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ALTERNATIVE POLYNOMIAL EQUATION APPROACH TO LQ DISCRETE–TIME FEEDBACK CONTROL

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The usual solution of the single-input, single-output, LQ discrete-time feedback control through the polynomial equations is modified. The way starts with a general solution of the "implied" closed-loop equation the free polynomial of which is then optimized.

At the same time the conditions are derived under which the implied equation minimum solution represents the LQ optimal one.The conclusions are obtained to be more general if compared with the former results.

1. INTRODUCTION

The polynomial and polynomial matrix equation approach to LQ and LQG control problems is well-known. Fundamental results for general, multi-input, multi-output, discrete-time systems can be found in [4]. Many other contributions have been written in recent years to extend the problem solution for various types of the control structures and acting signals or reduce it under special assumptions, e. g., [1, 2, 3, 5, 6, 7].

Although validity of the results presented below can be generalized, the simplest feedback structure of a single-input, single-output, deterministic control is considered according to Figure 1.

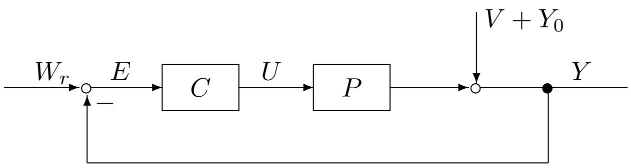

Fig. 1.

The output Y of a controlled process P should track a reference W_r, being affected simultaneously by a possible load disturbance V (referred to the output) and current nonzero conditions Y_0 at the control start. C, E and U denote a controller, error and control signal, respectively. All the signals are assumed here to be described in the discrete-time fashions and P denotes the discrete-time model of a continuous-time process including the zero-order hold.

The relations

$$E = W - PU \quad \text{and} \quad U = CE \tag{1}$$

are obvious, where

$$W = W_r - V - Y_0 \tag{2}$$

is a (generalized) reference as the single input.

Hereafter, polynomials and sequences in d (one step delay in the time domain or the inverse Z-transform complex variable in the complex frequency domain) as well as the usual symbols are used. Namely, $\deg a$, a^+, a^-, $a_* = a(d^{-1})$, $a^\sim = d^{\deg a} a_*$ concerning a polynomial a, (a, b), $b|a$ and $b \sim a$ for two polynomials, $F_*(d) = F(d^{-1})$ and $\langle F \rangle = \phi_0$ for a sequence $F = \cdots + \phi_{-1} d^{-1} + \phi_0 + \phi_1 d + \cdots$. Moreover the only a^c to emphasize a causal polynomial and the factorization $a = a^+ a^0 a^=$ are introduced, where all zeros d_i of $a^+(d)$, $a^0(d)$ and $a^=(d)$ have the property $|d_i| > 1$, $|d_i| = 1$ and $|d_i| < 1$, respectively.

The paper is organized into several parts. The standard way to solve the LQ discrete-time feedback control generally via the coupled polynomial equations is described briefly in Section 2. Then Section 3 deals with the other, alternative approach of the solution which starts with the "implied" equation. The second, related problem is treated in Section 4. The conditions are found under which the optimal LQ solution coincides with the minimum solution of the implied equation. The derived conclusions are more general if compared with the results reached by the other authors in [1] and [2].

2. USUAL SOLUTION OF LQ DISCRETE–TIME FEEDBACK CONTROL

Let us consider the control structure shown in Figure 1, where

$$P = \frac{b}{a}; \quad a,\, b \quad \text{coprime}, \quad a = a^c, \quad b = d^\beta b^c, \quad \beta > 0, \tag{3}$$

$$W = W_r - V - Y_0 = \frac{f}{h}; \quad h,\, f \quad \text{coprime}, \quad h = h^c, \tag{4}$$

and the optimal controller C described by

$$C = \frac{m}{n}; \quad n^-,\, m^- \quad \text{coprime}, \tag{5}$$

is sought such that the performance index

$$\vartheta = \psi \langle E E_* \rangle + \phi \langle U U_* \rangle \tag{6}$$

reaches its minimum; $\psi > 0$ and $\phi > 0$ are the chosen weighting constants.

Using the description (3) and (4) a controlled process P as well as an input generator W are supposed to be minimal realizations of their input-output behaviour. In this case the LQ control problem is generally solved by the minimum $\deg z$ solution m, n, z, $\deg z < \rho$, of the coupled equations

$$d^\rho s_* m + a h_a z = d^\rho b_* \psi p \tag{7}$$

and

$$d^\rho s_* n - b h_a z = d^\rho a_* \phi p \tag{8}$$

where

$$\rho \;=\; \max(\deg a,\, \deg b), \qquad h_a = \frac{h}{(a,h)} \qquad \text{and} \qquad a_h = \frac{a}{(a,h)}, \tag{9}$$

$$s = s^+ \qquad \text{follows from} \qquad s s_* = \psi b b_* + \phi a a_*, \tag{10}$$

and

$$p = p^+ p^0 \qquad \text{from} \qquad p p_* = a_h a_{h*} f f_*. \tag{11}$$

The problem becomes solvable if and only if $h_a = h_a^+$ and $p = p^+$ $(p^0 = a_h^0 f^0 \sim 1)$ and the optimal controller (5) is unique. The resulting error and corresponding control sequences are

$$E = \frac{a_h f n}{h_a s p} \qquad \text{and} \qquad U = \frac{a_h f m}{h_a s p}. \tag{12}$$

The polynomial c determining the closed-loop finite poles follows from the simple arrangement of equations (7) and (8) into so-called "implied" equation

$$c \;=\; an + bm \;=\; sp. \tag{13}$$

Usually the single equation (7) instead of the couple (7) with (8) may be solved for minimum $\deg z$, $\deg z < \rho$, to obtain the optimal m, [3]. Such a simplified solution holds whenever $\deg(d^\rho s_*, a h_a) = \deg(d^\rho s_*, a^-) = 0$. The remaining polynomial n follows from (13).

Notice that other current descriptions ([1, 2, 3, 4]) uses

$$P = \frac{B}{A} \qquad \text{and} \qquad W = \frac{F}{A} \tag{14}$$

where the couples A, B and A, F are not necessarily coprime but $(A, B, F) \sim 1$. (Capital letters stand for polynomials.)

Comparing (14) with (3) and (4) yields

$$A = a h_a = h a_h, \quad B = b h_a \quad \text{and} \quad F = f a_h. \tag{15}$$

3. ALTERNATIVE SOLUTION OF LQ DISCRETE–TIME FEEDBACK CONTROL

The other possible way to solve the LQ discrete-time, feedback control problem is presented in the following claim.

Claim 1. LQ discrete-time feedback control problem given by the relations (3) to (6) and (9) to (11) is solved by

$$m = m_p + at \qquad \text{and} \qquad n = n_p - bt \tag{16}$$

where m_p, n_p is any particular solution of the implied equation (13) and t belongs to the minimum $\deg z$ solution z, t, $\deg z < \rho$, of the polynomial equation

$$d^\rho s_* t + h_a z = r, \tag{17}$$

where introducing

$$q = \psi b n_{p*} - \phi a m_{p*} \tag{18}$$

yields $r = d^\rho q_*/s$ being always a polynomial.

The problem is solvable if and only if $h_a = h_a^+$ and $p = p^+$ and the solution is unique.

Proof. Suppose that polynomials m, n standing in (16) form just the optimal controller (5), i.e., they solve equations (7) and (8) with minimum $\deg z$, $\deg z < \rho$.

Substituting (16) into (7) and (8) we obtain

$$d^\rho s_* m_p + a(d^\rho s_* t + h_a z) = d^\rho b_* \psi p \tag{19}$$

and

$$d^\rho s_* n_p - b(d^\rho s_* t + h_a z) = d^\rho a_* \phi p. \tag{20}$$

If (19) multiplied by n_p and (20) by m_p are mutually subtracted we have

$$(an_p + bm_p)(d^\rho s_* t + h_a z) = pd^\rho(\psi b_* n_p - \phi a_* m_p) \quad \text{or} \quad sr = d^\rho q_* \tag{21}$$

when (17), (18) and the relation $an_p + bm_p = sp$ have been used. Hence (17) has to be solved for minimum $\deg z$, $\deg z < \rho$. Seeing (12) $h_a = h_a^+$ and $p = p^+$ are found to be necessary as well as sufficient conditions of the problem solvability. In this case (17) is always solvable and the solution in unique. □

4. OPTIMAL LQ FEEDBACK CONTROL SOLUTION VIA THE IMPLIED EQUATION ONLY

Using and analyzing the previous relations the sufficient conditions can be found under which the special particular solution of the single implied equation (13) is just the LQ optimal one. The following claim gives the resulting conclusions.

Claim 2. LQ discrete-time feedback control problem defined by relations (3) to (6) and (9) to (11) is solved uniquely by the minimum $\deg m$ solution m, n, $\deg m < \deg a$, of the equation (13), if simultaneously

$$\deg h_a = 0 \tag{22}$$

and

$$\deg a + \beta \ > \ \deg p\,. \tag{23}$$

P r o o f . Let us assume that m_p, n_p in (16) is the minimum $\deg m$ solution of (13) which should coincide with the optimal LQ solution m, n. Then $t = 0$ in (16) as well as (17) such that $h_a z = r$. Generally (but not necessarily always) h_a does not divide r and therefore condition (22) or $h_a = 1$ must be supposed to ensure the polynomial fashion of z; then $z = r$.

Since optimal polynomial z with $\deg z < \rho$ is unique, the desired identity $z = r$ follows simply from the degree relation

$$\deg z = \deg r < \rho\,. \tag{24}$$

Thus the conditions have to be found under which (24) is true.

Several preliminary relations are emphasized at first.

1. $\deg\,(d^\rho s_*) = \deg\,(d^\rho a_*) = \rho$ and $\deg\,(d^\rho b_*) = \rho - \beta.$ (25)

2. The following cases of the process model properties may be distinguished:

   i) $\deg a > \deg b^c$; then $\deg s = \deg a$ (26)

   ii) $\deg a = \deg b^c$; then either $\deg s = \deg a$ (27)
   or $\deg s < \deg a$ (28)

   iii) $\deg a < \deg b^c$; then $\deg s = \deg b^c$. (29)

3. The degrees of minimum $\deg m$ solution m_p, n_p of (13) are as follows:

$$\deg m_p < \deg a \tag{30}$$

and either

$$\deg n_p < \deg b \tag{31}$$

if

$$\deg a + \deg b > \deg s + \deg p \tag{32}$$

or

$$\deg n_p < \deg s + \deg p - \deg a + 1 \tag{33}$$

if

$$\deg a + \deg b \leq \deg s + \deg p\,. \tag{34}$$

Usual (generic) degrees are the upper limits in (30) and (31) or (33), i.e., $\deg m_p = \deg a - 1$ and $\deg n_p = \deg b - 1$ or $\deg n_p = \deg s + \deg p - \deg a$.

Seeing (18) and (21) and applying (25) we can write

$$\begin{aligned} \deg r \ &= \ \deg\,(d^\rho q_*) - \deg s \leq \max\,(\rho - \beta + \deg n_p,\ \rho + \deg m_p) \\ &\quad -\deg s = \rho - \deg s + \max\,(\deg n_p - \beta,\ \deg m_p). \end{aligned} \tag{35}$$

Provided (32) is valid the relation (35) obtains the form

$$\deg r < \rho - \deg s + \max\,(\deg b^c,\ \deg a).$$

Hence obviously (24) is fulfilled if (26) or (27) or (29) is true. In the case (28) we express

$$\begin{aligned} abd^\rho q_* &= d^\rho b_* \psi ban_p - d^\rho a_* \phi abm_p \\ &= d^\rho b_* \psi ban_p - d^\rho s_* sbm_p \ + \ d^\rho b_* \psi bbm_p \\ &= d^\rho b_* \psi bsp - d^\rho s_* sbm_p = sb(d^\rho b_* \psi p - d^\rho s_* m_p). \end{aligned}$$

Then

$$ad^\rho q_* = s(d^\rho b_* \psi p - d^\rho s_* m_p)$$

and

$$\begin{aligned} \deg r &= \deg \left(d^\rho b_* \psi p - d^\rho s_* m_p \right) - \deg a \\ &\leq \max \left(\rho - \beta + \deg p, \rho + \deg a - 1 \right) - \deg a \\ &= \rho + \deg p - \beta - \deg a, -1). \end{aligned}$$

Hence the condition (23) must be valid to ensure (24). Since this condition is always satisfied in the previous cases too, it is sufficient for the whole case (32).

If (34) is true then (35) has the form

$$\begin{aligned} \deg r &< \rho - \deg s + \max \left(\deg\, s + \deg p - \deg a \ + 1 - \beta,\, \deg a \right) \\ &= \rho + \max \left(\deg p - \deg a + 1 - \beta,\, \deg a - \deg s \right). \end{aligned}$$

Hence $\deg r < \rho$ if (26) or (27) is valid and (23) is satisfied at the same time.

Investigating the remaining cases (28) and (29) we find that (23) would be true again. But the contradictory relation $\deg a + \beta \leq \deg p$ follows from (34). Moreover $\deg a -$ $\deg s > 0$ breaks (23) too in the case (28). Therefore (23) can never be reached provided (28) or (29) along with (34) are valid.

Thus conditions (22) and (23) are verified to be the sufficient conditions for LQ optimality of the minimum $\deg m$ solution of the implied equation (13). □

The condition (23)

- is always true if (26) or (27) or (29) along with (32) hold
- can be true if either (28) with (32) or (26) or (27) with (34) hold
- is never true if either (28) or (29) along with (34) are valid.

5. CONCLUSIONS

The derived sufficient conditions (22) and (23) may be compared with the former results presented by the other authors.

The first investigation of LQ optimality of the implied equation minimum solution was given for matrix equations of multivariable systems in [1]. Based on this work the SISO case results have been formulated in [2] or [3]. The respective conditions, although considered there in the stochastic framework,may be expressed for the deterministic approach in the form using starting description (14) as

i) A and B coprime; (36)

ii) F stable;

iii) $W = F/A$ proper (with respect to d). (37)

If our assumptions and denotations are used these conditions obtain the form

i) $\deg h_a = 0$; (38)

ii) stability of $F = a_h f$ is always secured since F is replaced by p according to (11);

iii) $\deg a \geq \deg p$ or $\deg a + 1 > \deg p$. (39)

Comparing these relations with the results of Claim 2 one can see the difference in condition (37) or (39) and (23). They only coincide if $\beta = 1$, i. e., for processes and systems without deadtime longer than the sampling time. For $\beta > 1$, when a process having deadtime longer than a sampling time is controlled, properness of W is not necessary but (23). Therefore (23) represents the more general condition if compared with (37).

6. EXAMPLE

Let us solve the LQ control problem for $\psi = 1$ and $\phi = 0.75$, if

$$P = \frac{b}{a} = \frac{d^2(1-0.5d)}{1-2d}, \qquad W = \frac{f}{h} = \frac{1-0.1d-0.2d^2}{1-2d}.$$

At first we determine $h_a = 1,\ a_h = 0,\ p = f^+ = f = 1 - 0.1d - 0.2d^2$, $s = -2 + d,\ \rho = 3,\ \deg a = 1,\ \deg h_a = 0,\ \deg s = 1,\ \deg p = 2$ and $\beta = 2$.

Although W is not proper the conditions of Claim 2 are satisfied and the minimum $\deg m$ solution of the implied equation

$$(1-2d)\,n + d^2(1-0.5d)\,m = (-2+d)\,(1-0.1d-0.2d^2)$$

is

$$m = -7.2 \quad \text{and} \quad n = -2 - 2.8d + 1.9d^2$$

which is just the optimal one.

Writing a general solution

$$m = -7.2 + (1-2d)\,t \quad \text{and} \quad n = -2 - 2.8d + 1.9d^2 - d^2(1-0.5d)\,t$$

we can determine

$$q = 7.3 - 14.55d - 0.6d^2 + d^3 \quad \text{and} \quad r = -0.5 + 0.05d + 7.3d^2.$$

The equation (17)

$$d^2(1-2d)\,t + z = -0.5 + 0.05d + 7.3d^2$$

is solved for minimum $\deg z$ by $t = 0,\ z = r$, $\deg z = \deg r < \rho$.

Solving the problem by the usual couple (7) and (8)

$$d^2(1-2d)\,m + (1-2d)\,z = (-0.5+d)\,(1-0.1d-0.2d^2)$$

and

$$d^2(1-2d)\,n - d^2(1-0.5d)\,z = 0.75d^2(-2+d)\,(1-0.1d-0.2d^2)$$

yields the same results.

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