KYBERNETIKA — VOLUME *34* (1998), NUMBER 3, PAGES 317 – 333

# SPECTRUM OF RANDOMLY SAMPLED MULTIVARIATE ARMA MODELS

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The paper is devoted to the spectrum of multivariate randomly sampled autoregressive moving-average (*ARMA*) models. We determine precisely the spectrum numerator coefficients of the randomly sampled *ARMA* models. We give results when the non-zero poles of the initial *ARMA* model are simple. We first prove the results when the probability generating function of the random sampling law is injective, then we precise the results when it is not injective.

## 1. INTRODUCTION

Let $X = (X_t)_{t\in\mathbb{Z}}$ be a discrete-time second order stationary process with zero-mean and values in $\mathbb{R}^k$, satisfying an autoregressive moving-average model. Suppose that the process is sampled by a random walk $T = (T_n)_{n\in\mathbb{Z}}$ with values in $\mathbb{Z}$, independent of $X$. Denote the randomly sampled process by $\widetilde{X} = (X_{T_n})_{n\in\mathbb{Z}}$. Let us consider the situation where the available data are only from the process $\widetilde{X}$. The problem is to recover the covariance properties of the original process $X$. According to Shapiro and Silverman [17], we know that the univalence of the sampling probability generating function is sufficient to allow unique recovering of the covariance function of $X$. Hence the study of the model structure of the process $\widetilde{X}$ arises. Robinson [15] proves that when $X$ is an *ARMA* model, $\widetilde{X}$ is also an *ARMA*.

In a previous paper [12], we obtain the rational spectrum of the process $\widetilde{X}$, when $X$ is a univariate *ARMA* model. We give matrix representations for the spectrum numerator coefficients of $\widetilde{X}$. The *AR* part is given in Robinson [15] in the univariate case, and in Kadi et al [11] in the multivariate case. A functional relation between the poles of $X$ (the roots of the *AR* part) and those of $\widetilde{X}$ is derived. The problem of the zeros of $\widetilde{X}$ (the roots of the *MA* part) still arises in the multivariate case.

In the present paper, we examine the rational spectrum of $\widetilde{X}$ when $X$ is a multivariate *ARMA* model. The spectrum numerator coefficients of $\widetilde{X}$ are expressed through block-matrices. The non-zero poles of the initial model are assumed to be simple.

Another interesting problem in random sampling situation is the estimation of the second order characteristics of the process $X$ using directly the observations from the process $\widetilde{X}$. An extensive literature already exists for this statistical problem in the univariate case; see for instance Bloomfield [1], Brillinger [2], Dunsmuir [3, 4, 5], Dunsmuir and Robinson [6, 7, 8], Marshall [13], Parzen [14], Robinson [17], Toloi and Morettin [18].

The organization of the paper is as follows: In Section 2, we introduce some definitions and recall some results about randomly sampled multivariate *ARMA* models. In Section 3, we derive the spectrum of $\widetilde{X}$ for initial *ARMA* process with simple non-zero poles. We examine cases when the sampling probability generating function is injective and when it is non-injective. The numerator spectrum coefficients are given in terms of the initial *ARMA* model parameters and of the sampling distribution convolution law.

## 2. PRELIMINARIES

Let $\epsilon = (\epsilon_t)_{t\in\mathbb{Z}}$ be a zero-mean white noise, with values in $\mathbb{R}^k$ and $\Sigma_\epsilon$ its covariance matrix.

Let $X = (X_t)_{t\in\mathbb{Z}}$ be a zero-mean second-order stationary process with values in $\mathbb{R}^k$ satisfying the $ARMA(p, q)$ equation:

$$\sum_{j=0}^{p} \Phi_j X_{t-j} = \sum_{j=0}^{q} \Theta_j \epsilon_{t-j}, \quad \forall\, t \in \mathbb{Z}, \tag{1}$$

where $\Phi_j$ $(0 \le j \le p)$ and $\Theta_j = (0 \le j \le q)$ are the matrix coefficients with $\Phi_0 = \Theta_0 = I_k$.

Denote the $AR$ matrix polynomial by $\Phi(z) = \sum_{j=0}^{p} \Phi_j z^{p-j}$, the $MA$ matrix polynomial by $\Theta(z) = \sum_{j=0}^{q} \Theta_j z^{q-j}$, and, for any square matrix $A$, we write $|A|$ for the determinant of $A$ and $com\,A$ for the matrix of cofactors of $A$. We will refer to the roots of $|\Phi(z)|$ as the poles of the model and to the roots of $|\Theta(z)|$ as the zeros of the model. Denote the spectrum of the process $X$ by $\widehat{C}_X(z) = \sum_{h\in\mathbb{Z}} C_X(h) z^{-h}$, where $C_X(h) = E(X_0\, {}^tX_h)$, ${}^tX$ is the transpose of $X$.

Let $\|\cdot\|$ denotes any of the norms on the $k \times k$ matrices with complex coefficients. The sequence $\big(C_X(h)\big)$ is square summable, i. e.,

$$\sum_{h\in\mathbb{Z}} \|C_X(h)\|^2 < \infty.$$

Consider now a sampling process $T = (T_n)_{n\in\mathbb{Z}}$ where the random variables $(T_{n+1} - T_n)_{n\in\mathbb{Z}}$ are mutually independent and identically distributed. Denote by $L$ the distribution of $(T_{n+1} - T_n)$ and by $L_j = P(T_{n+1} - T_n = j)$. Let $\widehat{L}(z) = \sum_{j=\ell}^{\infty} L_j z^j$ be the probability generating function of $L$ which is assumed to be defined in a domain including the unit disk; $\ell$ is the smallest integer such that $L_\ell \neq 0$. Denote by $L^{*h}$ the convolution of the distribution function $L$ with itself, $h$ times.

The sampled process $\widetilde{X} = (X_n)_{n\in\mathbb{Z}}$ is defined by:

$$\widetilde{X}_n = X_{T_n}, \quad n \in \mathbb{Z}. \tag{2}$$

We assume the following assumptions:

$\mathcal{A}_1$) the poles of $X$ are inside the unit circle;

$\mathcal{A}_2$) the zeros of $X$ are inside or on the unit circle;

$\mathcal{A}_3$) $\Phi$ and $\Theta$ have no common left divisors;

$\mathcal{A}_4$) the matrix $\Phi_p$ is of full rank;

$\mathcal{A}_5$) $T_0 = 0$;

$\mathcal{A}_6$) the support of $L$ is $\mathbb{N}^*$;

$\mathcal{A}_7$) the sampling process $T$ is independent of $X$.

Let us now recall some results on randomly sampled multivariate *ARMA* models (see Kadi et al [11]):

i) The process $\widetilde{X}$ is an *ARMA*.

ii) Since $X$ has a rational spectrum, there exists $a$ in $]0,1[$ such that $\widehat{C}_X$ exists in the ring $]a, a^{-1}[$. Then, the spectrum of $\widetilde{X}$ exists for all $z$ in the ring $]a, a^{-1}[$

$$\widehat{C}_{\widetilde{X}}(z) = \left[\frac{1}{2i\pi}\int_{C_\gamma}\left(\frac{{}^t\widehat{C}_X(x)}{1 - z\widehat{L}(x)} + \frac{\widehat{C}_X(x)}{1 - z^{-1}\widehat{L}(x)}\right)\frac{\mathrm{d}x}{x}\right] - C_X(0), \tag{3}$$

$C_\gamma$ is the circle of radius $\gamma$ with $a < \gamma < \min(|z|, |z|^{-1})$.

iii) There exists a representation of $\widetilde{X}$ whose poles are the non-zero images by $\widehat{L}$ of the non-zero poles of $X$, with fewer or the same multiplicity orders.

Before stating the next section, we need two technical lemmas which will be useful in the proofs. Denote by $J$ the Jordan partitioned square matrice of finite dimension

$$J = \begin{pmatrix} 0_k & 0_k & 0_k & \dots & 0_k \\ I_k & 0_k & 0_k & \dots & 0_k \\ 0_k & I_k & 0_k & \dots & 0_k \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_k & 0_k & 0_k & I_k & 0_k \end{pmatrix}$$

where $O_k$ and $I_k$ have size $k \times k$. The dimension of $J$ will be specified in every case. Note that the matrix $J$ is nilpotent of order equal to its dimension.

We admit to set as notation

$$J^j = J^{-j}, \quad \text{if } j < 0.$$

**Lemma 1.** Let $P$ and $Q$ be two matrix polynomials of degree $m$, $P(z) = \sum_{j=0}^m A_j z^j$ and $Q(z) = \sum_{j=0}^m B_j z^j$, where the size of matrices $(A_j)$ and $(B_j)$ is $k \times k$.
Then the matrix coefficients $C_j$ of the expression $P(z)\,{}^tQ(z^{-1})$ are as

$$C_j = {}^tAJ^jB, \quad \text{if } j \geq 0$$

and

$$C_j = {}^tA^tJ^jB, \quad \text{if } j < 0$$

with $A = {}^t(A_0, A_1, \dots, A_m)$ and $B = {}^t(B_0, B_1, \dots, B_m)$.

**Lemma 2.** Let $P$ and $Q$ be two matrix polynomials of degree $m$, $P(z) = \sum_{j=0}^m A_j z^j$ and $Q(z) = \sum_{j=0}^m B_j z^j$, where the size of matrices $(A_j)$ and $(B_j)$ is $k \times k$. Then the matrix coefficients $C_j$ of the expression ${}^tP(z^{-1})Q(z)$ are as

$$C_j = {}^tA^tJ^jB, \quad \text{if } j \geq 0$$

and

$$C_j = {}^tAJ^jB, \quad \text{if } j < 0$$

with $A = {}^t({}^tA_0, {}^tA_1, \dots, {}^tA_m)$ and $B = {}^t({}^tB_0, {}^tB_1, \dots, {}^tB_m)$.

## 3. SPECTRUM OF RANDOMLY SAMPLED *ARMA* MODELS

Let us introduce the following notations.

$(r_j)$ are the simple non-zero poles of $X$, $|\Phi(x)| = \prod_{j=1}^{kp}(x - r_j)$. $\Phi_1(x) = x^p\Phi(x^{-1})$, $\Theta_1(x) = x^q\Theta(x^{-1})$, and $M(x) = {}^t\big(com\ \Phi(x)\big)\Theta(x)\Sigma_\epsilon{}^t\Theta_1(x)\big(com\ \Phi_1(x)\big)$. The elements of $M(x)$ are polynomials in the variable $x$.
Set $M = {}^t({}^tM_{q-p}, {}^tM_{q-p-1}, \dots, {}^tM_0)$ when $q - p \geq 0$. $M_j$ is the coefficient of $x^j$ in the matrix polynomial $M(x)$.

$$R_j = \frac{M(r_j)}{r_j^{q-p+1}\prod_{\ell=1}^{kp}(1 - r_jr_\ell)\prod_{\ell\neq j}(r_j - r_\ell)} \quad \text{and} \quad R = {}^t({}^tR_1, {}^tR_2, \dots, {}^tR_{kp}).$$

Set $\Psi = {}^t(\psi_0 I_k, \psi_1 I_k, \dots, \psi_{q-p}I_k)$; $\psi_j$ are the first coefficients of the series $\big[x^{kp}|\Phi(x)|\,|\Phi(x^{-1})|\big]^{-1}$.
$|\widetilde{\Phi}(z)| = \prod_{j=1}^{kp}(z - \widehat{L}(r_j)) = \sum_{i=0}^{kp}\widetilde{\phi}_i z^{kp-i}$ is the determinant of the $AR$ characteristic polynomial of $\widetilde{X}$ and set $\widetilde{\phi} = {}^t(\widetilde{\phi}_0 I_k, \widetilde{\phi}_1 I_k, \dots, \widetilde{\phi}_{kp}I_k)$.
$|\widetilde{\Phi}(z)|^{(j)} = \prod_{h\neq j}(z - \widehat{L}(r_h)) = \sum_{i=0}^{kp-1}\widetilde{\phi}_i^{(j)} z^{kp-1-i}$ with $\widetilde{\phi}_0^{(j)} = 1$, $\forall j \in \{1, 2, \dots, kp\}$.

In fact, the polynomial $|\widetilde{\Phi}(z)|^{(j)}$ coincides with the determinant of the characteristic polynomial of the randomly sampled process $\widetilde{X}$ without the root $\widehat{L}(r_j)$.

Let

$$A = \begin{pmatrix} \widetilde{\phi}_0^{(1)}I_k & \widetilde{\phi}_0^{(2)}I_k & \dots & \widetilde{\phi}_0^{(kp)}I_k \\ \widetilde{\phi}_1^{(1)}I_k & \widetilde{\phi}_1^{(2)}I_k & \dots & \widetilde{\phi}_1^{(kp)}I_k \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{\phi}_{kp-1}^{(1)}I_k & \widetilde{\phi}_{kp-1}^{(2)}I_k & \dots & \widetilde{\phi}_{kp-1}^{(kp)}I_k \\ 0_k & 0_k & \dots & 0_k \end{pmatrix}$$

$A$ is of dimension $\big(k(kp+1), k(kp)\big)$. $\widetilde{\varphi}_i$ $(-kp \le i \le kp)$ is the coefficient of $z^i$ in the product $|\widetilde{\Phi}(z)|\;|\widetilde{\Phi}(z^{-1})|$ and $\widetilde{\varphi}_i = \widetilde{\varphi}_{-i}$, $\forall i \in \{1, 2, \ldots, kp\}$.

Define $\Delta_L = \sum_{j=\ell}^{q-p} L_j J^j$. Our main result is as follows

**Theorem.** Assume that the poles of $X$ are simple and that $(q-p) \ge 0$. Then the spectrum of the process $\widetilde{X}$ is

$$\widehat{C}_{\widetilde{X}}(z) = \frac{\sum_{j=-(kp+n)}^{kp+n} \widetilde{V}_j(\Phi, \Theta, L) z^j}{\prod_{j=1}^{kp}(1 - z\widehat{L}(r_j))(1 - z^{-1}\widehat{L}(r_j))},$$

$n$ is the highest integer such that $n\ell \le q-p$, and

$$\widetilde{V}_j(\Phi, \Theta, L) = \begin{cases} \widetilde{V}_j^{(AR)}(\Phi, \Theta, L) + \widetilde{\Gamma}_{j+n+kp}(\Phi, \Theta, L), & \forall\, j \in \{0, 1, \ldots, kp\} \\ \widetilde{\Gamma}_{j+n+kp}(\Phi, \Theta, L), & \forall\, j \in \{kp+1, \ldots, kp+n\} \end{cases}$$

where

$$\widetilde{V}_j^{(AR)} = {}^tR^tAJ^j\widetilde{\phi} + {}^t\widetilde{\phi}\;{}^tJ^jA \times R - \frac{1}{2}\left[\sum_{h=1}^{kp}({}^tR_h + R_h)\right]{}^t\widetilde{\phi}J^j\widetilde{\phi}, \quad \forall\, j \in \{0, 1, \ldots, kp\},$$

and

$$\widetilde{\Gamma}_j(\Phi, \Theta, L) = \sum_{h=0}^{j}\big({}^tM\Delta_L^{h-n}\Psi\big)\widetilde{\varphi}_{kp-(i-h)}, \quad \forall j \in \{0, 1, \ldots, kp+n\}.$$

The numerator spectrum coefficients satisfy

$${}^t\widetilde{V}_{-j} = \widetilde{V}_j, \;\forall\, j \in \{1, 2, \ldots, n+kp\}.$$

Proof. We proceed to the calculation of $\widehat{C}_{\widetilde{X}}$ (see formula (3)) by residues.

As $X$ is an *ARMA* process,

$$\begin{aligned} \widehat{C}_X(x) &= \Phi^{-1}(x)\,\Theta(x)\Sigma_\epsilon{}^t\Theta\left(\frac{1}{x}\right){}^t\Phi^{-1}\left(\frac{1}{x}\right) \\ &= \frac{M(x)}{x^{q-p}\prod_{j=1}^{kp}(x - r_j)\,(1 - r_jx)}. \end{aligned}$$

We have then

$$\widehat{C}_{\widetilde{X}}(z) = \left\{\frac{1}{2i\pi}\int_{C_\gamma}\left[x^{q-p+1}\prod_{j=1}^{kp}(x - r_j)(1 - r_jx)\right]^{-1} \cdot\left(\frac{M(x)}{1 - z^{-1}\widehat{L}(x)} + \frac{{}^tM(x)}{1 - z\widehat{L}(x)}\right)\mathrm{d}x\right\} - C_X(0). \tag{4}$$

For $|x| \leq \gamma$,we have $|\widehat{L}(x)| \leq \gamma < \min(|z|, |z|^{-1})$; therefore $1 - z\widehat{L}(x) \neq 0$ and $1 - z^{-1}\widehat{L}(x) \neq 0$. So the expression under the integral sign has as poles the roots $r_j$ of $|\Phi(x)|$ and zero.

We have to compute the three terms in the right-hand side (RHS) of formula (4):

**a)**
– The residue at the simple pole $r_j$ is the constant term in the expansion of

$$M(x)\left[x^{q-p+1}\prod_{\ell=1}^{kp}(1-r_\ell x)\prod_{\ell\neq j}(x-r_\ell)\left(1-\frac{\widehat{L}(x)}{z}\right)\right]^{-1}$$

in powers of $(x - r_j)$; thus it is

$$\frac{M(r_j)}{r_j^{q-p+1}\prod_{\ell=1}^{kp}(1-r_jr_\ell)\prod_{\ell\neq j}(r_j-r_\ell)} \times \frac{1}{1-\frac{\widehat{L}(r_j)}{z}}.$$

– The residue at the pole zero (this occurs when $q - p + 1 > 0$) is the coefficient of $x^{q-p}$ in the expansion of

$$M(x)\left[\prod_{j=1}^{kp}(1-xr_j)(x-r_j)\big(1-\frac{\widehat{L}(x)}{z}\big)\right]^{-1} \tag{5}$$

in powers of $x$.

Since $(1 - z^{-1}\widehat{L}(x))^{-1}$ may be expanded into:

$$\sum_{h=0}^{\infty}(z^{-1}\widehat{L}(x))^h,$$

and as $\big(\widehat{L}(x)\big)^h = \widehat{L}^{*h}(x) = \sum_{j=h\ell}^{\infty} L_j^{*h}x^j$, the regular part of the expansion of $\big(1 - z^{-1}\widehat{L}(x)\big)^{-1}$ in powers of $x$ at order $(q - p)$ is

$$\sum_{h=0}^{n} z^{-h}\left(\sum_{j=h\ell}^{q-p} L_j^{*h}x^j\right),$$

where $n$ is the highest integer such that $n\ell \leq (q - p)$. Now, to obtain the regular part in the expansion of (5), we need only to express

$$\left(\sum_{j=0}^{q-p}\psi_j x^j\right)\left[\sum_{h=0}^{n} z^{-h}\left(\sum_{j=h\ell}^{q-p} L_j^{*h}x^{j+q-p}\right)\right]\left(\sum_{j=0}^{q-p} M_{-j+(q-p)}x^{-j}\right).$$

We find that the residue at the pole zero is a matrix polynomial in $z^{-1}$ of degree $n$, and using Lemma 1, the coefficient of $z^{-h}$ is

$$\sum_{j=h\ell}^{q-p} L^{*h}\ {}^t\Psi\ {}^tJ^j\ M$$

but

$$\sum_{j=h\ell}^{q-p} L_j^{*h}\ {}^tJ^j = \left(\sum_{j=\ell}^{q-p} L_j\ {}^tJ^j\right)^h = {}^t\Delta_L^h.$$

Let us denote by $\left(\widetilde{V}_j^{(0)}(\Phi,\Theta,L)\right)_{-n\le j<0}$ the coefficients in $z^{-1}$ of the residue at zero.

**b)** In order to compute the second term in the RHS of formula (4), we need to replace $1/z$ by $z$ and $M(x)$ by ${}^tM(x)$.

To derive $\left(\widetilde{V}_j^{(0)}(\Phi,\Theta,L)\right)$ for $0<j\le n$, we apply Lemma 2. We obtain

$$\widetilde{V}_j^{(0)}(\Phi,\Theta,L) = {}^tM\Delta_L^j\Psi.$$

These coefficients satisfy

$${}^t\widetilde{V}_{-j}^{(0)}(\Phi,\Theta,L) = \widetilde{V}_j^{(0)}(\Phi,\Theta,L), \quad \forall\, j\in\{1,2,\ldots,n\}$$

and

$$\widetilde{V}_0^{(0)}(\Phi,\Theta,L) = {}^tM\Psi + {}^t\Psi M.$$

**c)** The last term $C_X(0)$ in the RHS of formula (4) is equal to

$$\frac{1}{4i\pi}\int_{C_\gamma}\left({}^t\widehat{C}_X(x)+\widehat{C}_X(x)\right)\frac{\mathrm{d}x}{x}$$

and the residue of this integral is $\frac{1}{2}\sum_{j=1}^{kp}\left(R_j+{}^tR_j\right)$.

It follows from **a**), **b**) and **c**) that the spectrum of the process $\widetilde{X}$ is

$$\sum_{j=1}^{kp}\frac{R_j}{1-z^{-1}\widehat{L}(r_j)} + \sum_{j=1}^{kp}\frac{{}^tR_j}{1-z\widehat{L}(r_j)} - \frac{1}{2}\sum_{j=1}^{kp}\left(R_j+{}^tR_j\right) + \sum_{j=-n}^{n}\widetilde{V}_j^{(0)}(\Phi,\Theta,L)z^j.$$

Now, we express the difference

$$\left[\widehat{C}_{\widetilde{X}}(z) - \sum_{j=-n}^{n}\widetilde{V}_j^{(0)}(\Phi,\Theta,L)z^j\right]. \tag{6}$$

After reduction to the same denominator, we obtain as numerator in the matrix expression (6)

$$\begin{aligned}
&\left(\sum_{j=1}^{kp}{}^tR_j\prod_{h\ne j}(1-z\widehat{L}(r_h))\right)\prod_h(1-z^{-1}\widehat{L}(r_h)) \qquad (7)\\
&\quad+\left(\sum_{j=1}^{kp}R_j\prod_{h\ne j}(1-z^{-1}\widehat{L}(r_h))\right)\prod_h(1-z\widehat{L}(r_h))\\
&\quad-\frac{1}{2}\left[\sum_{j=1}^{kp}\left(R_j+{}^tR_j\right)\right]\prod_h(1-z\widehat{L}(r_h))(1-z^{-1}\widehat{L}(r_h)).
\end{aligned}$$

The matrix polynomial $\sum_{j=1}^{kp} {}^tR_j \prod_{h\neq j}(1 - z\widehat{L}(r_h))$ is equal to

$$\sum_{j=1}^{kp} {}^tR_j \prod_{h\neq j}(1 - z\widehat{L}(r_h)) = \sum_{j=1}^{kp} {}^tR_j \left[z^{kp-1}|\widetilde{\Phi}(z^{-1})|^{(j)}\right] = \sum_{i=0}^{kp-1}\left(\sum_{j=1}^{kp} {}^tR_j \widetilde{\phi}_i^{(j)}\right) z^i.$$

Set:

$$\widetilde{R}_i = \sum_{j=1}^{kp} R_j \widetilde{\phi}_i^{(j)}, \quad \forall\, i \in \{0, 1, \ldots, kp-1\},$$

the partitioned matrix ${}^t({}^t\widetilde{R}_0, {}^t\widetilde{R}_1, \ldots, {}^t\widetilde{R}_{kp-1}, 0_k)$ may be written in a matrix form: $A \times R$ (The matrices $A$ and $R$ are introduced in the notations).

Now we have

$$\begin{aligned}\left(\sum_{i=0}^{kp-1} {}^t\widetilde{R}_i z^i\right) \prod_h \left(1 - z^{-1}\widehat{L}(r_h))\right) &= \left(\sum_{i=0}^{kp-1} \widetilde{R}_i z^i\right)\left[z^{-kp}|\widetilde{\Phi}(z)|\right] \\ = \quad \left(\sum_{i=0}^{kp-1} \widetilde{R}_i z^i\right)\left(\sum_{i=0}^{kp} \widetilde{\phi}_i z^{-i}\right) &= \sum_{h=-kp}^{kp} \beta_h z^h\end{aligned}$$

where by Lemma 1

$$\beta_h = \begin{cases} {}^tR^tAJ^h\widetilde{\phi}, & \text{if } h \geq 0 \\ {}^tR^tA\ {}^tJ^h\widetilde{\phi}, & \text{if } h < 0 \end{cases}$$

In the same way

$$\begin{aligned} &\left[\sum_{j=1}^{kp} R_j \prod_{h\neq j}(1 - z^{-1}\widehat{L}(r_h))\right] \prod_h (1 - z\widehat{L}(r_h)) \\ = \quad &\left(\sum_{i=0}^{kp-1} \widetilde{R}_i z^{-i}\right)\left(\sum_{i=0}^{kp} \widetilde{\phi}_i z^i\right) = \sum_{h=-kp}^{kp} \beta'_h z^h,\end{aligned}$$

where

$$\beta'_h = \begin{cases} {}^t\widetilde{\phi}J^hA \times R, & \text{if } h \geq 0 \\ {}^t\widetilde{\phi}\ {}^tJ^hA \times R, & \text{if } h < 0. \end{cases}$$

and

$$\begin{aligned}\sum_{j=1}^{kp} {}^tR_j \prod_h (1 - z\widehat{L}(r_h))(1 - z^{-1}\widehat{L}(r_h)) \; &= \; \left[\sum_{j=1}^{kp} {}^tR_j z^{kp}|\widetilde{\Phi}(z^{-1})|\right]\left(\sum_{i=0}^{kp} \widetilde{\phi}_i z^{-i}\right) \\ = \left[\sum_{i=0}^{kp}\left(\sum_{j=1}^{kp} {}^tR_j\right)\widetilde{\phi}_i z^i\right]\left(\sum_{i=0}^{kp} \widetilde{\phi}_i z^{-i}\right) \; &= \; \sum_{h=-kp}^{kp} \beta''_h z^h\end{aligned}$$

it follows from Lemma 1 that the coefficients $\beta_h^{''}$ of this product are

$$\beta_h^{''} = \begin{cases} \sum_{j=1}^{kp} {}^tR_j {}^t\widetilde{\phi} J^h \widetilde{\phi}, & \text{if } h \geq 0 \\ \sum_{j=1}^{kp} {}^tR_j {}^t\widetilde{\phi}\ {}^tJ^h \widetilde{\phi}, & \text{if } h < 0. \end{cases}$$

Then it comes that

$$\widehat{C}_{\widetilde{X}}(z) - \sum_{j=-n}^{n} \widetilde{V}_j^{(0)}(\Phi, \Theta, L) z^j = \frac{\sum_{j=-kp}^{kp} \widetilde{V}_j^{(AR)}(\Phi, \Theta, L) z^j}{\prod_{j=1}^{kp} (1 - z\widehat{L}(r_j))(1 - z^{-1}\widehat{L}(r_j))},$$

with

$$\widetilde{V}_j^{(AR)}(\Phi, \Theta, L) = {}^tR^tAJ^j\widetilde{\phi} + {}^t\widetilde{\phi}\ {}^tJ^jA \times R - \frac{1}{2}\left[\sum_{h=1}^{kp} ({}^tR_h + R_h)\right] {}^t\widetilde{\phi} J^j \widetilde{\phi}, \quad j \in \{0, 1, \ldots, kp\}.$$

The coefficients $\widetilde{V}_j^{(AR)}(\Phi, \Theta, L)$ satisfy

$${}^t\widetilde{V}_{-j}^{(AR)}(\Phi, \Theta, L) = \widetilde{V}_j^{(AR)}(\Phi, \Theta, L), \quad \forall\, j \in \{1, \ldots, kp\}.$$

This leads to

$$\widehat{C}_{\widetilde{X}}(z) = \frac{\sum\limits_{j=-kp}^{kp} \widetilde{V}_j^{(AR)}(\Phi, \Theta, L) z^j + \left(\sum\limits_{j=-n}^{n} \widetilde{V}_j^{(0)} z^j\right) \prod\limits_{j=1}^{kp} (1 - z\widehat{L}(r_j))(1 - z^{-1}\widehat{L}(r_j))}{\prod\limits_{j=1}^{p} (1 - z\widehat{L}(r_j))(1 - z^{-1}\widehat{L}(r_j))}$$

and

$$\begin{aligned} &\left(\sum_{j=-n}^{n} \widetilde{V}_j^{(0)} z^j\right) \prod_{j=1}^{kp} (1 - z\widehat{L}(r_j))(1 - z^{-1}\widehat{L}(r_j)) \\ &= \frac{1}{z^{n+kp}} \left(\sum_{i=0}^{2n} \widetilde{V}_{n-i}^{(0)} z^{2n-i}\right) \left(\sum_{i=0}^{2kp} \widetilde{\varphi}_{kp-i} z^{2kp-i}\right) \\ &= \frac{1}{z^{n+kp}} \sum_{i=0}^{2(n+kp)} \widetilde{\Gamma}_i z^i = \sum_{i=-(n+kp)}^{n+kp} \widetilde{\Gamma}_{i+n+kp} z^i \end{aligned}$$

with $\widetilde{\Gamma}_i = \sum_{h=0}^{i} \widetilde{V}_{h-n}^{(0)} \widetilde{\varphi}_{kp-(i-h)}$.

Finally, it comes that

$$\widehat{C}_{\widetilde{X}}(z) = \frac{\sum_{j=-(kp+n)}^{kp+n} \widetilde{V}_j(\Phi, \Theta, L) z^j}{\prod_{j=1}^{kp} (1 - z\widehat{L}(r_j))(1 - z^{-1}\widehat{L}(r_j))}$$

with

$$\widetilde{V}_j(\Phi,\Theta,L)=\begin{cases}\widetilde{V}_j^{(AR)}(\Phi,\Theta,L)+\widetilde{\Gamma}_{j+n+kp}(\Phi,\Theta,L), & \text{if } j\in\{0,1,\ldots,kp\}\\ \widetilde{\Gamma}_{j+n+kp}(\Phi,\Theta,L), & \text{if } j\in\{kp+1,\ldots,kp+n\}.\end{cases}$$

This concludes the proof of the theorem. □

**Remark 1.** The matrix $J$ which we use to define $\Delta_L$ is of dimension $k(q-p+1)\times k(q-p+1)$ while it is of dimension $k(kp+1)\times k(kp+1)$ elsewhere in this theorem. We keep the notation $J$ for the same type of matrices.

**Remark 2.** The initial model $X$ has no zero pole owing to assumption $\mathcal{A}_4$. There is no further difficulty to replace this assumption by weaker one: the matrix $(\Phi_p,\Theta_q)$ is of full rank. In this case, we have to consider zero as a possible pole of $X$ with multiplicity $s_0$.

**Remark 3.** When $(q-p)<0$, there is no pole at 0 and the spectrum of $\widetilde{X}$ is simply given by:

$$\begin{aligned}\widehat{C}_{\widetilde{X}}(z) &= \sum_{j=1}^{kp}\frac{R_j}{1-z^{-1}\widehat{L}(r_j)}+\sum_{j=1}^{kp}\frac{{}^tR_j}{1-z\widehat{L}(r_j)}-\frac{1}{2}\sum_{j=1}^{kp}\left(R_j+{}^tR_j\right)\\ &= \frac{\sum_{j=-kp}^{kp}\widetilde{V}_j(\Phi,0,L)z^j}{\prod_{j=1}^{kp}(1-z\widehat{L}(r_j))(1-z^{-1}\widehat{L}(r_j))}\end{aligned}$$

where

$$\widetilde{V}_j(\Phi,0,L) = {}^tR\,{}^tAJ^j\widetilde{\phi}+{}^t\widetilde{\phi}\,{}^tJ^jA\times R-\frac{1}{2}\left[\sum_{h=1}^{kp}({}^tR_h+R_h)\right]{}^t\widetilde{\phi}J^j\widetilde{\phi},$$
$$j\in\{0,1,\ldots,kp\}.$$

Now, let us consider the situation where the probability generating function $\widehat{L}$ is not injective. In this case, the randomly sampled model may be reduced. Denote by $\widehat{L}_1$, $\widehat{L}_2$, …, $\widehat{L}_b$ the distinct values in the sequence $\left(\widehat{L}(r_1),\widehat{L}(r_2),\ldots,\widehat{L}(r_p)\right)$. Let us divide the set $\{1,2,\ldots,p\}$ into $b$ distinct and non-empty classes $E_1$, $E_2$, …, $E_b$ such that $E_j=\{h/\widehat{L}(r_h)=\widehat{L}_j\}$. Then clearly $\sum_{j=1}^{kp}\frac{R_j}{1-z^{-1}\widehat{L}(r_j)}=\sum_{j=1}^{b}\frac{R'_j}{1-z^{-1}\widehat{L}(r_j)}$ where $R'_j=\sum_{h\in E_j}R_h$.

This leads to Corollary 1 where the matrix $A$ is of dimension $(b+1)\times b$ and $R'={}^t({}^tR'_1,{}^tR'_2,\ldots,{}^tR'_b)$.

**Corollary 1.** Assume that the poles of $X$ are simple and that $(q-p) \geq 0$. If $b$ denotes the number of distinct and non zero values of $\widehat{L}$, then the spectrum of the process $\widetilde{X}$ is

$$\widehat{C}_{\widetilde{X}}(z) = \frac{\sum_{j=-(b+n)}^{b+n} \widetilde{V}_j(\Phi,\Theta,L) z^j}{\prod_{j=1}^{b}(1-z\widehat{L}_j)(1-z^{-1}\widehat{L}_j)},$$

$n$ is the highest integer such that $n\ell \leq q-p$, and

$$\widetilde{V}_j(\Phi,\Theta,L) = \begin{cases} \widetilde{V}_j^{(AR)}(\Phi,\Theta,L) + \widetilde{\Gamma}_{j+n+b}(\Phi,\Theta,L), & \forall\, j \in \{0,1,\ldots,b\} \\ \widetilde{\Gamma}_{j+n+b}(\Phi,\Theta,L), & \forall\, j \in \{b+1,\ldots,b+n\} \end{cases}$$

where:

$$\widetilde{V}_j^{(AR)} = {}^tR^tAJ^j\widetilde{\phi} + {}^t\widetilde{\phi}^tJ^jA \times R - \frac{1}{2}\left[\sum_{h=1}^{kp}({}^tR_h + R_h)\right]{}^t\widetilde{\phi}J^j\widetilde{\phi}, \quad \forall\, j \in \{0,1,\ldots,b\},$$

and

$$\widetilde{\Gamma}_j(\Phi,\Theta,L) = \sum_{h=0}^{j}\left({}^tM\Delta_L^{h-n}\Psi\right)\widetilde{\varphi}_{b-(i-h)}, \quad \forall\, j \in \{0,1,\ldots,b+n\}.$$

The numerator spectrum coefficients satisfy

$${}^t\widetilde{V}_{-j} = \widetilde{V}_j, \quad \forall\, j \in \{1,2,\ldots,n+b\}.$$

Let us consider the $AR(p)$ models.

**Corollary 2.** Assume that $q=0$ and that the poles of $X$ are simple. Then the spectrum of the process $\widetilde{X}$ is

$$\widehat{C}_{\widetilde{X}}(z) = \frac{\sum_{j=-kp}^{kp} \widetilde{V}_j(\Phi,0,L) z^j}{\prod_{j=1}^{kp}(1-z\widehat{L}(r_j))(1-z^{-1}\widehat{L}(r_j))},$$

where

$$\widetilde{V}_j(\Phi,0,L) = {}^tR^tAJ^j\widetilde{\phi} + {}^t\widetilde{\phi}\,{}^tJ^jA \times R - \frac{1}{2}\left[\sum_{h=1}^{kp}({}^tR_h + R_h)\right]{}^t\widetilde{\phi}J^j\widetilde{\phi}, \quad \forall\, j \in \{0,1,\ldots,kp\}.$$

The numerator spectrum coefficients satisfy

$${}^t\widetilde{V}_{-j} = \widetilde{V}_j, \quad \forall\, j \in \{1,2,\ldots,kp\}.$$

P r o o f. In this case, we have no pole at 0 and the spectrum of $\widetilde{X}$ is as in Remark 3. □

Let us now consider the $MA(q)$ models. Denote $\Omega = {}^t\left(\Sigma_\epsilon^{\frac{1}{2}}, \Theta_1\Sigma_\epsilon^{\frac{1}{2}}, \ldots, \Theta_q\Sigma_\epsilon^{\frac{1}{2}}\right)$.

**Corollary 3.** Assume that $p = 0$. Then the spectrum of the process $\widetilde{X}$ is

$$\widehat{C}_{\widetilde{X}}(z) = \sum_{j=-n}^{n} \widetilde{V}_j(0, \Theta, L) z^j$$

where $n$ is the highest integer such that $n\ell \le q$ and the coefficients $\widetilde{V}_j$ are quadratic in the parameter matrices $\Theta_j$,

$$\widetilde{V}_j(0, \Theta, L) = {}^t\Omega \Delta_L^j \Omega, \quad \forall\, j \in \{0, 1, 2, \ldots, n\}.$$

These coefficients satisfy

$${}^t\widetilde{V}_{-j}(0, \Theta, L) = \widetilde{V}_j(0, \Theta, L), \quad \forall\, j \in \{1, 2, \ldots, n\}.$$

P r o o f. For $p = 0$, we obtain

$$\begin{aligned} \widetilde{V}_j(0, \Theta, L) &= \widetilde{\Gamma}_{j+n}(0, \Theta, L), \quad \forall\, j \in \{1, \ldots, n\} \\ &= \sum_{h=0}^{j+n} \widetilde{V}_{h-n}^{(0)} \widetilde{\varphi}_{-(j+n-h)}. \end{aligned}$$

All the terms $\widetilde{\varphi}_{-(j+n-h)}$ vanish except when $h = j + n$. So

$$\widetilde{V}_j(0, \Theta, L) = \widetilde{V}_j^{(0)}(0, \Theta, L) = {}^tM \Delta_L^j \Psi.$$

Given that the matrix $\Delta_L^j$ has the form

$$\begin{pmatrix} 0_{j\ell k} & 0_{j\ell k} & \ldots & 0_k & 0_k \\ L_{j\ell}^{*j} I_k & 0_k & \ldots & 0_k & 0_k \\ L_{j\ell+1}^{*j} I_k & L_{j\ell}^{*j} I_k & 0_k & \ldots & 0_k \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ L_q^{*j} I_k & L_{q-1}^{*j} I_k & L_{j\ell+1}^{*j} I_k & L_{j\ell}^{*j} I_k & {}^t0_{j\ell k} \end{pmatrix},$$

where the matrix $0_{j\ell k}$ is as $0_{j\ell k} = {}^t(\underbrace{0_k, \ldots, 0_k}_{j\ell \text{ times}})$; we obtain,

$$\widetilde{V}_j(0, \Theta, L) = \sum_{h=j\ell}^{q} L_h^{*j}\, {}^tM_{q-h}.$$

The initial model is a $MA(q)$, so the matrix $M(x)$ is as follows

$$M(x) = \Theta(x) \Sigma_\epsilon {}^t\Theta_1(x) = x^q \Theta(x) \Sigma_\epsilon {}^t\Theta(x^{-1}) = x^q \sum_{j=-q}^{q} C(j) x^j.$$

Therefore the matrix coefficients $M_0, M_1, \dots, M_q$ are respectively equal to $C(-q)$, $C(-q+1), \dots, C(0)$ and the matrix covariances may be written by Lemma 1

$$C(j) = \begin{cases} {}^t\Omega J^j \Omega, & \text{if } j \geq 0 \\ \\ {}^t\Omega\ {}^tJ^j\Omega, & \text{if } j < 0. \end{cases}$$

Hence

$$\widetilde{V}_j(0,\Theta,L) = \sum_{h=j\ell}^{q} L_h^{*j}\ {}^tC(-h) = {}^t\Omega \left( \sum_{h=j\ell}^{q} L_h^{*j} J^h \right) \Omega = {}^t\Omega\Delta_L^j\Omega. \qquad \square$$

## 4. NUMERICAL EXAMPLES

In this section, let us examine some simple cases. The process considered is two-dimensional and the sampling law is such that $L_1 \neq 0$.

- Let $X$ be a first order moving average process: $X_t = \epsilon_t + \Theta\epsilon_{t-1}$, where $\Theta = \begin{pmatrix} 0.5 & -1 \\ 0 & 0.5 \end{pmatrix}$ and $\Sigma_\epsilon = I_2$.

The spectrum of the process $X$ is given by:

$$\widehat{C}_X(z) = C_X(-1)z^{-1} + C_X(0) + C_X(1)z = \Sigma_\epsilon{}^t\Theta z^{-1} + \left(\Sigma_\epsilon + \Theta\Sigma_\epsilon{}^t\Theta\right) + \Theta\Sigma_\epsilon z.$$

So

$$z\widehat{C}_X(z) = \begin{pmatrix} \frac{1}{2}z^2 + \frac{9}{4}z + \frac{1}{2} & -z^2 - \frac{1}{2}z \\ -\frac{1}{2}z - 1 & \frac{1}{2}z^2 + \frac{5}{4}z + \frac{1}{2} \end{pmatrix}.$$

The spectrum of the process $\widetilde{X}$ is obtained by applying Corollary 3:

$$\begin{aligned} \widehat{C}_{\widetilde{X}}(z) &= \widetilde{V}_{-1}(0,\Theta,L)z^{-1} + \widetilde{V}_0(0,\Theta,L) + \widetilde{V}_1(0,\Theta,L)z \\ &= L_1\Sigma_\epsilon{}^t\Theta z^{-1} + \left(\Sigma_\epsilon + \Theta\Sigma_\epsilon{}^t\Theta\right) + L_1\Theta\Sigma_\epsilon z. \end{aligned}$$

So

$$z\widehat{C}_{\widetilde{X}}(z) = \begin{pmatrix} \frac{1}{2}L_1z^2 + \frac{9}{4}z + \frac{1}{2}L_1 & -L_1z^2 - \frac{1}{2}z \\ -\frac{1}{2}z - L_1 & \frac{1}{2}L_1z^2 + \frac{5}{4}z + \frac{1}{2}L_1 \end{pmatrix}.$$

In Tables 1 and 2, we compute the zeros of the process $\widetilde{X}$ for different values of $L_1$.

a) When the sampling distribution is a Bernoulli law, we have
$L_j = p^{j-1}(1-p)^{2-j}, \quad j \in \{1,2\}$.

a) When the sampling distribution is a Poisson law, we have
$L_j = e^{-\lambda}\frac{\lambda^{j-1}}{(j-1)!}, \quad j \geq 1$.

**Remark.** In all the tables, we only report the models with zeros inside the unit disk.

**Table 1.** Bernoulli law.

| $L_1$ | zeros | modules of zeros |
|---|---|---|
| 0.1 | $-0.0245 \pm 0.0195i$ | 0.0313 |
| 0.2 | $-0.0494 \pm 0.0391i$ | 0.0630 |
| 0.3 | $-0.0755 \pm 0.0587i$ | 0.0956 |
| 0.4 | $-0.1033 \pm 0.0783i$ | 0.1296 |
| 0.5 | $-0.1338 \pm 0.0979i$ | 0.1658 |
| 0.6 | $-0.1684 \pm 0.1173i$ | 0.2053 |
| 0.7 | $-0.2096 \pm 0.1357i$ | 0.2497 |
| 0.8 | $-0.2621 \pm 0.1512i$ | 0.3026 |
| 0.9 | $-0.3381 \pm 0.1558i$ | 0.3722 |

We notice, from Table 1, that the zeros of the sampled process $\widetilde{X}$, are more stable than those of the process $X$. We also see, that the more $L_1$ is small, the more these zeros are stable. So the more the sampling process has increments of longer 2, the more the zeros of $\widetilde{X}$ are stable. In Kadi [10], we study by means of numerical examples the behaviour of the zeros of $\widetilde{X}$ in relation with the zeros and the poles of $X$ in the univariate case. The same properties are reported.

**Table 2.** Poisson law.

| $\lambda$ | $L_1$ | zeros | modules of zeros |
|---|---|---|---|
| 0.1 | 0.9048 | $-0.3427 \pm 0.1553i$ | 0.3764 |
| 0.3 | 0.7408 | $-0.2293 \pm 0.1426i$ | 0.2700 |
| 0.4 | 0.6703 | $-0.1965 \pm 0.1304i$ | 0.2358 |
| 0.6 | 0.5488 | $-0.1501 \pm 0.1074i$ | 0.1846 |
| 1 | 0.3679 | $-0.0941 \pm 0.0720i$ | 0.1185 |
| 3 | 0.0498 | $-0.0122 \pm 0.0097i$ | 0.0156 |
| 6 | 0.0025 | $(-06046 \pm 0.4837i) \times 10^{-3}$ | $0.7742 \times 10^{-3}$ |
| 9 | $1.2341 \times 10^{-4}$ | $(-0.3010 \pm 0.2408i) \times 10^{-4}$ | $0.3855 \times 10^{-4}$ |

We notice the same behaviour as in the case of the Bernoulli law.

- Let $X$ be a first order autoregressive process: $X_t + \Phi X_{t-1} = \epsilon_t$, where $\Phi = \begin{pmatrix} 0.25 & 1 \\ 0 & -0.5 \end{pmatrix}$ and $\Sigma_\epsilon = I_2$.

  The spectrum of the process $\widetilde{X}$ is obtained by applying Corollary 2:

$$\prod_{j=1}^{2} \left(1 - z\widehat{L}(r_j)\right) \left(1 - z^{-1}\widehat{L}(r_j)\right) \widehat{C}_{\widetilde{X}}(z) = \sum_{j=-2}^{2} \widetilde{V}_j(\Phi, 0, L) z^j.$$

To compute the coefficients $\widetilde{V}_j$, we need:

$$\begin{aligned}
M(x) &= \begin{pmatrix} -\frac{1}{2}x^2+\frac{9}{4}x-\frac{1}{2} & -1-\frac{1}{4}x \\ -x^2-\frac{1}{4}x & \frac{1}{4}x^2+\frac{17}{16}x-\frac{1}{4} \end{pmatrix} \\
|\widetilde{\Phi}(z)| &= \widetilde{\phi}_0 z^2+\widetilde{\phi}_1 z+\widetilde{\phi}_2 \\
&= z^2-\left(\widehat{L}(r_1)+\widehat{L}(r_2)\right)z+\widehat{L}(r_1)\widehat{L}(r_2) \\
A &= \begin{pmatrix} I_2 & I_2 \\ -\widehat{L}(r_2)I_2 & -\widehat{L}(r_1)I_2 \\ 0_2 & 0_2 \end{pmatrix}
\end{aligned}$$

$$R_1=\frac{M(r_1)}{(1-r_1^2)(1-r_1r_2)(r_1-r_2)} \quad \text{and} \quad R_2=\frac{M(r_2)}{(1-r_2^2)(1-r_1r_2)(r_2-r_1)}.$$

Then we obtain:

$$\begin{aligned}
\widetilde{V}_0(\Phi,0,L) &= \frac{1}{2}(R_1+R_2+{}^tR_1+{}^tR_2)-(\widetilde{\phi}_1\widehat{L}(r_2)+\frac{1}{2}\widetilde{\phi}_1^2+\frac{1}{2}\widetilde{\phi}_2^2)(R_1+{}^tR_1) \\
&\quad -(\widetilde{\phi}_1\widehat{L}(r_1)+\frac{1}{2}\widetilde{\phi}_1^2+\frac{1}{2}\widetilde{\phi}_2^2)(R_2+{}^tR_2) \\
\widetilde{V}_1(\Phi,0,L) &= -(\widehat{L}(r_2)+\frac{1}{2}\widetilde{\phi}_1+\frac{1}{2}\widetilde{\phi}_1\widetilde{\phi}_2)(R_1+{}^tR_1)-(\widehat{L}(r_1)+\frac{1}{2}\widetilde{\phi}_1 \\
&\quad +\frac{1}{2}\widetilde{\phi}_1\widetilde{\phi}_2)(R_2+{}^tR_2) \\
\widetilde{V}_2(\Phi,0,L) &= -\frac{1}{2}\widetilde{\phi}_2\left(R_1+R_2+{}^tR_1+{}^tR_2\right).
\end{aligned}$$

In Tables 3 and 4, we compute the poles and the zeros of the process $\widetilde{X}$ for different values of the Bernoulli parameter $p$ and the Poisson parameter $\lambda$.
a) When the sampling distribution is a Bernoulli law, we have $\widehat{L}(z)=(1-p)z+pz^2$.
b) When the sampling distribution is a Poisson law, we have $\widehat{L}(z)=z\exp(\lambda(z-1))$.

**Table 3.** Bernoulli law.

| $p$ | poles | zeros |
|---|---|---|
| 0.1 | −0.2188; 0.4750 | −0.1392; 0.1409 |
| 0.2 | −0.1875; 0.4500 | −0.1228; 0.1242 |
| 0.3 | −0.1563; 0.4250 | −0.1055; 0.1065 |
| 0.4 | −0.1250; 0.4000 | −0.0871; 0.0878 |
| 0.5 | −0.093; 0.3750 | −0.676; 0.0679; 0.7280 |
| 0.6 | −0.0625; 0.3500 | −0.466; 0.0468; 0.5559 |
| 0.7 | −0.0313; 0.3250 | −0.8662; −0.0242; 0.0242; 0.4575 |
| 0.8 | $1.387\times10^{-17}$; 0.3000 | −0.5467; 0.0000; 0.0000; 0.3862 |
| 0.9 | 0.0313; 0.2750 | −0.4207; −0.0261; 0.0260; 0.3295 |

**Table 4.** Poisson law.

| $\lambda$ | poles | zeros |
|---|---|---|
| 0.1 | $-0.2206$; 0.4756 | $-0.1401$; 0.1418 |
| 0.3 | $-0.1718$; 0.4304 | $-0.1139$; 0.1151 |
| 0.4 | $-0.1516$; 0.4094 | $-0.1224$; 0.1033 |
| 0.6 | $-0.1181$; 0.3704 | $-0.0822$; 0.0828; 0.9973 |
| 0.9 | $-0.0812$; 0.3188 | $-0.0585$; 0.0587; 0.5320 |
| 1 | $-0.0716$; 0.3033 | $-0.8435$; $-0.0521$; 0.0523; 0.4825 |
| 3 | $-0.0059$; 0.1116 | $-0.1472$; $-0.0046$; 0.0046; 0.1425 |
| 6 | $-1.3827 \times 10^{-4}$; 0.0249 | $-0.0314$; 0.0313 |
| 9 | $-3.2518 \times 10^{-6}$; 0.0056 | $-0.0063$; 0.0069 |

In the univariate case, we observe on many examples that the zeros of $\widetilde{X}$ are more stable than those of $X$. This property needs to be more studied to determine the conditions under which this property holds.

(Received September 19, 1996.)

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