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SCHWARZ–LIKE METHODS FOR APPROXIMATE SOLVING COOPERATIVE SYSTEMS*

Ivo Marek

This paper is dedicated to Prof. Dr. Dr.h.c. František Nožička on the occasion of his 85th birthday.

The aim of this contribution is to propose and analyze some computational means to approximate solving mathematical problems appearing in some recent studies devoted to biological and chemical networks.

Keywords: cooperative systems, steady states of evolution problems, Schwarz iterative solution

AMS Subject Classification: 65F10, 47B60

Basic mathematical tools for investigating some biological and chemical networks as presented in [11, 12] are recalled in Section 1. In Section 2 some variants of iterative Schwarz-like methods studied in [1, 16] are described and applied to problems discussed in Section 1. An analysis and comparison of two particular Schwarz-like methods is presented in Section 4.

1. COOPERATIVE SYSTEMS

In [11] a theory for linear problems of the type

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t) = Tw(t), \quad w(0) \text{ given}, \tag{1}$$

where T is a given infinitesimal generator of a semigroup of operators is developed. Several examples mainly from biology and chemistry where the state vectors $w(t)$ of the underlying chemical network follow an evolution (1) are presented there. In these cases the conservation of matter requires the existence of an element f such that the duality pairing $[w(t), f]$ is constant during all times of the evolution so that we have

$$[w(t), f] = [w(0), f], \quad t \geq 0. \tag{2}$$

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It should be noted that in case the space where the evolution is investigated is a Hilbert space the pairing just mentioned becomes a corresponding inner product.

More complicated networks (and some examples are shown [12]) are described by a state vector $u(t)$ which is formed by finitely many substates in the fashion

$$u(t) = \left(u^1(t), \dots, u^N(t)\right) \tag{3}$$

following an evolution

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u^j(t) = T^{(j)}(u(t))u^j(t) := B^{(j)}u^j(t) + G^{(j)}(u(t))u^j(t) \\ \left[u^j(t), f^j\right] = \left[u^j(0), f^j\right], \quad t > 0,\ j = 1, \dots, N. \end{cases} \tag{4}$$

Hence, the subsystems evolve like (1) and (2) for which we developed the theory in [11]. The dependence of the operators $T^{(j)}$ on the data is typically on the complete state (3) rather than only on some substates.

If $u^j \in X^j$, $j = 1, \dots, N$, for the states of the subsystems, the product $X = X^1 \times X^2 \times \cdots \times X^N$ can be formed and the "block"-diagonal operator

$$B = \operatorname{diag}\left\{B^{(1)}, \dots, B^{(N)}\right\}, \quad G(u) = \operatorname{diag}\left\{G^{(1)}(u), \dots, G^{(N)}(u)\right\} \tag{5}$$

can be defined on X. Now (3) evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Bu(t) + G(u(t))u(t), \quad [u(t), f] = [u(0), f], \quad t \geq 0, \tag{6}$$

where $f = (f^1, \dots, f^N) \in X$ and B is the infinitesimal generator of a semigroup of operators of class $\mathcal{C}_0$ [13, p. 321]. Note that structurally (6) is very similar to (1), (2). Only the infinitesimal generator $G = G(u)$ itself depends on the total state (3) so that our problem becomes nonlinear.

In [12] the evolution problem was assumed in the form (6) under very general conditions and an existence theorem (concerning mild solutions which in our applications become classical solutions) was proven for all times $t \geq 0$ and the question of its long run behaviour settled: In fact, any solution of (6) settles in the long run at a steady state. Defining equations to determine this steady state are given there. This theorem is important not only in its own right. It also provides the basis for singular perturbation techniques on such systems to obtain analytic expressions which characterize the speed of reaction systems, and we refer to [7, 22] for the pseudo-steady state process leading to the definition of the speed of the underlying chemical networks. Note that the theory does not need the diagonal form of (5). However, it relies heavily on (2). Relation (6) is only an easy way to fit (3), (4) under the general pattern (10). The original form of the examples is (4).

As in the earlier papers [11, 12] we need the theoretical basis for infinitesimal generators with monotonicity properties. The corresponding basic definitions are summarized in this section.

Let $\mathcal{E}$ be a Banach space over the field of real numbers. Let $\mathcal{E}'$ denote the dual space of $\mathcal{E}$. Let $\mathcal{F}$, $\mathcal{F}'$ be the corresponding complex extensions of $\mathcal{E}$, $\mathcal{E}'$ respectively and let $\mathbf{B}(\mathcal{E})$ and $\mathbf{B}(\mathcal{F})$ be the spaces of all bounded linear operators mapping $\mathcal{E}$

into $\mathcal{E}$ and $\mathcal{F}$ into $\mathcal{F}$ respectively. In fact, we are going to provide our investigations in a Hilbert space $\mathcal{E}$ equipped with an inner product $[\cdot,\cdot]$.

Let $\mathcal{K} \subset \mathcal{E}$ be a closed normal and generating cone, i. e. let

(i) $\mathcal{K} + \mathcal{K} \subset \mathcal{K}$,

(ii) $a\mathcal{K} \subset \mathcal{K}$ for $a \in \mathbb{R}_+$,

(iii) $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$,

(iv) $\overline{\mathcal{K}} = \mathcal{K}$,
where $\overline{\mathcal{K}}$ denotes the norm-closure of $\mathcal{K}$,

(v) $\mathcal{E} = \mathcal{K} - \mathcal{K}$,
and

(vi) there exists a $\delta > 0$ such that $||x + y|| \geq \delta||x||$, whenever $x,\ y \in \mathcal{K}$.

Property (vi) is called *normality of* $\mathcal{K}$.

We let

$$x \preceq y \text{ or equivalently } y \succeq x \Longleftrightarrow (y - x) \in \mathcal{K}$$

(vii) For every pair $x, y \in \mathcal{K}$ there exist $x \wedge y = \inf\{x, y\}$ and $x \vee y = \sup\{x, y\}$ as elements of $\mathcal{K}$.

A cone $\mathcal{K}$ satisfying condition (vii) is called a *lattice cone* and the partial order on $\mathcal{E}$ a *lattice order*. In the terminology of H. H. Schaefer [19] $\mathcal{E}$ is called a *Banach lattice*. Our theory is free of hypothesis (vii).

Let

$$\mathcal{K}' = \{x' \in \mathcal{E}' :\ x'(x) \geq 0 \text{ for all } x \in \mathcal{K}\}$$

and

$$\mathcal{K}^d = \{x \in \mathcal{K} : x'(x) > 0 \text{ for all } 0 \neq x' \in \mathcal{K}'\}\,.$$

We call $\mathcal{K}'$ the *dual cone* of $\mathcal{K}$ and $\mathcal{K}^d$ the *dual interior* of $\mathcal{K}$, respectively. If $\mathcal{E}$ happens to be a Hilbert space then $\mathcal{K}'$ is replaced by $\mathcal{K}^*$ a representation of $\mathcal{K}'$ in the sense of natural isomorphism of the dual $\mathcal{E}'$ with $\mathcal{E}$.

In the following analysis we assume that the dual interior $\mathcal{K}^d$ is nonempty.

A set $\mathcal{H}' \subset \mathcal{K}'$ is called $\mathcal{K}$-total if the following implication holds

$$x'(x) \geq 0 \quad \forall\, x' \in \mathcal{H}' \Longrightarrow x \in \mathcal{K}.$$

A linear form $\hat{x}' \in \mathcal{K}'$ is called *strictly positive*, if $\hat{x}'(x) > 0$ for all $x \in \mathcal{K}, x \neq 0$.

We write $[x, x']$ in place of $x'(x)$, where $x \in \mathcal{E}$ and $x' \in \mathcal{E}'$ respectively. If $\mathcal{E}$ happens to be a Hilbert space then $[x, x']$ denotes the appropriate inner product.

A bounded linear operator $T \in \mathbf{B}(\mathcal{E})$ is called $\mathcal{K}$-nonnegative if $T\mathcal{K} \subset \mathcal{K}$. We write in this case $T \succeq 0$ and equivalently $0 \preceq T$. If T and S both in $\mathbf{B}(\mathcal{E})$ satisfy $(S - T)\mathcal{K} \subset \mathcal{K}$ we write $T \preceq S$ or equivalently $S \succeq T$.

If $T \in \mathbf{B}(\mathcal{E})$ then T' denotes its dual and hence, $T' \in \mathbf{B}(\mathcal{E}')$. In case $\mathcal{E}$ is a Hilbert space, the dual operator T' is to be replaced by the adjoint operator T^* defined via relations $[Tx, y] = [x, T^*y]$ valid for all x in the domain of T and y in the domain of T^*.

Definition 1. Operator $T \in \mathbf{B}(\mathcal{E})$ is called *$\mathcal{K}$-stochastic* if there exists a vector $\hat{x}' \in \mathcal{K}'$ such that for the dual map T' the following relation

$$T'\hat{x}' = \hat{x}',$$

holds. We also say that T is a *transition operator of a Markov chain or process* and that *operator T corresponds to vector $\hat{x}'$*. If $\mathcal{E}$ is a Hilbert space the dual T' is to be replaced by its adjoint T^*.

Definition 2. A bounded linear operator T is $\mathcal{K}$-irreducible if for every pair of elements $0 \neq x \in \mathcal{E}, 0 \neq x' \in \mathcal{E}'$, there is a positive integer $p = p(x, x')$ such that $x'(B^p x) \neq 0$. This implies that in the Markov chain each state has access to every other state, i. e., the chain is ergodic [20]. The Perron–Frobenius theorem states that for $T \geq O$ irreducible, $r(T)$ is an isolated eigenvalue, and the corresponding eigenvector is positive; see, e. g., [3].

Let $T \in \mathbf{B}(\mathcal{F})$ and let $\sigma(T)$ denote its spectrum. Further, let $T \in \mathbf{B}(\mathcal{E})$. We introduce the operator $\tilde{T}$ by setting $\tilde{T}z = Tx + iTy$, where $z = x + iy$, $x, y \in \mathcal{E}$ and call it *complex extension* of T. By definition, we let $\sigma(T) := \sigma(\tilde{T})$. Similarly, we let $r(T) := r(\tilde{T})$, where $r(\tilde{T}) = \max\{|\mu| : \mu \in \sigma(\tilde{T})\}$ denotes the *spectral radius* of $\tilde{T}$.

In order to simplify notation we will identify T and its complex extension and will thus omit the tilde sign denoting the complex extension.

The set

$$\sigma_\pi(T) = \{\mu \in \sigma(T) : \ |\mu| = r(T)\}$$

is called *peripheral spectrum of T*. Note that $\sigma_\pi(T)$ is never empty.

If μ is an isolated singularity of $R(\lambda, T) = (\lambda I - T)^{-1}$ we have the following Laurent expansion of $R(\lambda, T)$ around μ [17, 21]

$$R(\lambda, T) = \sum_{k=0}^{\infty} A_k(\mu)(\lambda - \mu)^k + \sum_{k=1}^{\infty} B_k(\mu)(\lambda - \mu)^{-k}, \tag{7}$$

where A_{k-1} and B_k, $k = 1, 2, \ldots$, belong to $\mathbf{B}(\mathcal{F})$. Moreover, it holds [21]

$$B_1(\mu) = \frac{1}{2\pi i} \int_{\mathcal{C}_0} (\lambda I - T)^{-1} \mathrm{d}\lambda, \tag{8}$$

where

$$\mathcal{C}_0 = \{\lambda : |\lambda - \mu| = \rho_0\}$$

and ρ_0 is such that $\{\lambda : |\lambda - \mu| \leq \rho_0\} \cap \sigma(T) = \{\mu\}$.

Furthermore,

$$B_{k+1}(\mu) = (T - \mu I)B_k(\mu), \quad k = 1, 2, \ldots \tag{9}$$

If there is a positive integer $q = q(\mu)$ such that

$$B_q \neq 0, \text{ and } B_k = 0, \text{ for } k > q,$$

then μ is called *a pole of the resolvent operator* and q its *multiplicity.*

We define the symbol

$$\operatorname{ind}(\mu I - T) = q(\mu)$$

and call it the *index of* T *at* μ. In particular, we call $\operatorname{ind}(T)$ the *index of* T instead of index of T at 0.

The motivating examples are particular cases of Problem (P) defined as follows:

Problem (P) To find $\mathcal{K}$-positive solutions

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Bu(t) + G(u(t))u(t) = T(u(t))u(t),\ u(0) = u_0, \tag{10}$$

where B is generally an unbounded linear densely defined operator and $G(u)$ for every $u \in \mathcal{E}$ is a bounded linear map on $\mathcal{E}$, where $\mathcal{E}$ denotes the underlying space to be specified in each particular situation. We assume that we can identify situations in which Problem (P) as formulated above possesses solutions and we are aware of conditions guaranteeing the existence of them as well as some of their properties such as uniqueness, asymptotic behaviour etc. A rather typical representative of such problem is described in our study [12]. Since our aim in the present contribution is to propose some algorithms of computational nature and analyse their properties we do not go into much details referring the reader to [12] to consulting general aspects. All properties needed for a good understanding of the numerical processes studied will be presented here.

2. CONE PRESERVING ITERATIVE METHODS

In this section we present some notation, definitions, and preliminaries. Analogous concepts on nonnegative matrices (defined here for generally infinite dimensional spaces) can be found in the standard reference [3].

By $\sigma(C)$ we denote the spectrum of C and by $r(C)$ its spectral radius. By $\mathcal{R}(C)$ and $\mathcal{N}(C)$ we denote the range and null space of C, respectively.

Let $\lambda \in \sigma(C)$ be a pole of the resolvent operator $R(\mu, C) = (\mu I - C)^{-1}$. The multiplicity of λ as a pole of $R(\mu, C)$ is called the index of C with respect to λ and denoted $\operatorname{ind}_\lambda C$. Equivalently, $q = \operatorname{ind}_\lambda C$ if it is the smallest integer for which $\mathcal{R}((\lambda I - C)^{q+1}) = \mathcal{R}((\lambda I - C)^q)$. This happens if and only if $\mathcal{R}((\lambda I - C)^q) \oplus \mathcal{N}((\lambda I - C)^q) = \mathcal{E}$.

A very important concept in matrix theory is the notion of an M-matrix. For our purposes we need a slightly more general concept of $\mathcal{K} - M$-operator and its unbounded version of an approximate $\mathcal{K} - M$-operator.

Definition 3. A bounded linear operator $A \in \mathbf{B}(\mathcal{E})$ is called $\mathcal{K} - M$*-operator* if $A = bI - B$, where $B \in \mathbf{B}(\mathcal{E})$ and $b \geq r(B)$. Note that in case $\mathcal{E} = \mathcal{R}^N$ and $\mathcal{K} = \mathcal{R}^N_+$ *operator* A is called M-operator (matrix).

Let A be a densely defined bounded from above linear operator with its domain of definition $\Delta(A)$ and let $a \in \mathcal{R}^1$ be its (lower) bound. Assume that there exists a

system of operators $\{A_h : 0 < h \leq h_0\}$, $A_h \in \mathbf{B}(\mathcal{E})$, such that relation

$$\lim_{h\to 0} \|A_h x - Ax\| = 0 \tag{11}$$

holds for every $x \in \Delta(A)$. Operator A is called *approximate* $\mathcal{K} - M$*-operator* if each of the operators A_h in the collection mentioned has the form $A_h = bI - B_h$ with $b \geq r(B_h)$ and each B_h being $\mathcal{K}$-nonnegative. A pair of operators (M, W) is called a splitting of A if $A = M - W$ and M^{-1} exists as a bounded linear operator on $\mathcal{E}$. A splitting of an operator A is called *of* $\mathcal{K}$*-nonnegative type* if the operator $T = M^{-1}W$ is $\mathcal{K}$-nonnegative [15]. If, in particular, both operators M^{-1} and W are $\mathcal{K}$-nonnegative, the splitting is called *regular* [23]. If M^{-1} and $T = M^{-1}W$ are nonnegative, the splitting is called *weak regular* [18].

Note a weak regular splitting does require explicitly no conditions upon part W of the splitting of A.

Let T be a bounded linear operator. T is called *convergent* if $\lim_{k\to\infty} T^k$ exists and *zero-convergent*, if moreover $\lim_{k\to\infty} T^k = O$. Standard stationary iterations of the form

$$x^{k+1} = Tx^k + c, \quad k = 0, 1, \ldots, \tag{12}$$

converge if and only if either T is zero-convergent or, if $r(T) = 1$, T is convergent. A bounded linear operator T with unit spectral radius being an isolated pole of the resolvent operator is convergent if the following two conditions hold:

(i) if $\lambda \in \sigma(T)$ and $\lambda \neq 1$, then $|\lambda| < 1$.

(ii) $\mathrm{ind}_1 T = 1$.

Equivalent conditions for (ii) can be found in [3].

It is useful to write $T = Q + S$, where Q is the first term of the Laurent expansion of T, i. e., the eigenprojection onto the invariant subspace corresponding to $\lambda = 1$; see, e. g., [21]. Then $Q^2 = Q$, $QS = SQ = O$, and $1 \notin \sigma(S)$. This is called the *spectral decomposition* of T. The condition (i) above is equivalent to having $r(S) < 1$.

We state a very useful lemma; its proof can be found, e. g., in [9]. We note that when $r(T) = 1$, this lemma can be used to show condition (ii) above. To prove convergence one needs to show in addition that condition (i) also holds.

Lemma 1. Let T be a $\mathcal{K}$-nonnegative bounded linear operator such that $Tv \leq \alpha v$ with $v > 0$. Then $r(T) \leq \alpha$. If furthermore $r(T) = \alpha$ is a pole of the resolvent operator $(\lambda I - T)^{-1}$, then $\mathrm{ind}_\alpha T = 1$.

3. ALGEBRAIC FORMULATION OF SCHWARZ METHODS

In this section we want to generalize convergence of some procedures well known as algebraic Schwarz iteration techniques [1, 16]; our aim is however to analyze solving equations in infinite dimensional spaces as well. Given an initial approximation x^0 to the solution of

$$Ax = b, \quad b \in \mathcal{E}, \tag{13}$$

the (one-level) multiplicative Schwarz method can be written as the stationary iteration

$$x^{k+1} = Tx^k + c, \tag{14}$$

where

$$T = T_\mu = (I - P_p)(I - P_{p-1}) \cdots (I - P_1) = \prod_{i=p}^{1} (I - P_i) \tag{15}$$

and c is a certain vector in $\mathcal{E}$. Here

$$P_i = R_i^T (R_i A R_i^T)^{-1} R_i A, \tag{16}$$

where R_j is a suitable linear operator and R_j^* its adjoint with respect to the inner product in the Hilbert space $\mathcal{E}$. Note that each P_i, and hence each $I - P_i$, is a projection operator; i. e., $(I - P_i)^2 = I - P_i$. Each $I - P_i$ naturally has spectral radius equal to 1.

The additive Schwarz method for the solution of (13) is of the form (14), where

$$T = T_\theta = I - \theta \sum_{i=1}^{p} P_i = I - \theta \sum_{i=1}^{p} R_i^T A_i^{-1} R_i A, \tag{17}$$

where $0 < \theta \leq 1$ is a damping parameter.

The operator R_i corresponds to the restriction operator from the whole space to a subset of the state space (usually of finite dimension $n_j, j = 1, \ldots, p$; the dimension of the range $R(R_0)$ is infinite in general) in the domain decomposition setting, and the operator $A_i = R_i A R_i^T$ is the restriction of A to that subset. A solution using A_i is called a local solver as in the domain decomposition method as well as in the algebraic case.

We assume that our standard choice of operators R_j follows the same idea as does the choice of the rows of R_i as rows of the $n \times n$ identity matrix I in case of $\mathcal{E} = \mathcal{R}^n$, e. g.,

$$R_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let us assume that the Hilbert space $\mathcal{E}$ with which we provide our analysis is partially ordered by a closed normal cone generating $\mathcal{E}$, i. e. $\mathcal{E} = \mathcal{K} - \mathcal{K}$. Moreover, let subsets $\mathcal{H}'_j \subset \mathcal{K}'$ $j = 1, \ldots, p$, exist such that

$$\bigcup_{j=1}^{p} \mathcal{H}'_j = \mathcal{H}'$$

is $\mathcal{K}$-total and $x \in \mathcal{E}_j$ holds if and only if there exists an element $x'_x \in \mathcal{H}'_j$ such that $x'_x(x) \neq 0$. In order to allow overlaps of individual reduction maps R_j we do not assume that $\mathcal{E}_j \cap \mathcal{E}_k, j \neq k$, $\mathcal{E}_j$ =range(R_j) and $\mathcal{E}_k$ =range(R_k). Obviously, $0 \neq x \in \mathcal{E}_j \cap \mathcal{E}_k$ if and only if there are $x' \in \mathcal{H}'_j$ and $y' \in \mathcal{H}'_k$ such that $x'(x)y'(x) \neq 0$ and $\mathcal{E}_j \cap \mathcal{E}_k = \{0\}$ if and only if $x'(x)y'(x) = 0$ whenever $x' \in \mathcal{H}'_j, y' \in \mathcal{H}'_k$ and $x \in \mathcal{E}_j \cap \mathcal{E}_k$.

Formally, space $\mathcal{E}$ can be considered as a direct sum

$$\mathcal{E} = \mathcal{E}_j \oplus \mathcal{E}_{-j},$$

where $\mathcal{E}_{-j}$ is the "complementary" subspace to $\mathcal{E}_j$. We also have a corresponding decomposition of operator A given by formula

$$A = \begin{pmatrix} A_j & K_j \\ L_j & A_{-j} \end{pmatrix},$$

where A_j maps $\mathcal{E}_j \cap \Delta(A)$ into $\mathcal{E}_j$ and A_{-j} maps $\mathcal{E}_{-j} \cap \Delta(A)$ into $\mathcal{E}_{-j}$.

We assume that the maps

$$E_j = R_j^* R_j$$

are *diagonal operators*, i. e.

$$\mu_j I \preceq E_j \preceq \nu_j I \tag{18}$$

for some real $\mu_j, \nu_j, j = 1, \ldots, p$. In fact, we expect that similarly as in the finite dimensional case operators $\mathcal{E}_j$ will signal whether the method chosen does possess some overlaps and how extensive they are.

It is easy to see that as in case of finite dimensional situation both A_i and $A_{\neg i}$ are $\mathcal{K} - M$-operators [3]. For each $i = 1, \ldots, p$, we construct diagonal operators $E_i \in \mathrm{R}^{n \times n}$ associated with R_i chosen

$$E_i = R_i^T R_i. \tag{19}$$

If A is an $\mathcal{K} - M$-operator for each $i = 1, \ldots, p$, we construct a second collection of operators M_i associated with R_i as follows

$$M_i = \begin{bmatrix} A_i & O \\ O & D_{\neg i} \end{bmatrix}, \tag{20}$$

where

$$D_{\neg i} \geq O \tag{21}$$

is invertible diagonal operator representing a "diagonal" of A.

The following result comes from [1].

Proposition 1. Let A be a nonsingular $\mathcal{K} - M$-operator. Let M_i be defined as in (20). Then the splittings $A = M_i - W_i$ are regular (and thus weak regular and of nonnegative type).

In the cases considered in this paper, we always have that M_i defined in (20) are nonsingular. With the definitions (19) and (20) we obtain the following equality

$$E_i M_i^{-1} = R_i^T A_i^{-1} R_i, \quad i = 1, \ldots, p. \tag{22}$$

We can thus rewrite (15) as

$$T = T_\mu = (I - E_p M_p^{-1} A)(I - E_{p-1} M_{p-1}^{-1} A) \cdots (I - E_1 M_1^{-1} A). \tag{23}$$

Similarly, (17) can be rewritten as

$$T = T_\theta = I - \theta \sum_{i=1}^{p} E_i M_i^{-1} A. \tag{24}$$

This is how we interpret the multiplicative and additive Schwarz iterations.

In [1] it was shown that when A is nonsingular, $r(T_\mu) < 1$, and thus, the method (12) is convergent. Furthermore, there exists a unique splitting $A = M - W$ such that $T = T_\mu = M^{-1}W$. This splitting is a weak regular splitting.

In this paper we want explore the convergence of (12), using the iterations defined by (23), (24), when A is singular.

4. TWO APPROACHES TO APPROXIMATE SOLVING STATIONARY EQUATION

We are going to describe two ways for constructing solutions to stationary equations. The first one is based on an application of one of the Schwarz algorithms directly to the stationary equation, the second one utilizes the asymptotic behaviour of solutions of the appropriate evolution. The second alternative is suitable in particular for problems when no additional information is available such as irreducibility etc.

As classical properties of M-matrices suggest strict positive diagonals may strongly influence convergence and the speed of convergence of the investigated iterative processes. We recall a useful

Lemma 2. Let $T \in \mathbf{B}(\mathcal{E})$ satisfy $T\mathcal{K} \subset \mathcal{K}$ and $T \succeq \alpha I$ with some real $\alpha > 0$. Then $(1/r(T))T = S$ is convergent.

P r o o f. Obviously, $r(T) \geq \alpha$ and hence, $r(S) = 1$. For completing the proof we need to show that 1 is a unique spectral point with modulus $r(S)$. Let $T = \alpha I + V$. By hypothesis, $V\mathcal{K} \subset \mathcal{K}$ and $\lambda \in \sigma(T)$ can be written as $\lambda = \alpha + \mu, \mu \in \sigma(V)$. Hence, $r(T) = \alpha + r(V)$. Let $\mu = c + di$ with reals c, d and $i^2 = -1, c^2 + d^2 = r(V)^2$. It follows that

$$1 \geq \frac{|\lambda|^2}{(\alpha + r(V))^2} = \frac{\alpha^2 + 2\alpha c + r(V)^2}{\alpha^2 + 2\alpha r(V) + r(V)^2} = 1$$

only if $c = r(V)$ or, equivalently, if μ is real and hence λ positive. This completes the proof of the lemma. □

In [12] it has been shown that a solution $u = u(t)$ to Problem (P) gets stationary and $u(+\infty)$ is a solution to

$$B\left(\alpha u(+\infty) + \beta P u_0\right) + G(u(+\infty))u(+\infty) = 0, \tag{25}$$

where α and β are suitable nonnegative reals and u_0 the initial condition in (10). Since, by hypothesis,

$$-B_h = C_h - cI, \quad G(u)_{|h} = F(u)_{|h} - f(u)I,$$

$$C\mathcal{K} \subset \mathcal{K}, \quad F(u)_{|h}\mathcal{K} \subset \mathcal{K}, \quad c \geq r(C_h), \quad f(u) \geq r(H(u)_{|h}),$$

where C_h and $F(u)_{|h}$ denote appropriate discretizations of C and $F(u)$ respectively, it is an easy matter to show that

$$\frac{1}{r(C_h) + r(H(u)_{|h})} \left(C_h + H(u))_{|h}\right) = T(u)_{|h} \tag{26}$$

satisfies

$$T(u)^*_{|h} \hat{x}^*_h = \hat{x}^*_h, \quad \hat{x}^* \in \mathcal{K}^*.$$

Thus, in view of Definition 3, $T(u)$ is $\mathcal{K}$-stochastic corresponding to vector $\hat{x}^* \in \mathcal{K}^d$.

Theorem 1. Let $\mathcal{E}$ be a Hilbert space with an inner product $[\cdot, \cdot]$. Let $A = I - B$, where $B \in \mathbf{B}(E)$ is a $\mathcal{K}$-stochastic operator such that $Bv = v$ with $v \in \mathcal{K}^d$. Let $p \geq 1$ be a positive integer and $A = M_i - W_i$ be splittings of nonnegative type such that the diagonals of $T_i = M_i^{-1} N_i$, $i = 0, 1, \ldots, p$, are positive. Then

$$T = T_\mu = (I - E_p M_p^{-1} A)(I - E_{p-1} M_{p-1}^{-1} A) \cdots (I - E_1 M_1^{-1} A)$$

and $\hat{T} = T_p \ldots T_1$ are convergent operators. Furthermore, there is a splitting of nonnegative type

$$A = M - W \tag{27}$$

such that $T = M^{-1} W$, and the iteration operator T possesses the following properties:

$$T = Q + S, \quad Q^2 = Q, \quad QS = SQ = O, \quad r(S) < 1, \tag{28}$$

and

$$AQ = O. \tag{29}$$

The existence of a splitting of nonnegative type, and properties (28) and (29) also hold for $\hat{T}$.

P r o o f. We begin with the operator $\hat{T}$. Let $v > 0$ be such that $Bv = v$, i. e., $Av = 0$. For each splittings of $A = M_i - W_i$, we then have that $M_i v = N_i v$. This implies that $\hat{T} v = v$, and by Lemma 1 we have that $r(\hat{T}) = 1$ and that the index is 1. To show that $\hat{T}$ is convergent, we show that $T \succeq \alpha I$ for some real $\alpha > 0$. This follows from the fact that each of the operators T_i satisfies relation $T_j \succeq \alpha_j I$ with positive reals $\alpha_0, \ldots, \alpha_p$. We follow a similar logic for the multiplicative Schwarz iteration operator (23). Since $Av = 0$, $Tv = v$, and thus $r(T) = 1$ and $\mathrm{ind}_1 T = 1$. Each factor in (23) can be written as

$$I - E_i + E_i(I - M_i^{-1} A) = I - E_i + E_i M_i^{-1} W_i,$$

and since $O \leq E_i \leq I$ and $M_i^{-1} W_i \geq O$, each factor is nonnegative. For a row in which E_i is zero, the diagonal entry in this factor has value one. For a row in which E_i has value one, the diagonal entry in this factor is the positive diagonal entry of $M_i^{-1} W_i$. Thus, again, we have a finite product of $\mathcal{K}$-nonnegative operators, each dominanting a positive multiple of the identity operator, implying that the product

T does dominate a positive multiple of the identity operator too, and therefore it is convergent.

The rest of the proof applies equally to T and $\hat{T}$, we only detail it for T. The matrix T being convergent implies the spectral decomposition (28), where Q is the spectral projection onto the eigenspace of T corresponding to $r(T) = 1$. Furthermore since $T \geq O$, $Q = \lim_{k\to\infty} T^k \geq O$.

We show now that $\mathcal{N}(I - T) = \mathcal{N}(A)$. According to construction of T, the null spaces satisfy $\mathcal{N}(A) \subset \mathcal{N}(I - T)$. Any element of $y \in \mathcal{N}(I - T)$ which does not belong to $\mathcal{N}(A)$ has to have a form $y = Ax$ for some x and $y \neq 0$. Since $Q \geq O$, we have that $y \geq 0$. On the other hand $y^T e = x^T A^* e = 0$, a contradiction. Since we then have that $\mathcal{N}(I - T) = \mathcal{N}(A)$, the existence of a splitting of the form (27) follows similarly as does the finite dimensional analog of Theorem 1 from Theorem 2.1 of [2]. The fact that $T \geq O$ indicates that this splitting is of nonnegative type.

With this splitting, using (28) the following identity holds $AQ = M(I-T)Q = O$, so we also have (29). □

An example of splittings that lead to iteration matrices satisfying the hypotheses of Theorem 1 is described in the following proposition requiring no proof. It provides a possible modification to the local solvers, when the iteration operator defined by (20) does not dominate a positive multiple of the identity operator.

Proposition 2. Let $B \geq O$, $B^*\hat{x}' = \hat{x}'$. Let $\alpha_1, \ldots, \alpha_p$, be any positive real numbers. Let $A = I - B = M_i - N_i$, $i = 0, \ldots, p$, be defined by

$$M_i = \begin{bmatrix} \alpha_i I + A_i & 0 \\ 0 & \alpha_i I + D_{\neg i} \end{bmatrix} \tag{30}$$

and $W_i = M_i - A$, where $D_{\neg i}$ are defined in (20)–(21). Then, the splittings are regular, and iteration operator $T_i = M_i^{-1} W_i$ $i = 0, \ldots, p$ dominate a positive multiple of the identity operator.

Let us recall that we want to compute the stationary state, i. e. a solution of the system (25). This problem can be reformulated in terms of a new iterative process with the generating nonnegative operators C and $F(u(t))$:

$$u\left((k+1)\tau\right) = \frac{1}{c + f(u)} \left(C_h + F(u(k\tau))_{|h} u\left((k+1)\tau\right)\right), \quad u(0) = u_0. \tag{31}$$

An alternative to the method of approximate solving stationary equation of Problem (P) as described in Theorem 1 is to compute the limit $\lim_{k\to\infty} u(k\tau)$ using process (31).

From a variety of possible choices of rational approximations of the exponential we choose one from the class of limited Padé approximations, say

$$R(z) = \frac{P_j(z)}{(1 - \gamma z)^q}, \quad q \geq 2,$$

with appropriate real γ and polynomial P_j of degree j. Denoting

$$(L_k)_{|h} = \frac{1}{c + f(u)} \left(C_h + H(u(k\tau))_{|h}\right)$$

we want to compute according to (31)

$$\left(I - \gamma\tau(L_k)_{|h}\right)^q u((k+1)\tau) = P_j(k\tau)u((k+1)\tau), \quad k = 0, 1, \ldots \tag{32}$$

The above process can be implemented as follows. Let us omit the discretization parameter index and set $v^k = u(k\tau)$ and

$$v^{k+1/q} = (I - \gamma\tau L_k)\, u(k\tau), \ldots, v^{k+(q-1)/q} = (I - \gamma\tau L_k)^{q-1}\, u(k\tau), \quad k = 0, 1, \ldots$$

Then

$$\begin{aligned}
(I - \gamma\tau L_k)\, v^{k+1/q} &= P_j(\tau L_k)v^k(k\tau), \\
(I - \gamma\tau L_k)\, v^{k+2/q}(k\tau) &= P_j(\tau L_k)v^{k+1/q}(k\tau), \\
&\cdots \\
(I - \gamma\tau L_k)\, v^{k+1} &= P_j(\tau L_k)v^{k+(q-1)/q}(k\tau).
\end{aligned}$$

Convergence of the method just described is an easy consequence of the fact that the operators $\{(L_k)_{|h}\}, k = 1, 2, \ldots$ are nonsingular $\mathcal{K} - M$-operators in the spirit of Definition 3 and the convergence results of [1]. Actually, we have

Theorem 2. Let $\mathcal{E}$ be a Hilbert space over the reals generated by a closed normal cone $\mathcal{K}$. Assume B is a generally unbounded linear operator defined on a dense domain $\mathcal{D} \subset \mathcal{E}$ and its adjoint satisfies relation $B^*\hat{x}^* = 0$. We assume further that B generates a semigroup of operators of class $\mathcal{C}$ such that $T(t; -B)\mathcal{K} \subset \mathcal{K}, t > 0$. Finally assume that for any $u \in \mathcal{D} \cap \mathcal{K}$ operator $G(u) \in \mathbf{B}(\mathcal{E})$ is an $\mathcal{K} - M$-operator satisfying $[G(u)]^*\hat{x}^* = 0$.

Then the iteration process (32) returns a sequence $\{u(k\tau)\}$ of approximations to a unique solution to Problem (P) such that

$$\lim_{k\to\infty} u(k\tau) = u(+\infty).$$

5. CONCLUSIONS

Nowadays it is accepted by the community of numerical analysts that two- and generally multi-level iterative methods offer an essentially broader variety of tools to solve large scale computational problems. In this context the Schwarz and Schwarz-like methods play a quite important role. Many contributions of many authors document this statement as a rule by investigating problems characterized by non-singular operators. An emphasis of our approach is just the opposite that is to problems with singular operators. Another goal of our analysis is that we consider

the computational problems in their original form, i. e. we work with generally infinite dimensional objects and let discretizations to be made at an appropriate moment, e. g. at each iteration step.

We apply our Schwarz-like methods to a problem coming from stochastic modeling in biology and chemistry. We can thus profit from having nicely structured operators but suffer of approaching problems with hardly accessible data. We thus propose two methods each suitable in the corresponding situation. The first method assumes all data accessible and the second just the opposite. In particular, the method using auxiliary time evolution does require no a priori knowledge of location as well as access to each single data. As example let us mention irreducibility of the operators of the model and access to matrix elements of appropriate discretizations. The latter is compensated by ability of our method of an easy computation of the corresponding matrix actions.

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