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DISCUSSION OF THE STRUCTURE OF UNINORMS

Paweł Drygaś

The paper deals with binary operations in the unit interval. We investigate connections between families of triangular norms, triangular conorms, uninorms and some decreasing functions. It is well known, that every uninorm is build by using some triangular norm and some triangular conorm. If we assume, that uninorm fulfils additional assumptions, then this triangular norm and this triangular conorm have to be ordinal sums. The intervals in ordinal sum are depending on the set of values of a decreasing function.

Keywords: uninorm, triangular norm, triangular conorm, binary operation, increasing operation, idempotent operation, associative operation

AMS Subject Classification: 06F05, 03E72, 03B52

1. INTRODUCTION

Binary operations in the unit interval have many applications in fuzzy set theory as multivalued logical connectives (cf. [7]). On the other hand, they are examples of aggregation operators in the unit interval (cf. [11]). For that reason it is important to examine and characterize such operations.

We discuss the structure of associative operations $U : [0,1]^2 \to [0,1]$.

Definition 1. ([11]) Operation U is called a uninorm if it is commutative, associative, increasing with respect to both variables and has the neutral element $e \in [0,1]$.

Uninorms are the generalization of triangular norms (case $e = 1$) and triangular conorms (case $e = 0$). In the case $e \in (0,1)$ we obtain operations considered in [2] – [5], [8] – [11].

2. PRELIMINARY NOTES

First we start with basic definitions and some properties of triangular norms.

Definition 2. ([7]) Operation T (S) is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to both variables and has the neutral element $e = 1$ ($e = 0$).

Definition 3. ([7]) Operation T (S) is called a triangular subnorm (triangular superconorm) if it is commutative, associative, increasing with respect to both variables and fulfils the condition $T \le \min$ ($S \ge \max$).

Example 1. (cf. [7]) There are the basic triangular norms:

$$T_M(x,y) = \min(x,y), \quad x,\, y \in [0,1], \qquad \text{(minimum)}$$

$$T_P(x,y) = x \cdot y, \quad x,\, y \in [0,1], \qquad \text{(product)}$$

$$T_L(x,y) = \max(x+y-1,0), \quad x,\, y \in [0,1]. \qquad \text{(Łukasiewicz triangular norm)}$$

Of course every triangular norms are triangular subnorms. Operations

$$T_1(x,y) = 0, \quad x,\, y \in [0,1],$$

$$T_2(x,y) = \tfrac{1}{2} x \cdot y, \quad x,\, y \in [0,1],$$

are triangular subnorms, but not triangular norms.

These operations we can use to construct new triangular norms.

Lemma 1. (cf. [6]) Let $\{[a_k,b_k]\}_{k\in\mathcal{T}}$ be a countable family of nonoverlapping, closed, proper subintervals of $[0,1]$. Let T be an operation in $[0,1]$ defined by

$$T(x,y) = \begin{cases} a_k + (b_k - a_k) T_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & \text{if } x,\, y \in (a_k, b_k], \\ \min(x,y) & \text{otherwise,} \end{cases} \tag{1}$$

where T_i are triangular subnorms. Moreover, assume that operation T_k have neutral element $e = 1$ if $b_k = a_l$ and T_l is with zero divisor, or $b_k = 1$. Then T is a triangular norm.

Operation given by (1) is called the ordinal sum of $\{([a_i,b_i],T_i)\}_{i\in\mathcal{T}}$ and each T_i is called a summand.

Example 2. Operation

$$T(x,y) = \begin{cases} 0, & \text{if } x,\, y \in (0, \frac{1}{2}], \\ 2xy - x - y + 1, & \text{if } x,\, y \in (\frac{1}{2}, 1], \\ \min(x,y), & \text{otherwise,} \end{cases}$$

is an ordinal sum of operations $T_1 = 0$ and $T_2 = T_P$.

In this example we can see, that operation T_2 has no zero divisors and interval $(\frac{1}{2}, 1]$ is closed with respect to operation T, but interval $[\frac{1}{2}, 1]$ is not closed with respect to operation T.

Operation T_1 has zero divisors and $[0, \frac{1}{2}]$ is closed with respect to operation T.

Example 3. Operation

$$T(x,y) = \begin{cases} 0, & \text{if } x,\, y \in (0, \frac{1}{2}], \\ \max(x+y-1, \frac{1}{2}), & \text{if } x,\, y \in (\frac{1}{2}, 1], \\ \min(x,y), & \text{otherwise}, \end{cases}$$

is build by using (1), where $T_1 = 0$ and $T_2 = T_L$, but it is non-associative, because T_1 has no neutral element and T_L has zero divisors. E.g. for $x = \frac{1}{2}$, $y = z = \frac{3}{4}$, $T(\frac{1}{2}, T(\frac{3}{4}, \frac{3}{4})) = T(\frac{1}{2}, \frac{1}{2}) = 0$, $T(T(\frac{1}{2}, \frac{3}{4}), \frac{3}{4}) = T(\frac{1}{2}, \frac{3}{4}) = \frac{1}{2}$. Therefore $T(T(x,y),z) \neq T(x, T(y,z))$.

3. STRUCTURE OF UNINORMS

In general a uninorm is composed by using a triangular norm and a triangular conorm.

Theorem 1. (cf. [5]) If a uninorm U has the neutral element $e \in (0,1)$, then there exist a triangular norm T and a triangular conorm S such that

$$U = \begin{cases} T^* \text{ in } [0,e]^2 \\ S^* \text{ in } [e,1]^2, \end{cases} \tag{2}$$

where

$$\begin{cases} T^*(x,y) = \varphi^{-1}\left(T\left(\varphi(x), \varphi(y)\right)\right), \ \varphi(x) = x/e, & x,\, y \in [0,e] \\ S^*(x,y) = \psi^{-1}\left(S\left(\psi(x), \psi(y)\right)\right), \ \psi(x) = (x-e)/(1-e), & x,\, y \in [e,1]. \end{cases} \tag{3}$$

Conversely, directly by formula (3) we get triangular norm T and triangular conorm S associated with given uninorm U:

$$T(x,y) = \varphi\left(U\left(\varphi^{-1}(x), \varphi^{-1}(y)\right)\right), \ S(x,y) = \psi\left(U\left(\psi^{-1}(x), \psi^{-1}(y)\right)\right), \ x,\, y \in [0,1]. \tag{4}$$

We ask about additional properties of operations given by (4). The answer essentially depends on the domain complementary to that used in (2):

$$A(e) = [0,e) \times (e,1] \cup (e,1] \times [0,e). \tag{5}$$

Theorem 2. (cf. [8]) Let $e \in (0,1)$. If T is an arbitrary triangular norm and S is an arbitrary triangular conorm, then formula (2) with $U = \min$ (or $U = \max$) in $A(e)$ gives a uninorm.

Example 4. (cf. [5]) Formula

$$U(x,y)=\begin{cases}0, & x=0 \ \text{ or } \ y=0,\\ \frac{xy}{(1-x)(1-y)+xy}, & x>0 \ \text{ and } \ y>0,\end{cases}$$

gives uninorm with $e=\frac{1}{2}$, $T(x,y)=\frac{xy}{2-(x+y-xy)}$, $S(x,y)=\frac{x+y}{1+xy}$, x, $y\in[0,1]$. T and S are arbitrary in Theorem 2, but here, T and S are dual (cf. [7], p. 223).

The most general observation on uninorms in the domain $A(e)$ given by (5) was presented in [5].

Lemma 2. (cf. [5]) If U is increasing and has the neutral element $e\in(0,1)$, then

$$\min\le U\le\max \quad \text{in} \quad A(e). \tag{6}$$

Furthermore $U(0,1)\in\{0,1\}$.

The frame structure of uninorms after Theorem 1 and Lemma 2 can be depicted in Figure 1.

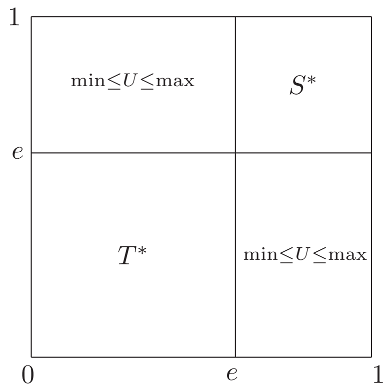

Fig. 1. Structure of uninorm.

Using additional assumption on U in $A(e)$ we get a representation of U by certain formula.

Lemma 3. (cf. [3]) If operation U is increasing with neutral element $e\in(0,1)$ and $U(x,y)\in\{x,y\}$ in $A(e)$, then the formula

$$g_U(x)=\begin{cases}\sup\{y:\ U(x,y)=x\}, & \text{if} \quad x\le e\\ \inf\{y:\ U(x,y)=x\}, & \text{if} \quad x>e,\end{cases} \tag{7}$$

gives a decreasing function $g_U : [0,1] \to [0,1]$ with fixed point e, such that

$$U(x,y) = \begin{cases} \min(x,y), & \text{if } \ y < g_U(x) \\ \max(x,y), & \text{if } \ y > g_U(x) \\ x \text{ or } y, & \text{if } \ y = g_U(x) \end{cases} \quad \text{in } \ A(e). \tag{8}$$

Example 5. Let $e \in (0,1)$. For the uninorm U given in Theorem 2 which is given by minimum in $A(e)$, the corresponding function is of the form

$$g_U(x) = \begin{cases} 1, & \text{if } \ x \in [0,e) \\ e, & \text{if } \ x \in [e,1]. \end{cases}$$

4. MAIN RESULTS

We ask about the structure of an operation which belong to the class

$$\mathcal{U}(e) = \{U : U \text{ is increasing, associative, binary operation in the unit interval, with the neutral element } e \in (0,1), \text{ and } U(x,y) \in \{x,y\} \text{ in } A(e)\}. \tag{9}$$

Problem 1. What can we say about properties of the function g_U given by formula (7), if $U \in \mathcal{U}(e)$?

The partial answer to the Problem 1 we describe in the next lemmas.

Lemma 4. (cf. [3, 10]) If $U \in \mathcal{U}(e)$, then function $g = g_U$ fulfils

$$\inf\{y : g(y) = g(x)\} \le g^2(x) \le \sup\{y : g(y) = g(x)\} \quad \text{for} \quad x \in [0,1] \tag{10}$$

and

$$U(x, g(x)) = \begin{cases} \min(x, g(x)), & \text{if } \ x < g^2(x) \\ \max(x, g(x)), & \text{if } \ x > g^2(x), \end{cases} \tag{11}$$

where $g^2(x) = g(g(x))$ for $x \in [0,1]$.

Moreover, U is commutative in $A(e)$, beyond the points $(x, g(x))$, such that $x = g^2(x)$.

Proof. Let $U \in \mathcal{U}(e)$. By Lemma 3 function $g = g_U$ is decreasing, with fixed point e and U is given by (8). First we show commutativity of the function U in $A(e)$, beyond the points belonging to the graph of the function g. Let $x < e < y$, $y \neq g(x)$. By monotonicity of g and U we have:

If $x < e < y < g(x)$, then by (8) $U(x,y) = \min(x,y) = x$ and $g(x) > e \geq g(y)$. Suppose, that $U(y,x) = y$ and let $c \in (y, g(x)) \subset [e,1]$. By (8) $U(x,c) = \min(x,c) = x$, and by associativity we have

$$y = U(y,x) = U(y,U(x,c)) = U(U(y,x),c) = U(y,c) \geq U(e,c) = c > y,$$

which is contradictory. So, $U(y,x) = x = U(x,y)$.

If $x < e < y$, $g(x) < y$, $g(y) < g(x)$, then by (8) we have $U(x,y) = y$. Suppose, that $U(y,x) = x$. By (8) $x \leq g(y)$. Let $c \in (g(x), y) \subset [e,1]$. Again by (8) and associativity we have

$$c = U(x,c) = U(U(y,x),c) = U(y,U(x,c)) = U(y,c) \geq U(y,e) = y > c,$$

which gives a contradiction. So, $U(y,x) = y = U(x,y)$.

If $x < e < y$ and $g(x) = g(y)$, then by the monotonicity of the function g we have $g(x) = g(y) = g(e) = e$. Take c, d, such that $x < c < e < d < y$ we have $g(c) = g(d) = e$, it means that $c < g(d)$ and $d > g(c)$, thus $U(d,c) = c$, $U(c,d) = d$. By associativity we obtain

$$d = U(c,d) = U(U(y,c),d) = U(y,U(c,d)) = U(y,d) \geq U(y,e) = y > d,$$

which leads to a contradiction, so this case is impossible.

The proof of the commutativity U on $A(e)$ in the case $x > e > y$ is similar. Therefore U is commutative on $A(e)$ beyond the points $(x, g(x))$.

Now we prove, that (10) holds. Let $x \in [0,1]$ and

$$a = \inf\{y : g(y) = g(x)\}, \quad b = \sup\{y : g(y) = g(x)\}.$$

Of course $a \leq x \leq b$ and $g(a) \geq g(x) \geq g(b)$. Suppose, that $g^2(x) < a$. Taking $c \in (g^2(x), a)$, we have $c < x$, thus $g(c) \geq g(x)$, and $g^2(x) < c$, so $(g(x), c) \in A(e)$ and does not belong to the graph of the function g. According to (8) we obtain, that $U(g(x), c) = \max(g(x), c)$. By the proved commutativity of the operation U we have $U(c, g(x)) = \max(c, g(x))$, and again by (8) $g(c) \leq g(x)$, therefore $g(c) = g(x)$. Which gives a contradiction with the assumption that a is the greatest lower bound of the set $\{y : g(x) = g(y)\}$. Suppose, that $g^2(x) > b$ and let $c \in (b, g^2(x))$. In a similar way we obtain a contradiction, which means, that (10) holds.

Now we show (11). Suppose, that $x < g^2(x)$. Obviously $x \neq e$ and $(g(x), x) \notin \{(y, g(y)) : \ y \in [0,1]\}$, but $(g(x), x) \in A(e)$, so, by the proved commutativity of U we have $U(x, g(x)) = U(g(x), x) = \min(x, g(x))$.

In a similar way, if $x > g^2(x)$, then we obtain $U(x, g(x)) = \max(x, g(x))$.

Now we show, that $U(x, g(x)) = U(g(x), x)$ for all $x \in [0,1]$, such that $x \neq g^2(x)$. By (11) we have:
if $x < g^2(x)$, then $U(g(x), x) = \min(g(x), x)$ and $U(x, g(x)) = \min(x, g(x))$,
if $x > g^2(x)$, then $U(g(x), x) = \max(g(x), x)$ and $U(x, g(x)) = \max(x, g(x))$.

So, U is commutative in $\overline{A(e)}$ beyond the points $(x, g(x))$, such that $x = g^2(x)$.□

Lemma 5. Let $e \in (0,1)$, $U \in \mathcal{U}(e)$ and $g = g_U$.

If g is strictly decreasing and continuous function on $(a,b) \subset [0,1]$ and $g((a,b)) = (c,d)$, then g is strictly decreasing and continuous function on (c,d), and $g^2(x) = x$ for $x \in (a,b) \cup (c,d)$.

If $s \in [0,1]$ is a point of discontinuity of the function g and

$$p := \begin{cases} \lim\limits_{x \to s^+} g(x), & \text{if} \quad s < 1 \\ 0, & \text{if} \quad s = 1, \end{cases} \qquad q := \begin{cases} \lim\limits_{x \to s^-} g(x), & \text{if} \quad s > 0 \\ 1, & \text{if} \quad s = 0, \end{cases} \tag{12}$$

then $g(x) = s$ for $x \in (p,q)$.

Let $s \in [0,1]$ and $B = \{x : g(x) = s\}$. If *card* $B \geq 2$ and $p = \inf B < \sup B = q$, then s is a point of discontinuity of the function g, $p \leq g(s) \leq q$ and (12) holds.

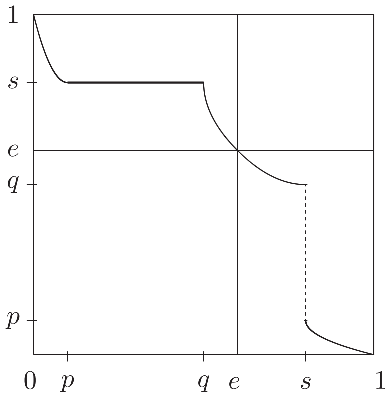

Fig. 2. The point of discontinuity of the function g.

P r o o f . We use results of Lemmas 3 and 4. Let $(a,b) \subset [0,1]$ be an interval, such that function g is strictly decreasing and continuous in (a,b). It means, that g is a bijection from (a,b) into $g((a,b)) = (c,d)$. By (10) $g^2(x) = x$ for $x \in (a,b)$, therefore $g((c,d)) = (a,b)$, and g is strictly decreasing and continuous in (c,d), moreover $g^2(x) = x$ in (c,d).

Let now s be a point of discontinuity of the function g, $\lim_{x \to s^-} g(x) = q$, $\lim_{x \to s^+} g(x) = p$ (cf. Figure 2).

For $s > e$ we have:

If $x \in (p,q)$, $e \leq y < s$, then $g(y) \geq q > x$ and by (8) $U(y,x) = \min(x,y) = U(x,y)$. It means, that $g(x) \geq y$ for all $y < s$. Thus $g(x) \geq s$ for $x \in (p,q)$. If $x \in (p,q)$, $y > s$, then $g(y) \leq p < x$ and by (8) $U(y,x) = \max(y,x) = U(x,y)$. Thus $g(x) \leq y$ for all $y > s$. Therefore $g(x) \leq s$ for $x \in (p,q)$. It means, that $g(x) = s$ in (p,q).

For $s < e$ we can use the same arguments.

For $s = e$ we have $(p,q) \subset [0,e]$ or $(p,q) \subset [e,1]$ and $g(x) \geq e$ for $x \in (p,q)$ or $g(x) \leq e$ for $x \in (p,q)$ respectively. Using arguments from two previous cases we have proved first part of the Lemma.

Let $s \in [0,1]$ be a point, such that $\operatorname{card} B \geq 2$ and $p = \inf B < \sup B = q$. The condition $p \leq g(s) \leq q$ we obtain by the monotonicity of the function g. Let $x \in (p,q)$, $y < s$. Then by (8) $U(x,y) = \min(x,y)$ and by commutativity of U we obtain $U(y,x) = \min(y,x)$. Again by (8) we obtain that $g(y) \geq x$ for $x \in (p,q)$, which means, that $g(y) \geq q$. Since $y < s$, thus $\lim_{x\to s^-} g(x) \geq q$. Supposing $r = \lim_{x\to s^-} g(x) > q$, then by previous part of lemma we have $g(x) = s$ for $x \in (p,r) \supsetneq (p,q)$, which gives a contradiction.

In the same way we show, that $\lim_{x\to s^+} g(x) = p$. Since $p < q$, then s is a point of discontinuity of the function g. □

Problem 2. What can we say about the influence of the properties of function g_U on the structure of operation U (and operations T, S given by formula (4))?

Example 6. (cf. [3]) Let $g(x) = 1 - x$, $x \in [0,1]$. Operation $U : [0,1]^2 \to [0,1]$ given by

$$U(x,y) = \begin{cases} 2xy, & \text{if } x,\, y \in [0, \frac{1}{2}], \\ \max(x,y), & \text{if } y > g(x), \\ \min(x,y), & \text{otherwise}, \end{cases}$$

is not associative. E.g. for $x = y = \frac{1}{4}$, $z = \frac{13}{16}$. $U\left(U\left(\frac{1}{4}, \frac{1}{4}\right), \frac{13}{16}\right) = U\left(\frac{1}{8}, \frac{13}{16}\right) = \frac{1}{8}$, $U\left(\frac{1}{4}, U\left(\frac{1}{4}, \frac{13}{16}\right)\right) = U\left(\frac{1}{4}, \frac{13}{16}\right) = \frac{13}{16}$. Therefore $U(U(x,y),z) \neq U(x,U(y,z))$. This shows, that triangular norm and conorm in (2) cannot be arbitrary, if we want to construct a uninorm.

The partial answer to the Problem 2 gives the next theorem.

Theorem 3. (cf. [3]) Let $g : [0,1] \to [0,1]$ be a decreasing involution ($g^2 = id$). Formula

$$U(x,y) = \begin{cases} T^*(x,y), & x,\, y \in [0,e] \\ S^*(x,y), & x,\, y \in [e,1] \\ \min(x,y), & x < e < y \leq g(x) \text{ or } y \leq g(x) < e < x \\ \max(x,y), & x < e < g(x) < y \text{ or } g(x) < y < e < x, \end{cases} \tag{13}$$

gives a uninorm, iff $T = \min$ and $S = \max$.

Lemma 6. Let $U \in \mathcal{U}(e)$, If $g = g_U$ is strictly decreasing and continuous function on $(a,b) \subset [0,e]$ and $g((a,b)) = (c,d)$, then (cf. Figure 3)

$$U = \min \quad \text{on} \quad [0,b] \times (a,e] \cup (a,e] \times [0,b], \tag{14}$$

$$U = \max \quad \text{on} \quad [e,d) \times [c,1] \cup [c,1] \times [e,d). \tag{15}$$

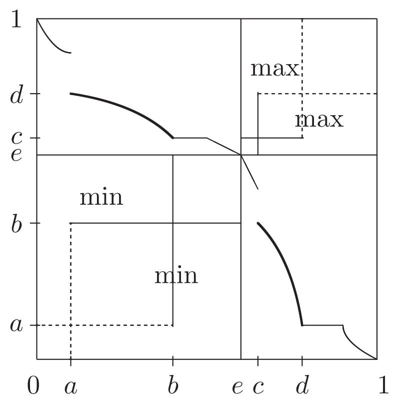

Fig. 3. Subinterval (a, b) with strictly decreasing g.

P r o o f. First we show, that all elements in $(a, b]$ are idempotent. Let $a < x \le b$, then $U(x, x) \le U(x, e) = x$. Suppose, that $U(x, x) < x$. Since $g(x) \ge e$ for $x \in [0, e]$ and $g(x) \le e$ for $x \in [e, 1]$, then by strict monotonicity of g we have $c \le g(x) < g(U(x, x))$. Let $z \in [e, 1]$, such that $g(x) < z < g(U(x, x))$. By associativity and (8) we obtain $U(x, x) = U(U(x, x), z) = U(x, U(x, z)) = U(x, z) = z > e$. This is contradictory, therefore we have $U(x, x) = x$.

To prove (14) we divide set $[0, b] \times (a, e] \cup (a, e] \times [0, b]$ into six parts.

If $a < x \le y \le e$, $x \le b$, then $x = U(x, x) \le U(x, y) \le U(x, e) = x$, thus $U(x, y) = x = \min(x, y)$.

If $a < y \le x \le e$, $y \le b$, then $y = U(y, y) \le U(x, y) \le U(e, y) = y$, thus $U(x, y) = y = \min(x, y)$.

If $x \le a < y < b$, then $U(x, y) \le \min(x, y)$ and let $z \in [e, 1]$, such that $g(x) > z > g(y)$. We have

$$U(U(x, y), z) = \min(U(x, y), z) = U(x, y),$$

$$U(x, U(y, z)) = U(x, \max(y, z)) = U(x, z) = \min(x, z) = x.$$

By associativity of U we have $U(x, y) = x = \min(x, y)$.

If $x \le a < b \le y \le e$, then for $z \in (a, b)$ we have $x = U(x, z) \le U(x, y) \le U(x, e) = x$. Therefore $U(x, y) = \min(x, y)$.

If $y \le a < x < b$, then $U(x, y) \le \min(x, y)$. If we fix $z \in (c, d)$, such that $x > g(z) > y$, then we have

$$U(z, U(x, y)) = \min(z, U(x, y)) = U(x, y),$$

$$U(U(z, x), y) = U(\max(z, x), y) = U(z, y) = \min(z, y) = y.$$

By associativity of U we obtain $U(x, y) = y = \min(x, y)$.

If $y \le a < b \le x \le e$, then for $z \in (a, b)$ we have $y = U(z, y) \le U(x, y) \le U(e, y) = y$. It means, that $U(x, y) = \min(x, y)$.

This proves (14). The proof of the formula (15) is similar. □

Lemma 7. Let $U \in \mathcal{U}(e)$ and s be a point of discontinuity of the function $g = g_U$. If $s \in [0, e]$, then

$$U = \min \quad \text{on} \quad [0, s] \times (s, e] \cup (s, e] \times [0, s]. \tag{16}$$

If $s \in [e, 1]$, then

$$U = \max \quad \text{on} \quad [e, s) \times [s, 1] \cup [s, 1] \times [e, s). \tag{17}$$

Proof. Let s be a point of discontinuity of the function g and p, q be given by (12).

Now we prove (16). If $s = 0$, then the domain in condition (16) reduces to the set, in which one of variables is equal zero, and second is less than e. Since e is neutral element, then $U(x, 0) \leq U(e, 0) = 0$ and $U(0, x) \leq U(0, e) = 0$. So $U(0, x) = U(x, 0) = 0 = \min(x, 0)$. Similar argument we can use for $s = e$.

Let $s \in (0, e)$ (see Figure 4).

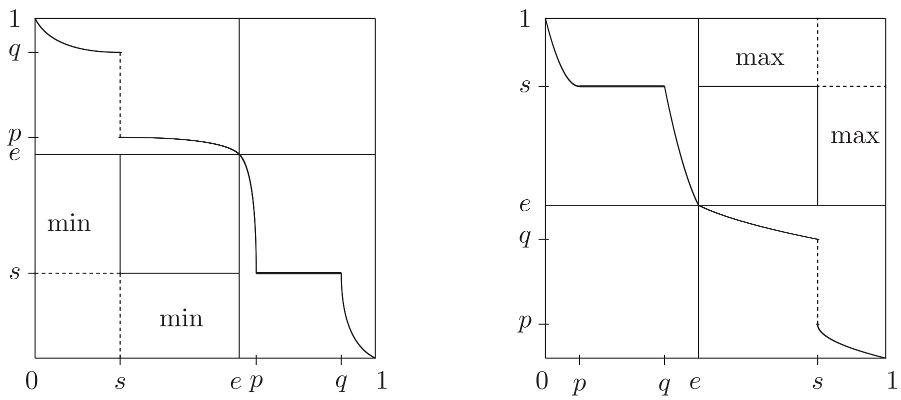

Fig. 4. Dependence of the points of operation U under discontinuity points of the function g_U.

If $0 \leq x < s < y \leq e$ and $z \in (p, q)$, then $g(x) > z > g(y)$, $U(x, y) \leq \min(x, y) < s$ and $g(U(x, y)) \geq q > z$. By (8) we have

$$U(U(x, y), z) = \min(U(x, y), z) = U(x, y),$$

$$U(x, U(y, z)) = U(x, \max(y, z)) = U(x, z) = \min(x, z) = x.$$

Now, by associativity of U we obtain $U(x, y) = x = \min(x, y)$.

If $0 \leq y < s < x \leq e$ and $z \in (p, q)$, then

$$U(z, U(x, y)) = \min(z, U(x, y)) = U(x, y),$$

$$U(U(z,x),y) = U(\max(z,x),y) = U(z,y) = \min(z,y) = y.$$

Therefore $U(x,y) = \min(x,y)$.

If $x = s$ and $y \in (s,e]$, then by above part of proof we have $U(x,y) \le \min(x,y) = x$ and for all $z < x$ we have $U(z,y) = \min(z,y) = z$. Since U is increasing, then $U(x,y) \ge \lim_{z\to x^-} U(z,y) = x$. So, $U(x,y) = x = \min(x,y)$. Case $y = s$ and $x \in (s,e]$ follows similarly. It shows (16).

The proof of the formula (17) is similar. □

Lemma 8. Let $U \in \mathcal{U}(e)$ and s be a point of discontinuity of the function $g = g_U$, p, q be given by (12) and $g(s) = r$, $p < r < q$.

If $s \in [0,e)$, then $r > e$ and (cf. Figure 5, left)

$$U = \max \quad \text{on} \quad (p,r)\times[r,q)\cup[r,q)\times(p,r). \tag{18}$$

If $s \in (e,1]$, then $r < e$ and (cf. Figure 5, right)

$$U = \min \quad \text{on} \quad (p,r]\times(r,q)\cup(r,q)\times(p,r]. \tag{19}$$

Proof. Let $s \in (e,1]$, $0 \le p < x < r < y < q \le e$. By (8) and commutativity of U in $A(e)$ beyond the graph of the function g we have $U(U(x,y),s) = U(x,y)$, $U(x,U(y,s)) = U(x,s) = x$, $U(U(s,y),x) = U(s,x) = x$, $U(s,U(y,x)) = U(y,x)$. By associativity of U we obtain $U(x,y) = x = \min(x,y)$.

If $x = r$, $y \in (r,q)$, then $U(x,y) \le \min(x,y) = x$ and $U(z,y) = \min(z,y)$ for all $z \in (p,r)$. Since U is increasing, then $U(x,y) = x = \min(x,y)$. It shows (19).

In a similar way we can prove (18). □

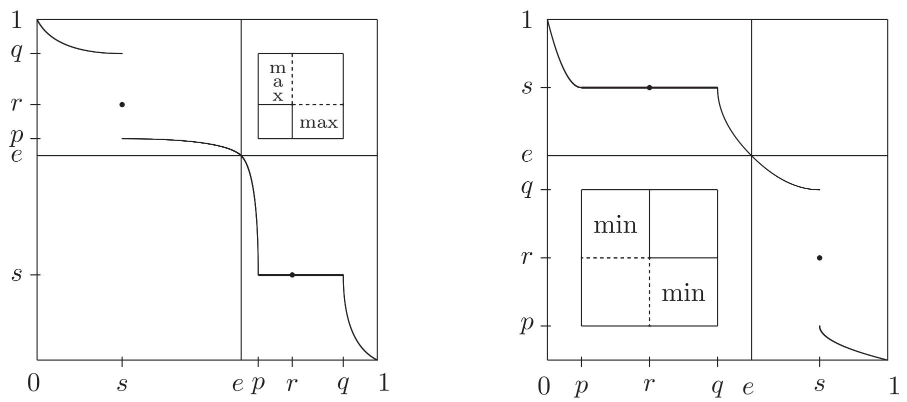

Fig. 5. The values of the function g in the points of discontinuity of the function g.

By this observation we obtain the fact that operation U on $[0,e]^2$ is an ordinal sum of $\{([a_i,b_i],T_i)\}_{i\in\mathcal{T}}$, (see Lemma 1), such that function g is constant on $(a_i,b_i)\subset[0,e]$ for all $i\in\mathcal{T}$. The operation U on $[e,1]^2$ is an ordinal sum of $\{([c_j,d_j],S_j)\}_{j\in\mathcal{S}}$, such that function g is constant on $(c_j,d_j)\subset[e,1]$ for all $j\in\mathcal{S}$.

The next theorem gives the characterization uninorm belonging to the set $\mathcal{U}(e)$.

Theorem 4. Let $e\in(0,1)$. If uninorm $U\in\mathcal{U}(e)$, then

- there exists a decreasing function $g:[0,1]\to[0,1]$ with fixed point e, such that (10) holds and U is given by formula

$$U(x,y)=\begin{cases}\min(x,y) & \text{if} \quad y<g(x) \quad \text{or} \quad y=g(x) \text{ and } x<g^2(x)\\ \max(x,y) & \text{if} \quad y>g(x) \quad \text{or} \quad y=g(x) \text{ and } x>g^2(x) \quad \text{in } A(e),\\ x \text{ or } y & \text{if} \quad y=g(x) \quad \text{and} \quad x=g^2(x)\end{cases} \tag{20}$$

- $U|_{[0,e]^2}$ is an ordinal sum of $\{([a_i,b_i],T_i)\}_{i\in\mathcal{T}}$, such that $(a_i,b_i)\subset[0,e]\setminus\{g(x): x\in[e,1]\}$ for all $i\in\mathcal{T}$,
- $U|_{[e,1]^2}$ is an ordinal sum of $\{([c_j,d_j],S_j)\}_{j\in\mathcal{S}}$, such that $(c_j,d_j)\subset[e,1]\setminus\{g(x): x\in[0,e]\}$ for all $j\in\mathcal{S}$.

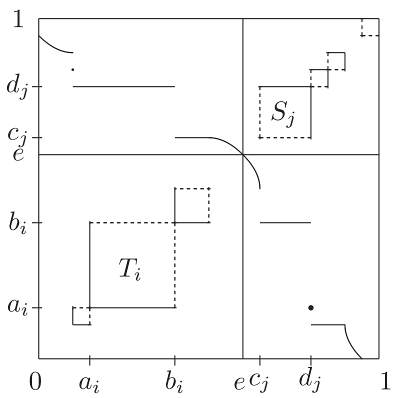

Fig. 6. Structure of uninorm belonging to the class $\mathcal{U}(e)$.

P r o o f. Let $e\in(0,1)$ and $U\in\mathcal{U}(e)$ be a uninorm. Directly by Lemma 3 the function $g=g_U$ is decreasing, with fixed point e. By Lemma 4 we obtain, that g fulfils condition (10). Mixing the formulas (8) and (11) we obtain, that U is given by (20) in $A(e)$. By Lemmas 6, 7, 8 operation U may differ from min only on the intervals $(a_i,b_i]$, such that $(a_i,b_i)\subset[0,e]\setminus\{g(x): x\in[e,1]\}$ (see Figure 6). Moreover, by Lemma 5 we have $g(x)=\mathit{const}$ for $x\in(a_i,b_i)$.

First we prove, that interval $(a_i, b_i]$ or $[a_i, b_i]$ is closed with respect to operation U. By Lemmas $6, 7, 8$ we have $U(a_i, x) = U(x, a_i) = a_i$ for $x \in (a_i, b_i]$.

If $U(x, y) > a_i$ for $x, y \in (a_i, b_i]$, then $(a_i, b_i]$ is stable under operation U.

If exist $x, y \in (a_i, b_i]$, such that $U(x, y) = a_i$, then by associativity of U we have $U(a_i, a_i) = U(U(x, y), a_i) = U(x, U(y, a_i)) = U(x, a_i) = a_i$. It means, that $[a_i, b_i]$ is a subset stable under operation U.

So, $U|_{[0,e]^2}$ is an ordinal sum of operations T_i which are isomorphic to the $U|_{(a_i,b_i]}$, where

$$T_i(x, y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0 \\ \frac{1}{b_i - a_i}\left(U((b_i - a_i)x + a_i, (b_i - a_i)y + a_i) - a_i\right), & \text{otherwise,} \end{cases}$$

In a similar way we can obtain, that $U|_{[e,1]^2}$ is an ordinal sum of operations S_i.□

5. CONCLUSION

In this paper we characterize uninorms with neutral element $e \in (0, 1)$, such that $U(x, y) \in \{x, y\}$ in $A(e)$.

We obtain, that triangular norm and conorm from Theorem 1 have the form of ordinal sum. The open problem is to characterize all uninorms, without assumption, that $U(x, y) \in \{x, y\}$ in $A(e)$. If triangular norm and triangular conorm given by (4) are continuous, then characterization we can find in [4]. The next problem is to characterize all uninorms such that all element of the set of values of the function g_U are idempotent.

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