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# EXTREME DISTRIBUTION FUNCTIONS OF COPULAS

Manuel Úbeda–Flores

In this paper we study some properties of the distribution function of the random variable $C(X,Y)$ when the copula of the random pair $(X,Y)$ is $M$ (respectively, $W$) – the copula for which each of $X$ and $Y$ is almost surely an increasing (respectively, decreasing) function of the other –, and $C$ is any copula. We also study the distribution functions of $M(X,Y)$ and $W(X,Y)$ given that the joint distribution function of the random variables $X$ and $Y$ is any copula.

*Keywords:* copula, diagonal section, distribution function, Lipschitz condition, opposite diagonal section, ordering, Spearman's footrule

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## 1. INTRODUCTION

Let $H_1$ and $H_2$ be two bivariate distribution functions with common continuous one-dimensional margins $F$ and $G$ – the distribution functions considered are taken to be right-continuous. Let $(X,Y)$ be a random pair – all the random variables considered are defined on the same probability space $(\Omega,\mathcal{F},P)$ – whose joint distribution function is $H_2$, and let $\langle H_1|H_2\rangle(X,Y)$ denote the random variable $H_1(X,Y)$. The $H_2$ *distribution function of* $H_1$, which we denote by $(H_1|H_2)$, is given by

$$\begin{aligned}(H_1|H_2)(t) &= \Pr[\langle H_1|H_2\rangle(X,Y) \le t] \\ &= \mu_{H_2}(\{(x,y)\in\mathbb{R}^2 \mid H_1(x,y)\le t\}),\ \ t\in[0,1],\end{aligned}$$

where $\mu_{H_2}$ denotes the measure on $\mathbb{R}^2$ induced by $H_2$ [7, 12]. In this paper we study some properties of the distribution function of the random variable $H_1(X,Y)$ when each variable of the random pair $(X,Y)$ is almost surely an increasing (respectively, decreasing) function of the other.

Since our methods involve the concept of a copula, we review this notion and some of its properties. A (bivariate) *copula* is the restriction to $[0,1]^2$ of a continuous (bivariate) distribution function whose margins are uniform on $[0,1]$. The importance of copulas stems largely from the observation that the joint distribution $H$ of the random pair $(X,Y)$ with respective margins $F$ and $G$ can be expressed by $H(x,y)=C(F(x),G(y))$, for all $(x,y)\in[-\infty,\infty]^2$, where $C$ is a copula that is uniquely determined on Range $F\times$Range $G$ (*Sklar's Theorem*) [17, 18]. Let $\Pi$ denote the copula for independent random variables, i. e., $\Pi(u,v)=uv$ for all $(u,v)\in[0,1]^2$. For a complete survey on copulas, see [11].

By Sklar's Theorem, if $C_1$ and $C_2$ are two copulas and $(U,V)$ is a pair of uniform $[0,1]$ random variables with copula $C_2$, and $\langle C_1|C_2\rangle(U,V)$ denotes the random variable $C_1(U,V)$ – written $\langle C_1|C_2\rangle$ when the meaning is clear –, then the $C_2$ distribution function of $C_1$ is given by

$$\begin{aligned}(C_1|C_2)(t) &= \Pr[\langle C_1|C_2\rangle(U,V)\le t]\\ &= \mu_{C_2}(\{(u,v)\in[0,1]^2\,|\,C_1(u,v)\le t\}),\ \ t\in[0,1].\end{aligned}$$

Every copula $C$ of the random pair $(X,Y)$ satisfies the following inequalities:

$$\begin{aligned}\max(u+v-1,0) &= W(u,v)\le C(u,v)\le M(u,v)\\ &= \min(u,v),\ \ \forall\ (u,v)\in[0,1]^2.\end{aligned}$$

$M$ (respectively, $W$) is the copula for which each of $X$ and $Y$ is almost surely an increasing (respectively, decreasing) function of the other.

In the sequel, we shall use the following notation: For any pair of random variables $X$ and $Y$ with respective distribution functions $F$ and $G$, "$\le_{st}$" denotes the stochastic inequality, i. e., $X\le_{st}Y$ if, and only if, $F\ge G$; and $X\overset{d}{=}Y$ denotes the equality in distribution.

Distribution functions of copulas are employed – among other purposes – to construct orderings on the set of copulas (see [12]). If $C$, $C_1$ and $C_2$ are copulas, two of those orderings are: (a) $C_1$ is $C$-*larger* than $C_2$ if $\langle C_1|C\rangle\ge_{st}\langle C_2|C\rangle$; and (b) $C_1$ is $C$-*larger in measure* than $C_2$ if $\langle C|C_1\rangle\ge_{st}\langle C|C_2\rangle$. As a consequence, two equivalences are given, namely: (c) $C_1$ is $C$-*equivalent* to $C_2$ (written $C_1\equiv_C C_2$) if $\langle C_1|C\rangle\overset{d}{=}\langle C_2|C\rangle$; and (d) $C_1$ is $C$-*equivalent in measure* to $C_2$ if $\langle C|C_1\rangle\overset{d}{=}\langle C|C_2\rangle$.

It is known that if $F$ is a right-continuous distribution function such that $F(0^-)=0$ and $F(t)\ge t$ for all $t$ in $[0,1]$, then there exists a copula $C$ such that $(C|C)(t)=F(t)$ for all $t$ in $[0,1]$ (see [13, 16]). We now wonder whether this result can be generalized (in some sense) to other distribution functions of copulas. To be exact: if $C_0$ is a copula, and $F$ is a distribution function such that $(M|C_0)(t)\le F(t)\le(W|C_0)(t)$ for all $t$ in $[0,1]$, does there exist a copula $C$ such that $(C|C_0)(t)=F(t)$ for all $t$ in $[0,1]$? The answer is affirmative when $C_0=M$. We will also provide some additional properties of the distributions $(C|M)$ and $(C|W)$ for any copula $C$.

## 2. THE $M$ DISTRIBUTION FUNCTION OF A COPULA

The *diagonal section* $\delta_C$ of a copula $C$ is the function given by $\delta_C(t)=C(t,t)$ for all $t$ in $[0,1]$. A *diagonal* is a function $\delta\colon[0,1]\longrightarrow[0,1]$ which satisfies the following properties:

(i) $\delta(1)=1$,

(ii) $\delta(t)\le t$ for all $t$ in $[0,1]$,

(iii) $0 \leq \delta(t') - \delta(t) \leq 2(t'-t)$ for all $t, t'$ in $[0,1]$ such that $t \leq t'$ – i. e., $\delta$ is increasing and 2-Lipschitz.

The diagonal section of any copula is a diagonal; and for any diagonal $\delta$, there always exist copulas whose diagonal section is $\delta$ [5] (see also [4, 14, 15]): for instance, the Bertino copula $B_\delta$ [6], which is given by

$$\begin{aligned} B_\delta(u,v) &= \min(u,v) - \min(s - \delta(s) \mid \min(u,v) \\ &\leq s \leq \max(u,v)), \quad (u,v) \in [0,1]^2. \end{aligned}$$

The diagonal section $\delta_C$ of a copula $C$ is the restriction to $[0,1]$ of the distribution function of $\max(U,V)$, whenever $(U,V)$ is a random pair distributed as $C$. Let $\delta_C^{(-1)}$ denote the *cadlag* inverse of $\delta_C$, i. e., $\delta_C^{(-1)}(t) = \sup\{u \in [0,1] \mid \delta_C(u) \leq t\}$ for $t$ in $[0,1]$.

The following result gives a (partial) answer to the question posed at the end of Section 1.

**Theorem 1.** Let $F$ be a right-continuous distribution function such that $F(0^-) = 0$, $F(t) \geq t$ for all $t$ in $[0,1]$, and $F'(t) \geq 1/2$ for almost every $t$ in $[0,1]$. Then there exists a copula $C$ such that $(C|M)(t) = F(t)$ for all $t$ in $[0,1]$.

Proof. We know that $\delta_C^{(-1)}$ is the restriction to the interval $[0,1]$ of a distribution function with support on $[0,1]$ and such that $(M|M)(t) \leq (C|M)(t) \leq (W|M)(t)$ for all $t$ in $[0,1]$. Since

$$(C|M)(t) = \delta_C^{(-1)}(t), \ \ \forall\, t \in [0,1]$$

(see [12]), and $\delta_C$ is 2-Lipschitz, we have that $\delta_C^{(-1)}$ must be a strictly increasing function (not necessarily continuous) whose derivative is greater or equal to $1/2$ for almost every point in $[0,1]$. Since the Bertino copula $B_\delta$ associated with $\delta$ satisfies $(B_\delta|M)(t) = \delta^{(-1)}(t) = F(t)$ for all $t$ in $[0,1]$ (see [12]), this completes the proof. □

If $C_1$ and $C_2$ are two copulas, then we say that $C_1 \equiv_M C_2$ if $(C_1|M)(t) = (C_2|M)(t)$ for all $t$ in $[0,1]$. The next example provides a class in this equivalence relation which contains more than one copula.

**Example 1.** Let $C$ be the copula given by $C(u,v) = \max(0, u+v-1, \min(u, v - 1/2))$, $(u,v) \in [0,1]^2$. $C$ is a *shuffle of Min* [9], whose mass is spread uniformly on two line segments on $[0,1]^2$: one joining the points $(0,1/2)$ and $(1/2,1)$, and the second one joining the points $(1/2,1/2)$ and $(1,0)$. Then it is easy to verify that $(C|M)(t) = (W|M)(t) = (1+t)/2$ for all $t$ in $[0,1]$.

As a consequence of Theorem 1, we have the following

**Corollary 2.** Each equivalence class of the equivalence relation $\equiv_M$ on the set of copulas contains a unique Bertino copula.

Consider *Spearman's footrule coefficient* [19], whose population version for a random pair $(X, Y)$ with copula $C$, is given by

$$\varphi_C = 1 - 3\int_0^1\int_0^1 |u - v|\,\mathrm{d}C(u, v)$$

(see [11]). In terms of the $M$ distribution function of the copula $C$, this measure can be rewritten as

$$\varphi_C = 4 - 6\int_0^1 (C|M)(t)\,\mathrm{d}t$$

(see [12]). Given two copulas $C_1$ and $C_2$, $\langle C_1|M\rangle \leq_{st} \langle C_2|M\rangle$ implies that $\varphi_{C_1} \leq \varphi_{C_2}$. However, the converse result is not true in general, as the following example shows.

**Example 2.** Let $C$ be the shuffle of Min given by $C(u, v) = \min(u, v, \max(1/3, u+v-2/3))$, $(u, v) \in [0, 1]^2$. Its mass is spread uniformly on three line segments in $[0, 1]^2$: one joining the points $(0, 0)$ and $(1/3, 1/3)$, another one joining the points $(1/3, 2/3)$ and $(2/3, 1/3)$, and the third one joining the points $(2/3, 2/3)$ and $(1, 1)$. Then we have $(\Pi|M)(t) = \sqrt{t}$ for all $t$ in $[0, 1]$, and $(C|M)(t) = 2/3$ if $t \in [1/3, 2/3]$ and $(C|M)(t) = t$ otherwise. Thus, $\varphi_\Pi = 0 < 2/3 = \varphi_C$, but $(\Pi|M)(1/3) \simeq 0.577 < 0.67 \simeq (C|M)(1/3)$.

The "$M$-larger" ordering has several applications. For example, if $(U_i, V_i)$ are two uniform $[0, 1]$ random variables with copula $C_i$, $i = 1, 2$, then $C_1$ is $M$-larger than $C_2$ if, and only if, the order statistics of $U_1$ and $V_1$ are stochastically "inside" the interval determined by the order statistics of $U_2$ and $V_2$ [12]. The next result shows the relationship between the $M$-larger and the $M$-larger in measure orderings. To this end, we first note that, for any pair $(U, V)$ of random variables with associated copula $C$, the $C$ distribution function of $M$ is given by

$$\begin{aligned}(M|C)(t) &= \Pr[\min(U, V) \leq t] = \Pr[U \leq t] + \Pr[U > t, V \leq t] \\ &= t + \int_t^1 \Pr[V \leq t|U = u]\,\mathrm{d}u = t + \int_t^1 \frac{\partial C}{\partial u}(u, t)\,\mathrm{d}u \\ &= t + t - C(t, t) = 2t - \delta_C(t)\end{aligned}$$

for every $t$ in $[0, 1]$.

**Proposition 3.** Let $C_1$ and $C_2$ be two copulas. Then $\langle M|C_1\rangle \leq_{st} \langle M|C_2\rangle$ if, and only if, $\langle C_1|M\rangle \leq_{st} \langle C_2|M\rangle$.

P r o o f. Let $\delta_{C_1}$ and $\delta_{C_2}$ be the respective diagonal sections of $C_1$ and $C_2$. Then $C_1$ is $M$-larger in measure than $C_2$ if, and only if, $2t - \delta_{C_1}(t) \leq 2t - \delta_{C_2}(t)$ for all $t$

in $[0,1]$, i. e., $\delta_{C_2} \le \delta_{C_1}$, which is equivalent to $\delta_{C_1}^{(-1)} \le \delta_{C_2}^{(-1)}$, that is, $(C_1|M)(t) \le (C_2|M)(t)$ for all $t$ in $[0,1]$. □

As a consequence of Proposition 3, the $M$-equivalence in measure coincides with the $M$-equivalence. We now show that the equality $\langle M|C\rangle \overset{d}{=} \langle C|M\rangle$ only holds when $C = M$.

**Proposition 4.** Let $C$ be a copula. Then $\langle M|C\rangle \overset{d}{=} \langle C|M\rangle$ if, and only if, $C = M$.

Proof. Suppose $\langle M|C\rangle \overset{d}{=} \langle C|M\rangle$, i. e., $2t - \delta_C(t) = \delta_C^{(-1)}(t)$ for all $t$ in $[0,1]$. Thus, $\delta_C^{(-1)}(t) = \sup\{u \in [0,1] \,|\, \delta_C(u) \le t\} \le 2t$ for all $t$ in $[0,1]$, which implies that $\delta_C(t) \ge t/2$ for all $t$ in $[0,1]$. Hence, $2t - \delta_C^{(-1)}(t) \ge t/2$ for all $t$ in $[0,1]$, i. e., $\delta_C^{(-1)}(t) \le 3t/2$ for all $t$ in $[0,1]$, which implies that $\delta_C(t) \ge 2t/3$ for all $t$ in $[0,1]$. After $n$ iterations, we have that $\delta_C(t) \ge nt/(n+1)$ for all $t$ in $[0,1]$. Therefore, if $n$ tends to infinity, we have that $\delta_C(t) \ge t$, and hence, $\delta_C(t) = t$ for all $t$ in $[0,1]$. Thus, we obtain that $C = M$; otherwise, if there exists a point $(u,v)$ in $[0,1]^2$ such that $C(u,v) < M(u,v)$ with $u \le v$ (the case $u \ge v$ is similar), then $C(u,u) \le C(u,v) < M(u,v) = u$, that is, there exists $u$ in $[0,1]$ such that $\delta_C(u) < u$, which is absurd. The converse is trivial, completing the proof. □

Let $C_1$ and $C_2$ be two copulas. We say that $C_1$ is df-*larger* than $C_2$ if $\langle C_1|C_1\rangle \ge_{st} \langle C_2|C_2\rangle$ [2, 12, 13]. The following example shows that the df-larger and the M-larger orderings are not comparable.

**Example 3.**

(a) Consider the copulas $\Pi$ and the shuffle of Min given by $C(u,v) = \min(u, v, \max(0, u-0.3, v-0.612, u+v-0.912))$, $(u,v) \in [0,1]^2$, whose mass is spread on three line segments in $[0,1]^2$: one joining the points $(0, 0.612)$ and $(0.3, 0.912)$, the second one joining the points $(0.3, 0)$ and $(0.912, 0.612)$, and the third one joining the points $(0.912, 0.912)$ and $(1,1)$. For every $t$ in $[0,1]$, we have $(\Pi|\Pi)(t) = t - t\ln t$, $(\Pi|M)(t) = \sqrt{t}$, $(C|C)(t) = \max(t, \min(2t, t+0.3, 0.912))$, and $(C|M)(t) = \max(t, \min(t+0.3, (t+0.912)/2))$ for all $t$ in $[0,1]$. Then, it is easy to check that $\langle\Pi|\Pi\rangle \le_{st} \langle C|C\rangle$; however, we have $(\Pi|M)(0) = 0 < 0.3 = (C|M)(0)$ and $(\Pi|M)(0.912) \simeq 0.955 > 0.912 = (C|M)(0.912)$.

(b) Consider now the copulas $\Pi$ and $A = (M+W)/2$ – recall that the convex linear combination of two copulas is again a copula. The mass distribution of $A$ is spread uniformly on two line segments in $[0,1]^2$: one connecting the points $(0,0)$ to $(1,1)$, and the second one connecting $(0,1)$ to $(1,0)$. Then, for every $t$ in $[0,1]$, we have $(A|A)(t) = \min(3t, (2+t)/3)$ and $(A|M)(t) = \min(2t, (2t+1)/3)$. Thus, it is easy to verify that $\langle\Pi|M\rangle \le_{st} \langle A|M\rangle$; but $(\Pi|\Pi)(0.25) \simeq 0.5966 < 0.75 = (A|A)(0.25)$ and $(\Pi|\Pi)(0.75) \simeq 0.9658 > 0.9167 = (A|A)(0.75)$.

## 3. THE $W$ DISTRIBUTION FUNCTION OF A COPULA

The *opposite diagonal section* $\omega_C$ of a copula $C$ is the function given by $\omega_C(t) = C(t, 1-t)$ for all $t$ in $[0,1]$. An *opposite diagonal* is a function $\omega\colon [0,1] \longrightarrow [0,1]$ which satisfies the following properties:

(i) $\omega(1) = 0$,

(ii) $\omega(t) \leq \min(t, 1-t)$ for all $t$ in $[0,1]$,

(iii) $\omega(t') - \omega(t) \leq t' - t$ for all $t, t'$ in $[0,1]$ such that $t \leq t'$ – i. e., $\omega$ is 1-Lipschitz.

The opposite diagonal section of any copula is an opposite diagonal; and for any opposite diagonal $\omega$, there exist copulas whose opposite diagonal section is $\omega$: for instance, the copula $J_\omega$ given by

$$J_\omega(u,v) = \max\left(0, u+v-1, \frac{u+v-1+\omega(u)+\omega(1-v)}{2}\right)$$

for all $(u,v)$ in $[0,1]^2$ (see [3]).

The following result provides a probabilistic interpretation of the opposite diagonal section of a copula (in the sequel, we will denote the distribution function of a random variable $X$ either by $\mathrm{df}(X)$ or a letter such as $F$).

**Proposition 5.** Let $(U,V)$ be a pair of random variables with associated copula $C$. Then

$$\omega_C(t) = \frac{1}{2} \cdot (\Pr[\min(U, 1-V) \leq t < \max(U, 1-V)]).$$

P r o o f. The copula $C'$ associated with the random pair $(U, 1-V)$ is given by $C'(u,v) = u - C(u, 1-v)$ for every $(u,v)$ in $[0,1]^2$ (see [11]). Then we have that

$$\begin{aligned} \mathrm{df}(\min(U,1-V))(t) &= \Pr[\min(U,1-V) \leq t] \\ &= \Pr[U \leq t] + \Pr[1-V \leq t] - \Pr[U \leq t, 1-V \leq t] \\ &= t + t - C'(t,t) = t + C(t, 1-t) = t + \omega_C(t), \end{aligned}$$

and

$$\begin{aligned} \mathrm{df}(\max(U,1-V))(t) &= \Pr[\max(U,1-V) \leq t] = \Pr[U \leq t, 1-V \leq t] \\ &= C'(t,t) = t - C(t,1-t) = t - \omega_C(t). \end{aligned}$$

whence the result easily follows. □

Let $(U,V)$ be a random pair with copula $C$. The $W$ distribution function of $C$ is given by

$$\begin{aligned} (C|W)(t) &= \Pr[C(U,V) \leq t] = \Pr[C(U,1-U) \leq t] = \Pr[\omega_C(U) \leq t] \\ &= \lambda(\{u \in [0,1] \,|\, \omega_C(u) \leq t\}), \end{aligned}$$

where $\lambda$ denotes the Lebesgue measure in $\mathbb{R}$.

Distribution functions of copulas are also employed in constructing new measures of association. Thus, for instance, given a copula $C$, it seems reasonable to obtain a measure $\chi_C$ – in the same sense than Spearman's footrule coefficient $\varphi_C$ – based on the $W$ distribution function of $C$, and given by the linear expression

$$\chi_C = a \int_0^1 (C|W)(t)\,\mathrm{d}t + b$$

where $a$ and $b$ are two real numbers. If we consider $\chi_W = -1$ and $\chi_\Pi = 0$ for this measure – for the Spearman's footrule coefficient we have $\varphi_M = 1$, $\varphi_\Pi = 0$, and $\varphi_W = -1/2$ –, since $(\Pi|W)(t) = 1 - \sqrt{\max(0, 1-4t)}$ and $(M|W)(t) = \min(2t, 1)$ for all $t$ in $[0,1]$, then we obtain

$$\chi_C = 5 - 6 \int_0^1 (C|W)(t)\,\mathrm{d}t.$$

The coefficient $\chi_C$ can be also written as

$$\chi_C = 6 \int_0^1 C(t, 1-t)\,\mathrm{d}t - 1 = 3 \int_0^1\int_0^1 |1-u-v|\,\mathrm{d}C(u,v) - 1.$$

This coefficient – which first appeared in this last form in [1] – satisfies $\chi_M = 1/2$. Observe also that the population version $\gamma_C$ of the known *Gini's rank correlation coefficient* [8, 10, 11] of a copula $C$ can be written as $\gamma_C = 2(\varphi_C + \chi_C)/3$.

Unlike the relationship between the $M$-larger and the $M$-larger in measure orderings, there is no analogue to Proposition 3 for the $W$-larger and the $W$-larger in measure orderings, as the next example shows. The example also provides a class in the equivalence relation $\equiv_W$ – recall that if $C_1$ and $C_2$ are two copulas, then $C_1 \equiv_W C_2$ if $(C_1|W)(t) = (C_2|W)(t)$ for all $t$ in $[0,1]$ – which contains more than one copula. First note that, if $(U,V)$ is a random pair with copula $C$, then the $C$ distribution function of $W$ is given by

$$\begin{aligned}(W|C)(t) &= \Pr[U+V-1 \le t] = \Pr[U \le t] + \Pr[U > t, V \le 1+t-U]\\ &= t + \int_t^1 \Pr[V \le 1+t-u \,|\, U=u]\,\mathrm{d}u\\ &= t + \int_t^1 \frac{\partial C}{\partial u}(u, 1+t-u)\,\mathrm{d}u\end{aligned}$$

for every $t$ in $[0,1]$.

**Example 4.** Let $C$ be the shuffle of Min given by $C(u,v) = \min(u, v, \max(1/2, u+v-1))$, $(u,v) \in [0,1]^2$. Its mass is spread uniformly on two line segments in $[0,1]^2$: one joining the points $(0,0)$ and $(1/2,1/2)$, and the second one joining the points $(1/2,1)$ and $(1,1/2)$. Then it is easy to verify that $(C|W)(t) = (M|W)(t) = \min(2t,1)$ for all $t$ in $[0,1]$. But, on the other hand, we have $(W|C)(t) = 1/2$ if $t \in [0,1/2)$ and $(W|C)(t) = 1$ if $t \in [1/2,1]$, and $(W|M)(t) = (1+t)/2$.

Hence, $(W|C)(1/4) = 1/2 < 5/8 = (W|M)(1/4)$ and $(W|C)(3/4) = 1 > 7/8 = (W|M)(3/4)$.

To see the "utility" of the $C$ distribution function of $W$, where $C$ is the copula of the random pair $(U, V)$, we provide the following result, which describes the relationship between this distribution function and the distribution function of the random variable $U + V$. In what follows, we will use some notation. Let $f$ be a real function defined on $[a, b]$ (or on a dense subset of $[a, b]$, including $a$ and $b$) having only removable or jump discontinuities. Then $\ell^+ f$ and $\ell^- f$ are the functions defined on $[a, b]$ via $\ell^+ f(x) = f(x^+)$ and $\ell^- f(x) = f(x^-)$, where $f(x^+)$ (respectively, $f(x^-)$) denotes the limit – if it exists – by the right (respectively, left) of $f$ in $x$. Let $\hat{C}$ denote the *survival* copula of $C$, i. e., $\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$ for every $(u, v) \in [0, 1]^2$ (see [11]).

**Proposition 6.** Let $(U, V)$ be a pair of random variables with associated copula $C$. Then we have

$$\mathrm{df}(U+V)(t) = \begin{cases} \ell^+(1 - (W|\hat{C})(1-t)), & \text{if } t \in [0,1] \\ (W|C)(t-1), & \text{if } t \in [1,2]. \end{cases}$$

Proof. Let $t \in [0, 1]$. Then we have

$$\begin{aligned} \mathrm{df}(U+V)(t) &= \mu_C(\{(u,v) \in [0,1]^2 \mid u + v \le t\}) \\ &= \mu_C(\{(u,v) \in [0,1]^2 \mid (1-u) + (1-v) - 1 \ge 1 - t\}) \\ &= \mu_C(\{(1-u', 1-v') \in [0,1]^2 \mid u' + v' - 1 \ge 1 - t\}) \\ &= \mu_{\hat{C}}(\{(u',v') \in [0,1]^2 \mid u' + v' - 1 \ge 1 - t\}) \\ &= \mu_{\hat{C}}(\{(u',v') \in [0,1]^2 \mid W(u',v') \ge 1 - t\}) \\ &= 1 - \mu_{\hat{C}}(\{(u',v') \in [0,1]^2 \mid W(u',v') < 1 - t\}) \\ &= 1 - \ell^-((W|\hat{C})(1-t)) \\ &= \ell^+(1 - (W|\hat{C})(1-t)), \end{aligned}$$

where we have done the transformations $u' = 1 - u$, $v' = 1 - v$. On the other hand, for every $t \in [1, 2]$, we have

$$\begin{aligned} (W|C)(t) &= \mu_C(\{(u,v) \in [0,1]^2 \mid u + v - 1 \le t\}) \\ &= \mu_C(\{(u,v) \in [0,1]^2 \mid u + v \le t + 1\}) \\ &= \mathrm{df}(U+V)(t+1), \end{aligned}$$

which completes the proof. □

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*Manuel Úbeda-Flores, Departamento de Estadística y Matemática Aplicada, Universidad de Almería, Carretera de Sacramento s/n, 04120 La Cañada de San Urbano, Almería. Spain.*
*e-mail: mubeda@ual.es*