var}(\Phi_u(B_a\backslash B))$ and $\operatorname{var}(\Phi_u(B\backslash B_a))$ tend to 0 as a tends to 0 by application of the theorem of dominated convergence.

Finally

$$\|\Psi_a(B) - \Xi_a(B)\| = (E(\Psi_a(B) - \Xi_a(B)))^2 + \operatorname{var}(\Xi_a(B) - \Phi_u(B)) \to 0 \quad \text{when } a \text{ tends to } 0. \tag{12}$$

b) Developing the expressions of $\Xi_a(B)$ and $L_a(B)$, one gets $\Xi_a(B) - L_a(B) = a\sum_{i\in K_a}(\Psi_a(ia) - N_{ia})$. $E(\Xi_a(B) - L_a(B)) = 0$ leads to

$$\begin{aligned} \|\Xi_a(B) - L_a(B)\| &\leq a^2 \sum_{i,j\in K_a} \operatorname{cov}(N_{ia} - \Psi_a(ia), (N_{ja} - \Psi_a(ja))) \\ &\leq a^2 \sum_{i,j\in K_a} \sqrt{\operatorname{var}(N_{ia} - \Psi_a(ia))\operatorname{var}(N_{ja} - \Psi_a(ja))} \end{aligned}$$

B satisfying the necessary conditions for $\Psi_a(ia)$ to tend uniformly to N_{ia}, then

$$\|\Xi_a(B) - L_a(B)\| \to 0 \text{ when } a \to 0 \tag{13}$$

and finally $\|L_a(B) - \Phi_u(B)\| \leq \|L_a(B) - \Xi_a(B)\| + \|\Xi_a(B) - \Phi_u(B)\|$ tends to 0 when a tends to 0. □

Theorem 2. Under the assumptions of Theorem 1, suppose that B is a compact convex set then for $a \to 0$,

$$L_a(B) \to \Phi_u(B) \text{ in quadratic mean.}$$

P r o o f. Let B be a convex compact set of R^2.

There exists a series D_n of squares with a horizontal face such that

$$B = \bigcup_n D_n \cup D_\infty$$

with $\nu(D_\infty) = 0$. Let us denote $B_n = \cup_{i \le n} D_i$.

Let $n > 0$, $\|L_a(B_n) - \Phi_u(B_n)\| = \| \sum_{i=1}^n L_a(D_i) - \Phi_u(D_i)\| \le \sum_{i=1}^n \|L_a(D_i) - \Phi_u(D_i)\| \to 0$ when $a \to 0$, each D_i satisfying conditions of Theorem 2 and the sum being finite.

Let B'_n be the dilation of B by the horizontal vector of length $1/n$. B'_n fulfils conditions of Lemma 4 and

$$B_n \subset B \subset B'_n$$

so that

$$L_a(B_n) - \Phi_u(B) \le L_a(B) - \Phi_u(B) \le L_a(B'_n) - \Phi_u(B).$$

It is $\|L_a(B_n) - \Phi_u(B)\| \le \|L_a(B_n) - \Phi_u(B_n)\| + \|\Phi_u(B_n) - \Phi_u(B)\|$; $\nu(B \backslash B_n) \to 0$ so that $\Phi_u(B_n) \to \Phi_u(B)$ in quadratic mean and there exists N such that $|\Phi_u(B_N) - \Phi_u(B)| \le \epsilon$.

$L_a(B_N) \to \Phi_u(B_N)$ in quadratic mean so that there exists $A > 0$ such that, if $0 < a \le A$,

$\|L_a(B_n) - \Phi_u(B_n)\| \le \epsilon$ and $\|L_a(B_n) - \Phi_u(B)\| \le 2\epsilon$.

The same reasoning applied to $\|L_a(B'_n) - \Phi_u(B)\|$ leads to the result. □

Lemma 5. Suppose that $B = [0, X] \times [0, Y]$ is a rectangle in R^2 with edge length Y parallel to u. Suppose furthermore that $p_u(r, \theta) - 1 = \frac{c(\theta)}{r} + g(\theta)$ where c satisfies the conditions of Lemma 1 and g is continuous with

- $g(\theta) = D_+(\pi/2 - \theta)^{\beta_+}$ when $\theta \to \pi/2$
- $g(\theta) = D_-(\theta + \pi/2)^{\beta_-}$ when $\theta \to -\pi/2$

for some real constants $D_+, D_-, \beta_+, \beta_-$ then,

- if $\alpha_+ > 0$, $\alpha_- > 0$, $\beta_+ > 0$, $\beta_- > 0$, $\text{cov}(N_y, N_{y+z})$ is continuous at $z = 0$ for any y,
- if $\alpha_+ > 1$, $\alpha_- > 1$, $\beta_+ > 1$, $\beta_- > 1$, the derivative $\frac{\partial \text{cov}(N_y, N_{y+z})}{\partial z}$ exists at $z = 0$ for any y and is equal to

$$L^2 \mathcal{F}_R^2(u) \left(X \int_{-\pi/2}^{\pi/2} \frac{g(\theta)}{\cos^2(\theta)} \mathrm{d}\theta - \int_{-\pi/2}^{\pi/2} \frac{|\sin(\theta)| c(\theta)}{\cos^2(\theta)} \mathrm{d}\theta \right).$$

P r o o f. Denote $\theta_l = \arctan\frac{X}{b}$. It holds

$$\begin{aligned}
\mathrm{cov}(N_y, N_{y+b}) &= L^2\mathcal{F}_R^2(u)\int_{-X}^{X}(X-|x|)(p_u(x,b)-1)\mathrm{d}x \\
&= L^2\mathcal{F}_R^2(u)\int_{-\theta_l}^{\theta_l}\frac{X-b|\tan(\theta)|}{\cos^2(\theta)}(\frac{c(\theta)\cos(\theta)}{b}+g(\theta))b\mathrm{d}\theta
\end{aligned}$$

and let $b \to 0$:

$$\begin{aligned}
&\int_0^{\theta_l}\frac{X-b|\tan(\theta)|}{\cos^2(\theta)}(\frac{c(\theta)\cos(\theta)}{b}+g(\theta))b\mathrm{d}\theta = \\
&= I_1(b)+I_2(b)+I_3(b)+I_4(b),
\end{aligned}$$

where

$$\begin{aligned}
I_1(b) &= X\int_0^{\theta_l}\frac{c(\theta)}{\cos\theta}\mathrm{d}\theta, & I_2(b) &= -b\int_0^{\theta_l}\frac{c(\theta)|\sin\theta|}{\cos^2\theta}\mathrm{d}\theta, \\
I_3(b) &= Xb\int_0^{\theta_l}\frac{g(\theta)}{\cos^2\theta}\mathrm{d}\theta, & I_4(b) &= -b^2\int_0^{\theta_l}\frac{g(\theta)|\sin\theta|}{\cos^3\theta}\mathrm{d}\theta.
\end{aligned}$$

Now using the assumptions of Lemma 1 we get that $\lim_{b\to 0} I_1(b)$ exists for $\alpha_+ > 0,\ \alpha_- > 0$. Denoting $I_1'(b) = \frac{dI_1(b)}{db}$ it follows

$$\begin{aligned}
\lim_{b\to 0} I_1'(b) = K_+X\lim_{b\to 0}\tfrac{1}{b}(\tfrac{\pi}{2}-\arctan\tfrac{X}{b})^\alpha \quad &= 0 \text{ for } \alpha > 1 \\
&= \infty \text{ for } \alpha < 1.
\end{aligned}$$

For $\alpha = 1$ it is $\lim_{b\to 0_+} I_1'(b) \neq \lim_{b\to 0_-} I_1'(b)$.

Similarly the other integrals are treated to get the result. □

Theorem 3. Under the conditions of Lemma 5, if B is a rectangle with one horizontal edge of length X, if Y is the length of the projection of B onto u, if $\alpha_+ > 1$, $\alpha_- > 1$, $\beta_+ > 1$, $\beta_- > 1$ then the speed of convergence of $L_a(B)$ to $\Phi_u(B)$ is given by:

$$\begin{aligned}
&E\left((L_a(B)-\Phi_u(B))^2\right) = \\
&= -\frac{Ya^2}{6}L^2\mathcal{F}_P^2(u)\left(X\int_{-\pi/2}^{\pi/2}\frac{g(\theta)}{\cos^2(\theta)}\mathrm{d}\theta - \int_{-\pi/2}^{\pi/2}\frac{|\sin(\theta)|c(\theta)}{\cos^2(\theta)}\mathrm{d}\theta\right) + o(a^2)
\end{aligned} \tag{14}$$

P r o o f. It is derived from a Matheron [3] result:

The estimation variance $\sigma_n^2 = \mathrm{var}\left(\frac{1}{Y}\int_Y N_y\mathrm{d}y - \frac{1}{K_a}\sum_{k\in K_a}N_{ka}\right)$ is equivalent to $\frac{1}{6}\gamma'(0)\frac{a}{K_a}$ as soon as the covariogram of N_y defined as $\gamma(z) = \mathrm{var}(N_0) - \mathrm{cov}(N_0, N_z)$ is derivable around 0.

Then,

$$E\left((L_a(B)-\Phi_u(B))^2\right)=Y^2\sigma_n^2=\frac{1}{6}Y^2\frac{a}{K_a}\gamma'(0)$$

and use

$$Y=aK_a(1+o(1))$$

and

$$\gamma'(0)=-\frac{\partial\mathrm{cov}(N_0,N_z)}{\partial z}=-L^2\mathcal{F}_P^2(u)\left(X\int_{-\pi/2}^{\pi/2}\frac{g(\theta)}{\cos^2(\theta)}\mathrm{d}\theta-\int_{-\pi/2}^{\pi/2}\frac{|\sin(\theta)|c(\theta)}{\cos^2(\theta)}\mathrm{d}\theta\right)$$

from Lemma 5 to obtain (14). □

3. EXAMPLES

In the following two examples the variances of the estimators $L_i,\ i=1,2,3$ of intensity L from Section 1 will be expressed. $B\subset R^2$ is a rectangle with edge lengths X,Y, where X is parallel to x-axis and Y to fixed direction $u\in M=\langle 0,\pi)$. Q is a projection measure on M. If necessary to integrate over $M_1=\langle -\pi,\pi)$, we extend functions $p_Q,\mathcal{F}_Q$ to this domain being even in R^2, e. g. $p_u(r,\pi-\theta)=p_u(r,-\theta),\ \theta\in M$.

3.1. The Poisson line process

The stationary isotropic Poisson line process in the plane is derived from the stationary Poisson point process on the cylinder surface when lines are parametrized by their orientation and distance from the origin, see Stoyan et al [7]. For this special fibre process it holds

$$p_Q(r,\theta)=1+\frac{\mathcal{F}_Q^2(\theta)\pi}{4rL},$$

specially

$$p(r,\theta)=1+\frac{1}{\pi rL}\ \text{and}\ p_u(r,\theta)=1+\frac{\pi\cos^2(\theta)}{4rL}.$$

For the estimators $L_i,\ i=1,2,3,$ of L defined in Section 1 we obtain

$$\mathrm{var}L_1=\frac{L}{\pi\nu(B)^2}\int_{(r,\theta)}\nu(B\cap B_{-(r,\theta)})\mathrm{d}r\mathrm{d}\theta \tag{15}$$

and

$$\mathrm{var}L_3=\frac{L\pi}{4\nu(B)^2}\int_{(r,\theta)}\nu(B\cap B_{-(r,\theta)})\mathcal{F}_Q^2(\theta)\mathrm{d}r\mathrm{d}\theta. \tag{16}$$

Let us suppose that $H_i\cap B,\ i=1,...,n$ are parallel sections of B of length X with common normal u, then for $\mathrm{var}L_2$ the covariances $\mathrm{cov}(N_i,N_j)$ are desired. In the model (4) we have $c(\theta)=\frac{\pi\cos^2(\theta)}{4L}$ and $h(\theta,r)=0$.

Applying Corollary 1, one gets

$$\begin{aligned} \operatorname{cov}(N_0, N_y) &= \frac{2Ly^2}{\pi}\int_0^X \frac{X-t}{(t^2+y^2)^{\frac{3}{2}}}\mathrm{d}t = \frac{2LX^2}{\pi\sqrt{X^2+y^2}}, \\ \operatorname{var}(N_0) &= \frac{2}{\pi}XL. \end{aligned} \tag{17}$$

As $\alpha_+ = \alpha_- = 2$ the covariance is derivable at 0 and $L_a(B)$ in (11) converges in quadratic mean to $\Phi_u(B)$ with

$$E(L_a(B) - \Phi_u(B))^2 = \frac{Ya^2L}{3\pi} + o(a^2),$$

see Theorem 3, where a is the distance between two consecutive planes.

For an anisotropic Poisson line process with probability density ρ of the rose of directions P we have similarily

$$p_u(r,\theta) = 1 + \frac{2\rho(\theta)\cos^2\theta}{rL\mathcal{F}_P^2(u)}, \quad \text{i.e.} \quad c(\theta) = \frac{2\rho(\theta)\cos^2\theta}{L\mathcal{F}_P^2(u)}.$$

3.2. The Boolean segment process

An anisotropic Boolean segment process in R^2 is a union of lines segments S the centres of which form a stationary Poisson point process with intensity λ. Let us suppose that the orientation distribution P of segments is independent of the distribution H of segment lengths, and suppose that these two distributions admit densities ρ and h. Let $\theta \in M$ and $r \in R^+$, then it holds (Beneš et al [1]):

- $L = \lambda\bar{H}$ where $\bar{H}$ is the mean segment length,
- $p_Q(r,\theta) = 1 + \frac{\mathcal{F}_Q^2(\theta)\rho(\theta)}{rL\mathcal{F}_{PQ}^2}\frac{\partial f(r)}{\partial r}$, specially
- $p(r,\theta) = 1 + \frac{\rho(\theta)}{Lr}\frac{\partial f(r)}{\partial r}$,

where $f(r) = \frac{1}{\bar{H}}\left(\int_0^r x^2\mathrm{d}H(x) + \int_r^\infty (2xr - r^2)\mathrm{d}H(x)\right)$ is the mean length of $S \cap D(0,r)$ under the condition that a random segment S hits the origin 0. Using $\frac{\partial f(r)}{\partial r} = \frac{2}{\bar{H}}\int_r^\infty (x-r)\mathrm{d}H(x)$ one gets:

$$\begin{aligned} p(r,\theta) &= 1 + \frac{2\rho(\theta)}{\lambda\bar{H}^2 r}\int_r^\infty (x-r)\mathrm{d}H(x), \\ p_Q(r,\theta) &= 1 + \frac{2\rho(\theta)\mathcal{F}_Q^2(\theta)}{\lambda\bar{H}^2 r\mathcal{F}_{PQ}^2}\int_r^\infty (x-r)\mathrm{d}H(x), \end{aligned}$$

so that

$$\begin{aligned} \operatorname{var}(L_1(B)) &= \frac{2\lambda}{\nu(B)^2}\int_0^\infty\int_M g_B(r,\theta)\rho(\theta)\int_r^\infty (x-r)\mathrm{d}H(x)\mathrm{d}r\mathrm{d}\theta, \\ \operatorname{var}(L_3(B)) &= \frac{2\lambda}{\mathcal{F}_{PQ}^2\nu(B)^2}\int_0^\infty\int_M g_B(r,\theta)\mathcal{F}_Q^2(\theta)\rho(\theta)\int_r^\infty (x-r)\mathrm{d}H(x)\mathrm{d}r\mathrm{d}\theta. \end{aligned}$$

Let $H_i \cap B$ be parallel sections of B as above, then

$$p_u(r,\theta) = 1 + \frac{2\rho(\theta)\cos^2(\theta)}{\lambda \bar{H}^2 r \mathcal{F}_P^2(u)} \int_r^\infty (x-r)\mathrm{d}H(x).$$

For simplicity assume that the length of segments is fixed and equal to $q = \bar{H}$ and the process is isotropic (i. e. $\rho(\theta) = \frac{1}{2\pi}$, $\theta \in M_1$). Then

$$\begin{aligned} \int_r^\infty (x-r)\,h(x)\mathrm{d}x \quad &= 0 \quad \text{for} \quad r \ge q \\ &= q - r \qquad r < q. \end{aligned}$$

We obtain $p_u(r,\theta) = 1 + \dfrac{\pi\cos^2\theta}{4Lr} - \dfrac{\pi\cos^2\theta}{4Lq}$ for $r < q$, $p_u(r,\theta) = 1$ for $r \ge q$, i. e.

$$c(\theta) = \frac{\pi\cos^2\theta}{4L}, \; g(\theta) = -\frac{\pi\cos^2\theta}{4Lq} \text{ for } r < q$$

with $\alpha_+ = \alpha_- = \beta_+ = \beta_- = 2$ in the model of Theorem 1 and Lemma 5. For $r \ge q$ it is $c(\theta) = g(\theta) = 0$. Then

$$\begin{aligned} \mathrm{var}N_y \quad &= \frac{2LX}{\pi}, \\ \mathrm{cov}(N_0, N_y) \quad &= 0 \text{ for } y \ge q, \\ \mathrm{cov}(N_0, N_y) \quad &= \frac{2y^2L}{\pi} \int_0^{\min(X,\sqrt{q^2-y^2})} (X-t)\left(\frac{1}{(t^2+y^2)^{3/2}} - \frac{1}{q(t^2+y^2)}\right)\mathrm{d}t \\ &\quad \text{for } y < q, \end{aligned}$$

which enables us to evaluate $\mathrm{var}L_2$. Finally formula (14) yields

$$E(L_a(B) - \Phi_u(B))^2 = \frac{Ya^2L}{6}\left(\frac{X}{q} + \frac{2}{\pi}\right) + o(a^2),$$

when $a \to 0$. Strictly speaking a modification of Lemma 3 and 5 is necessary for these results, which considers function g of a more general type $g(r,\theta) = g_1(\theta)(1 + o(1))$, $r \to 0$.

This modification covers also e. g. the anisotropic case with exponentially distributed segments ($H(x) = 1 - e^{-\frac{x}{q}}$, $x, q > 0$), where

$$p_u(r,\theta) = 1 + \frac{2\rho(\theta)\cos^2(\theta)e^{-\frac{r}{q}}}{Lr\mathcal{F}_P^2(u)}$$

and put $e^{-\frac{r}{q}} = 1 - \frac{r}{q} + o(r)$.

ACKNOWLEDGEMENTS

We thank to the referee for helpful comments and suggestions. The work of the second author was supported by Grant Agency of the Czech Republic, project no. 201/93/2172.

(Received March 3, 1994.)

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