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DECOUPLING IN SINGULAR SYSTEMS: A POLYNOMIAL EQUATION APPROACH

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In this paper, the row by row decoupling problem by static state feedback is studied for regularizable singular square systems. The problem is handled in matrix polynomial equation setting. The necessary and sufficient conditions on decouplability are introduced and an algorithm for calculation of feedback gains is presented. A structural interpretation is also given for decoupled systems.

1. INTRODUCTION

The input-output decoupling problem reduces a multi-input multi-output system to a set of single-input single-output systems. This problem was first introduced by Morgan [14] and the first major results were established by Falb and Wolovich [7]. Since then, there have been many contributions to this subject [4, 6, 9, 17]. Solutions for the decoupling problem of singular square systems via modified proportional and derivative feedback were given by Christodoulou and Paraskevopoulos [3] and Christodoulou [2]. The necessary and sufficient conditions for decoupling of singular systems via static state feedback were reported by Ailon [1] and the results on the same subject with the structure of the closed loop system were also established by Paraskevopoulos and Koumboulis [15]. In [1], the decoupling problem of a singular system has been investigated for different rank conditions defined on the system and the problem has been solved as an extension of the results of Falb and Wolovich. In [15], it has been shown that the decoupling problem of a singular system by static state feedback can be recasted to the decoupling problem of an ordinary system via pure derivative state feedback.

In this study, the decoupling problem for square systems is investigated using polynomial equation approach and a theorem on decouplability of regularizable singular systems by static state feedback is given. An algorithm for the construction of feedback gains and some structural properties of decoupled system are also presented. Although the results given in this study are developed for regularizable singular systems, they are also true for both regular and ordinary systems.

2. PRELIMINARIES

A generalized state-space system is defined by the equations,

$$E\dot{x} = Fx + Gu \tag{1a}$$

$$y = Hx \tag{1b}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$ and $E(n \times n)$, $F(n \times n)$, $G(n \times m)$ and $H(p \times n)$ are matrices with entries in R, the field of real numbers. The classification of a system whether it is ordinary or singular depends on the singularity of matrix E. The dynamics of (1) are completely characterized by the pencil $sE - F$. We assume that system (1) is regularizable, i.e. there exists a matrix $K \in R$ with compatible dimension such that rank $\{sE - F + GK\} = n$. The application of admissible static state feedback,

$$u = -Kx + Lv \tag{2}$$

where $v \in R^m$ and K, L are matrices over R, results in closed loop system transfer function,

$$T_{K,L}(s) = H[sE - F + GK]^{-1}\, GL. \tag{3}$$

Let $D(s)$ and $N(s)$ be a pair of matrices in $R[s]$, the ring of polynomial matrices, such that

$$[sE - F \quad -G] \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0 \tag{4}$$

where $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ is irreducible and column reduced.The matrices $D(s)$ and $N(s)$ are said the right external description of system (1a), [13]. The regularizability of system (1a) implies that rank $[sE - F \quad -G] = n$ and rank $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = m$. Lemma 1 of Kučera [10] can be arranged as follows for admissible state feedback of regularizable systems.

Lemma 1. Let $[sE - F]$ and G be regularizable and $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ be defined by (4). Then, for any compatible matrix K with entries in R, $(sE - F + GK)$ is nonsingular if and only if $[D(s) + KN(s)]$ is nonsingular.

Proof. We write (4) as,

$$[sE - F \quad -G] \begin{bmatrix} I_n & 0 \\ -K & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ K & I_m \end{bmatrix} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0$$

and so,

$$[sE - F + GK]\, N(s) = G[D(s) + KN(s)]. \tag{5}$$

If $[D(s) + KN(s)]$ is nonsingular, then

$$X(s) = N(s)\, [D(s) + KN(s)]^{-1}$$

is a rational solution of the equation

$$[sE - F + GK]\, X(s) = G$$

in $R(s)$, the field of rational functions. Hence, over the field $R(s)$,

$$\begin{aligned} \text{rank}\,[sE - F + GK] &= \text{rank}\,[sE - F + GK \quad G] \\ &= \text{rank}\,[sE - F \quad G] \begin{bmatrix} I_n & 0 \\ K & I_m \end{bmatrix} \\ &= \text{rank}\,[sE - F \quad G] = n \end{aligned} \tag{6}$$

so that $(sE - F + GK)$ is nonsingular.

If $(sE - F + GK)$ is nonsingular, then

$$Y(s) = (sE - F + GK)^{-1} G$$

is a rational solution of the equation

$$Y(s)\,[D(s) + KN(s)] = N(s) \tag{7}$$

Hence, over the field $R(s)$,

$$\begin{aligned} \text{rank}\,[D(s) + KN(s)] &= \text{rank} \begin{bmatrix} D(s) + KN(s) \\ N(s) \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} I_m & K \\ 0 & I_n \end{bmatrix} \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \text{rank} \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = m \end{aligned} \tag{8}$$

so that $[D(s) + KN(s)]$ is nonsingular. □

Throughout the paper, we assume that $p = m$, i.e. the system has the same number of inputs and outputs. When the closed loop transfer function $T_{K,L}(s)$ is diagonal and nonsingular, the system is single input-single output or the row by row decoupled and its transfer function can be given as,

$$T_{K,L}(s) = W^{-1}(s), \quad W(s) = \text{diag}[w_i(s)] \tag{9}$$

where, $w_i(s)$'s are in $R(s)$ $(i = 1, 2, \ldots, m)$. However, in this study, we shall consider that $w_i(s)$'s are in $R[s]$. Thus, the closed loop system designed will be proper. As it is known, this is desirable for practical reasons, in order to avoid an impulsive response for every initial condition.

The closed loop system transfer function can be written in terms of the right external description, i.e. $\begin{bmatrix} N(s) \\ D(s) + KN(s) \end{bmatrix}$ which comes from (5), as,

$$W^{-1}(s) = HN(s)\,[D(s) + KN(s)]^{-1} L \tag{10}$$

by using Lemma 1 and (3). Since $W(s)$ is defined as an invertible matrix, it is clear that $HN(s)$ should be invertible. Relation (10) can be arranged in the form of a polynomial matrix equation as,

$$W(s)HN(s) = XD(s) + YN(s) \tag{11}$$

where,

$$X = L^{-1}, \quad Y = L^{-1}K. \tag{12}$$

Thus, the problem of decoupling with making proper is considered as under what conditions there exists a diagonal $W(s)$ which ensures that (11) has a solution pair X, Y with entries in R, such that X nonsingular. In fact, this problem is closely related with the existence conditions of the constant solutions of a matrix polynomial equation.

3. CONSTANT SOLUTIONS OF A MATRIX POLYNOMIAL EQUATION

In order to derive a necessary and sufficient condition for the constant solutions of a matrix polynomial equation, the following definition is presented.

Definition 1. For a matrix $A(s)$ over $R[s]$ with dimension $(p \times q)$, $\deg A(s)$ denotes the maximum degree of the elements of $A(s)$. Let a given integer be $n \geq \deg A(s)$ and a vector be

$$\psi_n(s) := [s^n \ s^{n-1} \ \cdots \ s \ 1]^T \tag{13}$$

then, the coefficient matrix $\overline{A}$ of size $[p \times q(n+1)]$ associated with $A(s)$ can be defined as,

$$A(s) := \overline{A}\Psi_n(s) \tag{14}$$

where

$$\Psi_n(s) := \text{blockdiag } [\psi_n(s)] \tag{15}$$

with size $[q(n+1) \times q]$, [8].

It should be noted that the coefficient matrix of a polynomial matrix is uniquely determined by the polynomial matrix, for a given integer n. The following property of this representation demonstrates a characteristic of a polynomial matrix, which is reflected on the coefficient matrix.

Property 1. The rows of a polynomial matrix $A(s)$ are linearly dependent in R if the rows of $\overline{A}$ are linearly dependent and vice versa.

Proof. Since $\psi_n(s)$ and so $\Psi_n(s)$ have no constant kernel space, the property can be proven by the existence of a nonsingular matrix T with entries in R such that,

$$T\ A(s) = \begin{bmatrix} A_1(s) \\ 0 \end{bmatrix} \tag{16}$$

and then by Definition 1,

$$T\overline{A}\ \Psi_n(s) = \begin{bmatrix} \overline{A_1} \\ 0 \end{bmatrix} \Psi_n(s) \quad ; \quad \begin{bmatrix} \overline{A_1} \\ 0 \end{bmatrix} = T\overline{A}. \tag{17}$$

□

A result of this property, which will be used later, is that if a polynomial matrix is nonsingular, then its coefficient matrix has independent rows. A lemma can be given by using Definition 1 and Theorem 1 of [12].

Lemma 2. Let $P(s)$, $Q(s)$ and $R(s)$ be matrices of size $(m\times m)$, $(p\times m)$ and $(m\times m)$ respectively with their entries in $R[s]$, and $n = \max\{\deg P(s),\ \deg Q(s),\ \deg R(s)\}$. Then the equation,

$$X\ P(s) + Y\ Q(s) = R(s) \tag{18}$$

has a solution pair X, Y with entries in R, such that X is nonsingular, if and only if the row space of the coefficient matrix of $R(s)$ for n, i. e. $\overline{R}$, is a subspace of that of the coefficient matrix of $\begin{bmatrix} P(s) \\ Q(s) \end{bmatrix}$ for n, i. e. $\begin{bmatrix} \overline{P} \\ \overline{Q} \end{bmatrix}$, and also the rank of $\begin{bmatrix} \overline{P} \\ \overline{Q} \end{bmatrix}$ is equal to the rank of $\begin{bmatrix} \overline{R} \\ \overline{Q} \end{bmatrix}$.

Proof. Let X, Y be matrices with entries in R, then equation (18) can be written as,

$$\left(\begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} \overline{P} \\ \overline{Q} \end{bmatrix} - \overline{R} \right) \Psi_n(s) = 0 \tag{19}$$

by using Definition 1. Since $\psi_n(s)$ and so $\Psi_n(s)$ have no constant kernel space, equation (19) is satisfied if and only if the following matrix equation has a solution pair X, Y,

$$\begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} \overline{P} \\ \overline{Q} \end{bmatrix} = \overline{R}. \tag{20}$$

As it is known, the equation (20) has a solution if and only if the row space of $\overline{R}$, is a subspace of that of $\begin{bmatrix} \overline{P} \\ \overline{Q} \end{bmatrix}$. When equation (20) can be written as,

$$\begin{bmatrix} X & Y \\ 0 & I_p \end{bmatrix} \begin{bmatrix} \overline{P} \\ \overline{Q} \end{bmatrix} = \begin{bmatrix} \overline{R} \\ \overline{Q} \end{bmatrix} \tag{21}$$

then, it is evident that the matrix X is nonsingular, if and only if the rank of $\begin{bmatrix} \overline{P} \\ \overline{Q} \end{bmatrix}$ is equal to the rank of $\begin{bmatrix} \overline{R} \\ \overline{Q} \end{bmatrix}$. □

Let $P(s)$ be a column degree ordered polynomial matrix with column degrees of $n_1 \geq n_2 \geq \cdots \geq n_m$ with size of $(p \times m)$. By using Definition 1 the following representation can be given for $P(s)$,

$$P(s) = \overline{P}\ \Psi(s) \tag{23}$$

$$\Psi(s) := \text{blockdiag } [\psi_{n_i}(s)] \quad (i = 1, 2, \cdots, m) \tag{24}$$

with size $(m+q) \times m$ $(q := \sum_1^m n_i)$ and $\overline{P}$ is the coefficient matrix of $P(s)$ with size $[p \times (m+q)]$.

Lemma 3. Let $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ be an irreducible and column reduced matrix in $R[s]$ and $n_1 \geq n_2 \geq \cdots \geq n_m$ be its column degrees. If there exists a polynomial matrix $M(s)$ such that

$$\begin{bmatrix} X & Y \\ 0 & I_n \end{bmatrix} \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} M(s) \\ N(s) \end{bmatrix} \tag{25}$$

where X, Y are matrices with entries in R and X is nonsingular, then the following holds,

a) The column degrees of $M(s)$ are equal to or less than n_i respectively.

b) The rank of the matrix which corresponds to the columns with degree n_i of the coefficient matrix of $\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}$ is equal to m, i. e.,

$$\text{rank}\left\{ \lim_{s\to\infty} \begin{bmatrix} M(s) \\ N(s) \end{bmatrix} \left[\text{ diag } [s^{-n_i}] \ \right] \right\} = m. \tag{26}$$

Proof. The condition (a) is easily proven when the equation (25) is considered column by column. From Lemma 2 and (25), the following equation has also a nonsingular solution,

$$\begin{bmatrix} X & Y \\ 0 & I_n \end{bmatrix} \begin{bmatrix} \overline{D} \\ \overline{N} \end{bmatrix} = \begin{bmatrix} \overline{M} \\ \overline{N} \end{bmatrix}. \tag{27}$$

Furthermore, $\begin{bmatrix} \overline{M} \\ \overline{N} \end{bmatrix}$ contains the coefficient matrix that corresponds to the columns with degrees n_i. So Lemma 2 proves (b) when (27) is considered column by column. □

The conditions given in Lemma 3 are regarded as the necessary conditions for the existence of nonsingular constant solution pair of a polynomial matrix equation. In order to extend the above analysis to decoupling problem, a property of Definition 1 is presented as follows.

Property 2. Let $a(s)$ be a polynomial, $\deg a(s)$ denotes the degree of $a(s)$ and $\mu \geq 0$ be an integer. Then, the coefficient matrix $[s^\mu a(s)]$ is determined as

$$s^\mu a(s) = \overline{a} D^\mu \psi_n(s) \tag{28}$$

where D is a constant matrix defined as,

$$D = \left\{ \begin{array}{ll} D(j,k) = 1 & \text{if } j > 1 \text{ and } k = j-1 \\ D(j,k) = 0 & \text{else} \end{array} \right\}; \ j = 1, \cdots, n+1; \ k = 1, \cdots, n+1 \tag{29}$$

and $n \geq \mu + \deg a(s)$, $a(s) = \overline{a}\psi_n(s)$ as in Definition 1.

P r o o f . Let $a(s)$ be a polynomial such as

$$a(s) = [0\ 0\ \cdots\ 0\ a_{n-\lambda}\ a_{n-\lambda-1}\ \cdots\ a_1\ a_0]\,\psi_n(s) \tag{30}$$

Furthermore, the polynomial $[sa(s)]$ can be written as,

$$s\ a(s) = [0\ \cdots\ 0\ a_{n-\lambda}\ a_{n-\lambda-1}\ \cdots\ a_1\ a_0\ 0]\,\psi_n(s) \tag{31}$$

where $\lambda = n - \deg a(s)$. It is easily seen that the effect of the multiplication by 's' appears as a left shifting operation on the coefficient matrix. This effect can be represented by the matrix D given in (29). Moreover, it can be extended to D^μ for multiplication by s^μ. □

The expression given in Property 2 is also true for a row vector of polynomials by using the following definition,

$$D_\mu^\star := \text{blockdiag}\ [D^\mu] \tag{32}$$

and the coefficient matrix of a row vector $s^\mu A(s)$ is then easily obtained as,

$$s^\mu A(s) = \overline{A}\ D_\mu^\star\ \Psi_n(s). \tag{33}$$

It should be noted that μ can at most be equal to $\{n - \deg A(s)\}$.

4. MAIN RESULTS

In order to solve the decouplability problem in singular systems, first, we have to determine the degrees of $w_i(s)$, $(i = 1, 2, \ldots, m)$ defined in (9), such that they enable the existence of a nonsingular constant solution pair of the equation,

$$\begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = W(s)\ HN(s). \tag{34}$$

Let the degrees of $w_i(s)$ be represented as μ_i, then the condition (a) in Lemma 3 implies that μ_i's, $i = 1, 2, \ldots, m$ have the values of

$$\mu_i \leq d_i = \min_{j=1}^{m}\{d_{ij}\} \tag{35}$$

where,

$$\begin{aligned} d_{ij} &= n_j - k_{ij}\ \ ;\ \ k_{ij} = \deg hn_{ij} \\ n_j &= \text{the } j\text{th column degree of } \begin{bmatrix} D(s) \\ N(s) \end{bmatrix}. \end{aligned} \tag{36}$$

In this expression, $hn_{ij}(s)$'s are the entries of $HN(s)$. As it will be explained later, d_i's are also related with the infinite structure orders of the ith row of $T(s)$. Let $w_i(s)$'s have the following structure,

$$w_i(s) = w_{id_i}s^{d_i} + \cdots + w_{i1}s + w_{i0} \neq 0. \tag{37}$$

Thus, by using Definition 1 and Property 2 and (32), the following representations can be given as,

$$\overline{W} = \text{blockdiag}\,[\overline{w_i}]; \quad HN^\star = \begin{bmatrix} HN_1^\star \\ \vdots \\ HN_m^\star \end{bmatrix}; \quad HN_i^\star = \begin{bmatrix} \overline{HN_i} D_{d_i}^\star \\ \vdots \\ \overline{HN_i} D_0^\star \end{bmatrix} \tag{39}$$

such that, $\overline{HN_i}$ is the coefficient vector of the ith row of $HN(s)$. So the coefficient matrix of $W(s)\,HN(s)$, i. e. $\overline{WHN}$, is easily derived as,

$$\overline{WHN} = \overline{W}HN^\star \tag{40}$$

in terms of the coefficient vector of $w_i(s)$. As a consequence of the above analysis, the problem of decoupling with making proper has been reduced to the solvability problem of the following equation,

$$X\overline{D} + Y\overline{N} = \overline{W}\ HN^\star \tag{41}$$

where the sizes of the matrices mentioned above are as follows,

$$\begin{aligned} &\overline{D} \cdots m \times (n_1 + 1 + \ldots + n_m + 1); \overline{N} \cdots n \times (n_1 + 1 + \ldots + n_m + 1); \\ &\quad \overline{W} \cdots m \times (d_1 + 1 + \ldots + d_m + 1) \\ &HN^\star \cdots m \times (d_1 + 1 + \ldots + d_m + 1) \times (n_1 + 1 + \ldots + n_m + 1); \\ &\quad HN_i^\star \cdots (d_i + 1) \times (n_1 + 1 + \ldots + n_m + 1) \\ &\overline{HN_i} \cdots 1 \times (n_1 + 1 + \ldots + n_m + 1); D_j^\star \cdots (n_1 + 1 + \ldots + n_m + 1) \\ &\quad \times (n_1 + 1 + \ldots + n_m + 1); D^j \cdots (n_j + 1) \times (n_j + 1). \end{aligned}$$

It is clear that the above equation has a solution pair $\{X, Y\}$, if and only if the row space of $\overline{W}HN^\star$ is a subspace of that of $\begin{bmatrix} \overline{D} \\ \overline{N} \end{bmatrix}$. So, the notation below is given.

Notation 1. The matrix composed of the rows of $HN_i^\star$ which are linearly dependent on the rows of $\begin{bmatrix} \overline{D} \\ \overline{N} \end{bmatrix}$ is denoted by $HN_i^\diamond$. In other words, the row space of $HN_i^\diamond$ is a subspace of the row space of $HN_i^\star$ while the row space of $HN_i^\diamond$ is a subspace of that of $\begin{bmatrix} \overline{D} \\ \overline{N} \end{bmatrix}$.

Consequently, when the matrix $HN^\diamond$ composed of $HN_i^\diamond$'s is defined as,

$$HN^\diamond = \begin{bmatrix} HN_1^\diamond \\ \vdots \\ HN_m^\diamond \end{bmatrix} \tag{42}$$

the row space of $HN^\diamond$ is a subspace of that of $\begin{bmatrix} \overline{D} \\ \overline{N} \end{bmatrix}$. It can be shown that an appropriate matrix $W^\diamond$ can be derived from (40) as,

$$W^\diamond = \text{blockdiag}\ [w_i^\diamond] \quad ; \quad w_i^\diamond \neq 0 \tag{43}$$

in which $w_i^\diamond$'s are row vectors.

As a summary of the above analysis it has been shown that the following statements are equivalent;

a) For a system given in terms of H, $N(s)$ and $D(s)$ the problem of decoupling with making proper by static state feedback has a solution.

b) There exists a $W(s)$ defined in (9) such that the equation $X\ D(s) + Y\ N(s) = W(s)HN(s)$ has a solution pair X, Y in R, with X nonsingular.

c) There exists a $W^\diamond$ defined in (43) such that the equation $X\ \overline{D} + Y\ \overline{N} = W^\diamond HN^\diamond$ has a solution pair X, Y in R, with X nonsingular.

In order to present a theorem on the decoupling problem, the following notation and the remark are given.

Notation 2. From Notation 1 denote the $k_i's$, $(i = 1, 2, \ldots, m)$ as

$$k_i = \left\{ \begin{array}{ll} 1 \quad \text{if rank} \begin{bmatrix} HN_i^\diamond \\ \overline{N} \end{bmatrix} > \text{rank}[\overline{N}] \\ 0 \quad \text{if rank} \begin{bmatrix} HN_i^\diamond \\ \overline{N} \end{bmatrix} = \text{rank}[\overline{N}] \end{array} \right\}. \tag{44}$$

Remark 1. For $W^\diamond$ defined in (43), the rank of $\begin{bmatrix} W^\diamond HN^\diamond \\ \overline{N} \end{bmatrix}$ is equal to,

$$\text{rank} \begin{bmatrix} W^\diamond HN^\diamond \\ \overline{N} \end{bmatrix} = \sum_{i=1}^{m} k_i + \text{rank}[\overline{N}]. \tag{45}$$

P r o o f. From Notation 1 and 2, the following relation can be stated as,

$$\text{rank} \begin{bmatrix} w_i^\diamond HN_i^\diamond \\ \overline{N} \end{bmatrix} = \text{rank}[\overline{N}] + k_i. \tag{46}$$

Furthermore, since $W(s)HN(s)$ is an invertible polynomial matrix, Property 1 implies that each $w_i^\diamond HN_i^\diamond$ in $[W^\diamond HN^\diamond]$ brings an independent row from $w_j^\diamond HN_j^\diamond$ for $i \neq j$. So, when (46) is evaluated for $i = 1, 2, \ldots, m$, (45) is proven. □

As a consequence of the previous results, the following theorem can be given on decouplability of system (1).

Theorem. Let system (1) be regularizable and $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ be its external description. Then, the problem of decoupling with making proper by static state feedback for system (1) has a solution, if and only if,

$$\text{rank}\begin{bmatrix} \overline{D} \\ \overline{N} \end{bmatrix} = \sum_{i=1}^{m} k_i \ + \ \text{rank}\ [\overline{N}] \tag{47}$$

where k_i's are given in Notation 2 and $\begin{bmatrix} \overline{D} \\ \overline{N} \end{bmatrix}$ is the coefficient matrix of $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$.

Proof. It is mentioned before that, the problem of decoupling with making proper by static state feedback is the same as the solvability problem of,

$$X\overline{D} + Y\overline{N} = W^{\diamond}HN^{\diamond} \tag{48}$$

such that X nonsingular. So, Lemma 2 proves both necessity and sufficiency by using Remark 1, Notation 1 and Notation 2. □

5. CONSTRUCTION

The proof of Theorem provides a procedure for the construction of gains K and L. The procedure can briefly be given as follows,

(a) Find the external description of system (1a) as

$$[sE - F \quad -G]\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0$$

such that $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ is irreducible and column proper.

(b) Calculate d_i's, $HN_i^{\diamond}$'s, k_i's by using (35), Notation 1 and (42), and Notation 2 respectively.

(c) Check the following condition,

$$\text{rank}\begin{bmatrix} \overline{D} \\ \overline{N} \end{bmatrix} = \sum_{i=1}^{m} k_i + \text{rank}[\overline{N}]$$

if it fails, the system is not decouplable by static state feedback.

(d) If the system is decouplable, solve the following equation,

$$X\overline{D} + Y\overline{N} = W^{\diamond}HN^{\diamond}.$$

Finally, find the feedback gains as $K = X^{-1}Y,\ L = X^{-1}$.

An example. In (1), let,

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

follow the construction

step (a), $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ s & 0 \\ 0 & 0 \end{bmatrix};$

step (b), $d_1 = 1, \quad d_2 = 0, \quad HN_1^\diamond = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad k_1 = 1;$

$HN_2^\diamond = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}; \quad k_2 = 0$

step (c), $\operatorname{rank} \begin{bmatrix} \overline{D} \\ \overline{N} \end{bmatrix} = 3; \quad \operatorname{rank} \overline{N} = 2;$

$\operatorname{rank} \begin{bmatrix} \overline{D} \\ \overline{N} \end{bmatrix} = \operatorname{rank} \overline{N} + \sum_{i=1}^{2} k_i = 2 + 1 = 3$

so that the system is decouplable,

step (d), $X = \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix}, \; Y = \begin{bmatrix} \varepsilon & 0 & \zeta \\ \eta & \eta & \theta \end{bmatrix}, \; W^\diamond = \begin{bmatrix} \alpha & \varepsilon & 0 \\ 0 & 0 & \eta \end{bmatrix}$

i. e. $w_1(s) = \alpha s + \varepsilon \;\; w_2(s) = \eta.$

6. STRUCTURAL PROPERTIES OF DECOUPLED SYSTEMS

As it is known, the decouplability of ordinary systems is completely determined with their infinite zero structure orders, [5, 16]. In this section, we shall try to obtain some structural results for singular systems only in regular case, i. e. $(sE - F)$ is nonsingular. For this aim, we call the Smith McMillan factorization at infinity of rational functions.

Definition 2. [11] Let $T(s)$ be a given $(p \times m)$ rational matrix. Then the Smith McMillan form of $T(s)$ at infinity gives,

$$\Lambda(s) = \begin{bmatrix} \Lambda_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \Lambda_r(s) = \operatorname{diag}\,[s^{-\delta_1} s^{-\delta_2} \dots \; s^{-\delta_r}] \tag{49}$$

where r is the rank of $T(s)$ and $\delta_1 \leq \delta_2 \leq \ldots \leq \delta_r$ are integers. These integer are called the infinite structure orders of $T(s)$. This form is uniquely determined from $T(s)$ by Smith McMillan at infinity, or in other words in $R_c(s)$, the ring of causal (proper) rational functions, as following

$$T(s) = B_1(s) \begin{bmatrix} \Lambda_r(s) & 0 \\ 0 & 0 \end{bmatrix} B_2(s) \tag{50}$$

where $B_1(s)$ and $B_2(s)$ with size $(p \times p)$ and $(m \times m)$ respectively are defined as units of the ring of causal rational matrices, namely bicausal (biproper) rational matrices. It will be useful to note that $B(s)$ is a bicausal matrix if and only if $\det[\lim_{s\to\infty} B(s)] \neq 0$.

Proposition 1. Let system (1) be regular and invertible, and $T(s)$ be its transfer function, and $t_i^T(s)$ be the rows of $T(s)$, $i = 1, 2, \ldots, m$. Then, system (1) is decouplable, only if the infinite structure orders of $T(s)$ are respectively equal to the infinite structure order of $t_i^T(s)$ for $i = 1, 2, \ldots, m$.

Proof. If the system is decouplable, then there exists a precompensator $C(s)$ which decouples the system, and we can write,

$$T(s)\,C(s) = \mathrm{diag}\,[s^{-d_i}]\;B_0(s) \tag{51}$$

where $B_0(s)$ is a bicausal and diagonal matrix, and d_i's are integers. Let us define a set of indices δ_i, $i = 1, 2, \ldots, m$, such that,

$$\delta_i = \min\left\{\sigma, \lim_{s\to\infty} s^{\sigma} t_i^T(s) \neq 0, \infty; \sigma \in I\right\} \tag{52}$$

and a matrix with entries in R, such that,

$$\lim_{s\to\infty} \mathrm{diag}\,[s^{\delta_i}]\;T(s) = T. \tag{53}$$

We shall assume that the δ_i's are increasingly ordered; otherwise the outputs are renumbered. It should be noted that δ_i is the infinite structure order of ith row of $T(s)$ and T is invertible. When the relation (51) is arranged by using δ_i's, we have,

$$\mathrm{diag}[s^{\delta_i}]\;T(s)\;C(s)\;\mathrm{diag}[s^{\nu_i}] = B_0(s) \tag{54}$$

from diagonality of $B_0(s)$, here $\nu_i = d_i - \delta_i$. Since the matrix '$\mathrm{diag}[s^{\delta_i}]\,T(s)$' is a bicausal matrix, we obtain,

$$T\;\lim_{s\to\infty} C(s)\;\mathrm{diag}[s^{\nu_i}] = B_0 \tag{55}$$

when (54) is evaluated at infinity. Here $B_0 = \lim_{s\to\infty} B_0(s)$ and $\mathrm{rank} B_0 = m$. As a result of (55),

$$\det\left\{\lim_{s\to\infty} C(s)\;\mathrm{diag}\;[s^{\nu_i}]\right\} \neq 0. \tag{56}$$

Let, a bicausal matrix $B(s)$ be obtained from $C(s)$ as follows,

$$B(s) = C(s) \text{ diag } [s^{\nu_i}]. \tag{57}$$

Thus, from (54) and (57), we write,

$$T(s) = \text{diag } [s^{-\delta_i}] \ B_0(s)B^{-1}(s) \tag{58}$$

and finally by Definition 2 we say that δ_i's are also the infinite structure orders of $T(s)$. □

When the closed loop system is required to be proper, i.e. d_i's are defined as nonnegative integers, a natural result is that if $\delta_i < 0$, i.e. it is an infinite pole, then $\nu_i = -\delta_i, \ d_i = 0$ and if $\delta_i \geq 0$, i.e. it is an infinite zero, then $\nu_i = 0$, $d_i = \delta_i$, since each δ_i is either positive or zero or negative integer. So it is clear that d_i's defined in (35) coincides with d_i's defined in (51).

7. CONCLUSION

In this study, the row by row decoupling problem of regularizable singular square systems is investigated by polynomial equation approach. The necessary and sufficient conditions of the problem of decoupling with making proper by static state feedback is given in Theorem, such that the closed loop system is proper. A procedure for the construction of feedback gains K, L is also proposed. Moreover a structural interpretation is given in Proposition 1 which provides a necessary condition for decouplability of regular singular systems in terms of their infinite structure orders. The results presented here are not limited in the case of singular systems, i.e., they are also true for ordinary square systems.

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