Kybernetika

VOLUME *44* (2008), NUMBER 5

The Journal of the Czech Society for Cybernetics and Information Sciences

Published by:

Institute of Information Theory and Automation of the AS CR

Kybernetika is a bi-monthly international journal dedicated for rapid publication of high-quality, peer-reviewed research articles in fields covered by its title.

Kybernetika traditionally publishes research results in the fields of Control Sciences, Information Sciences, System Sciences, Statistical Decision Making, Applied Probability Theory, Random Processes, Fuzziness and Uncertainty Theories, Operations Research and Theoretical Computer Science, as well as in the topics closely related to the above fields.

The Journal has been monitored in the Science Citation Index since 1977 and it is abstracted/indexed in databases of Mathematical Reviews, Zentralblatt für Mathematik, Current Mathematical Publications, Current Contents ISI Engineering and Computing Technology.

K y b e r n e t i k a . Volume *44* (2008) ISSN 0023-5954, MK ČR E 4902.

Published bimonthly by the Institute of Information Theory and Automation of the Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 4, 182 08 Praha 8. — Address of the Editor: P. O. Box 18, 182 08 Prague 8, e-mail: kybernetika@utia.cas.cz. — Printed by PV Press, Pod vrstevnicí 5, 140 00 Prague 4. — Orders and subscriptions should be placed with: MYRIS TRADE Ltd., P. O. Box 2, V Štíhlách 1311, 142 01 Prague 4, Czech Republic, e-mail: myris@myris.cz. — Sole agent for all "western" countries: Kubon & Sagner, P. O. Box 34 01 08, D-8 000 München 34, F.R.G.

Published in November 2008.

KYBERNETIKA — VOLUME 44 (2008), NUMBER 5, PAGES 705 – 714

KERMACK–McKENDRICK EPIDEMICS VACCINATED

Jakub Staněk

This paper proposes a deterministic model for the spread of an epidemic. We extend the classical Kermack–McKendrick model, so that a more general contact rate is chosen and a vaccination added. The model is governed by a differential equation (DE) for the time dynamics of the susceptibles, infectives and removals subpopulation.

We present some conditions on the existence and uniqueness of a solution to the non-linear DE. The existence of limits and uniqueness of maximum of infected individuals are also discussed.

In the final part, simulations, numerical results and comparisons of the different vaccination strategies are presented.

Keywords: SIR epidemic models, vaccination, differential equation

AMS Subject Classification: 92D25, 37N25

1. INTRODUCTION

Models that describe behaviour of highly infective diseases are frequently studied in the recent literature (see [2, 3, 5, 9]). There are not so many models that would enable to understand an optimal control of the infection by means of vaccination evaluate it (see [4, 9]). This paper extends the classical Kermack–McKendrick differential equation for epidemics (see [3, 5, 6, 7]) and suggests its control by a continuous vaccination.

The presented model describes the behaviour of highly infective diseases (e. g. flu) with short healing time (few days) and very short incubation time, which is omitted in the model.

We consider a population that consists of N individuals and it is divided into three subpopulations. The first one is called *Susceptibles* (the individuals who are not infected, but who can be infected by the disease), its size at time t is denoted by x_t. The second one is made by *Infectives* (the infected individuals, who are able to spread the disease), its size being denoted by y_t. The *Removals* subpopulation consists of individuals who were infected, but who are not able to spread the infection further or get themselves infected again, because they are either isolated or cured and became immune. It follows that its size z_t equals to $N - x_t - y_t$ at arbitrary time t.

The model is described by the following differential equations (see [8]):

$$\begin{aligned} \mathrm{d}x_t &= -\beta(z_t)y_t \ [x_t - \vartheta(z_t)]^+ \,\mathrm{d}t, & x_0 > 0, \\ \mathrm{d}y_t &= \beta(z_t)y_t \ [x_t - \vartheta(z_t)]^+ \,\mathrm{d}t - \gamma y_t \mathrm{d}t, & y_0 > 0, \\ \mathrm{d}z_t &= \gamma y_t \mathrm{d}t, & z_0 = 0, \end{aligned} \tag{R1}$$

where $\beta(z_t)$ is a *Susceptibles–Infectives* contact rate that is time dependent through z_t, where γ is a recovery rate of the infection and, finally, $\vartheta(z_t)$ is the size of vaccinated susceptibles subpopulation controlled by z_t again.

We shall assume that $\beta(z) : \mathbb{R} \to \mathbb{R}^+$ is a nonincreasing continuous function, $\gamma > 0$ and that $\vartheta(z) : \mathbb{R} \to \mathbb{R}^+$ a nondecreasing continuous function. From the assumptions, we know that $x_t + y_t + z_t = N = x_0 + y_0$ for all $t \geq 0$.

It means that the size of individuals newly infected during the time interval $(t, t+\triangle)$ is approximately equal to the product $y_t[x_t - \vartheta(z_t)]^+\beta(z_t)\triangle$, where $y_t[x_t - \vartheta(z_t)]^+$ is the number of all possible contacts between infective and susceptible nonvaccinated people (i. e. the number of all possible pairs) in time t, and $\beta(z_t)$ is a probability that a randomly chosen susceptible nonvaccinated person is infected by a randomly chosen infective person during the time interval $(t, t+1)$. Because the population consists of people with different rate of immunity (e. g. children are more inclined to diseases then adults) and people with weaker immunity fall ill more easily than strong immune people, the rate of immunity of susceptibles grows with increasing z_t. Therefore, the Susceptibles–Infectives contact rate β is nonincreasing function of Removals.

After the consultation with practitioners in medicine, the function of vaccinated susceptibles ϑ is also considered to be a function of the removals, because the number of the removals is usually known and also because it is an indicator of the extent of the epidemic used in practice. Moreover, if we choose ϑ as an increasing linear function of vaccinated individuals, then we vaccinate more people in the case when we have more infected individuals, because the increment of Removals is proportional to the number of infectives.

2. THEORETICAL RESULTS

In this section, we will speak about a solution to DEs. By the solution to DE we mean the classical solution, i. e. we use the definition of solution as introduced in [1], p. 67.

The following lemma follows from more general results, e. g. Corollary 16.10, p. 219, in [1], but it could be unnoticed when using it for our case. Therefore we show more intuitive proof without using any special theorems.

Lemma 2.1. If $l_t = (x_t, y_t, z_t)$ is a solution to (R1), then $l_t \in [0, N]^3$ for all $t \geq 0$. Moreover, x_t is a nonincreasing function and z_t is an increasing function.

P r o o f . From (R1), we can get

$$y_t = y_0 \exp\left\{\int_0^t \beta(z_s)[x_s - \vartheta(z_s)]^+ - \gamma \,\mathrm{d}s,\right\}, \tag{1}$$

therefore $y_t > 0$ for all t.

Further, $z_t = z_0 + \int_0^t \gamma y_s \,\mathrm{d}s$, therefore we get z_t as a nonnegative increasing function as $y_t > 0$.

The size of susceptibles x_t is obviously a nonincreasing function.

If we denote by $\tau_v := \inf\{t \in \mathbb{R}^+ : x_t \leq \vartheta(z_t)\}$ the first time, when susceptibles are completely vaccinated, then

$$\mathrm{d}x_t = -\beta(z_t)y_t[x_t - \vartheta(z_t)]\,\mathrm{d}t \tag{2}$$

for $t \in [0, \tau_v]$. Moreover, as x_t is nonincreasing and z_t is increasing, we have for all $t > \tau_v$ that $x_t \leq \vartheta(z_t)$ and $\mathrm{d}x_t = 0$.

Solving equation (2) we get

$$x_t = \left[x_0 + \int_0^t \beta(z_s)\vartheta(z_s)y_s \exp\left\{\int_0^s \beta(z_u)y_u \,\mathrm{d}u\right\} \mathrm{d}s\right] \exp\left\{-\int_0^t \beta(z_s)y_s \,\mathrm{d}s\right\} \geq 0.$$

Since $x_t = x_{\tau_v}$ for all $t \in (\tau_v, \infty)$, x_t is nonnegative function.

We proved that any solution l_t to (R1) maps $[0, \infty)$ into the first octant. As $x_t \geq 0$, $y_t \geq 0$ and $z_t \geq 0$ for all $t \geq 0$ and $x_t + y_t + z_t = N$ it follows that $l_t \in [0, N]^3$. □

Hence ϑ is nondecreasing and $x(.)$ is nonincreasing, then using τ_v from the previous proof we can rewrite (R1) to the form

$$\begin{aligned} \mathrm{d}x_t &= -\beta(z_t)y_t\,[x_t - \vartheta(z_t)]\,\mathrm{d}t, & x_0 &> 0, \\ \mathrm{d}y_t &= \beta(z_t)y_t\,[x_t - \vartheta(z_t)]\,\mathrm{d}t - \gamma y_t\,\mathrm{d}t, & y_0 &> 0, \\ \mathrm{d}z_t &= \gamma y_t\,\mathrm{d}t, & z_0 &= 0, \end{aligned} \tag{R2}$$

for $t \in [0, \tau_v]$ and

$$\begin{aligned} \mathrm{d}x_t &= 0, \\ \mathrm{d}y_t &= -\gamma y_t\,\mathrm{d}t, \\ \mathrm{d}z_t &= \gamma y_t\,\mathrm{d}t, \end{aligned} \tag{R3}$$

for $t \in [\tau_v, \infty)$.

Lemma 2.2. Let β and ϑ be Lipschitz bounded functions. Then the equation $(R2)$ has a unique solution on the interval $[0, \tau_v]$.

P r o o f . Denote

$$f(x, y, z) = (-\beta(z)[x - \vartheta(z)]y, \beta(z)[x - \vartheta(z)]y - \gamma y, \gamma y)$$

and

$$\tilde{f}(l) = f(\tilde{x}, \tilde{y}, \tilde{z}),$$

where $\tilde{x} = (x \vee -2N) \wedge 2N$. Then, using Lemma 2.1 and the fact that the unique solution to

$$\mathrm{d}l = \tilde{f}(l),\ l_0 = (x_0, y_0, z_0)$$

is the unique solution to (R2) on the interval $[0, \tau_v]$, Lemma 2.2 follows from more general theorem 7.6 in [1], p. 100. □

Define $\tau_Y := \arg\max y_t := \{t \in [0, \infty) : y_t = \max_{s \in [0,\infty)} y_s\}$ the time of culmination of the epidemic (below we show that the time τ_Y is unique).

The following theorem is our main result.

Theorem 2.3. Let β and ϑ satisfy the conditions of Lemma 2.2. Then

(i) the equation (R1) has a unique solution on the time interval $[0, \infty)$,

(ii) there exist limits of x, y, z at infinity, $y_\infty = 0$. If $\tau_v = \infty$ then z_∞ is a solution to the equation $z = N - X(z)$, where

$$X(z) = \left[x^0 + \int_0^z \frac{\beta(u)}{\gamma} \vartheta(u) \exp\left\{ \frac{\int_0^u \beta(s)\,\mathrm{d}s}{\gamma} \right\} \mathrm{d}u \right] \exp\left\{ \frac{-\int_0^z \beta(u)\,\mathrm{d}u}{\gamma} \right\}.$$

(iii) the size of infectives subpopulation y_t has a unique maximum y_{τ_Y}.
If $\beta(z_0)[x_0 - \vartheta(z_0)] - \gamma > 0$, then $\beta(z_{\tau_Y})[x_{\tau_Y} - \vartheta(z_{\tau_Y})] = \gamma$.
If $\beta(z_0)[x_0 - \vartheta(z_0)] - \gamma \leq 0$, then $\tau_Y = 0$.

Proof.

(i) The existence and uniqueness of a solution to (R2) on the time interval $[0, \tau_v]$ follows from Lemma 2.2. Therefore we need to prove its existence and uniqueness on the time interval $[\tau_v, \infty]$ in the case $\tau_v < \infty$. Because the equation (R3) with the initial conditions $x(\tau_v) = \tilde{x}(\tau_v), y(\tau_v) = \tilde{y}(\tau_v), z(\tau_v) = \tilde{z}(\tau_v)$, where $(\tilde{x}, \tilde{y}, \tilde{z})$ is a solution to (R2) on the time interval $[0, \tau_v]$, has a unique solution, it follows that

$$\begin{aligned} x(t) &= x_{\tau_v}, \\ y(t) &= y_{\tau_v} - \mathrm{e}^{-\gamma\tau_v} + \mathrm{e}^{-\gamma t}, \\ z(t) &= N - x_{\tau_v} - y_{\tau_v} + \mathrm{e}^{-\gamma\tau_v} - \mathrm{e}^{-\gamma t} \end{aligned}$$

holds.

Joining these solutions, we get a unique solution to (R1) on the time interval $[0, \infty)$. Indeed, if we denote

$$\begin{aligned} \hat{l}_t &= (\hat{x}_t, \hat{y}_t, \hat{z}_t) = (\tilde{x}_t, \tilde{y}_t, \tilde{z}_t) \qquad t \in [0, \tau_v] \\ &= (x_t, y_t, z_t) \qquad\qquad\qquad\quad t \in (\tau_v, \infty), \end{aligned}$$

then

$$\hat{x_t} = \hat{x_0} - \int_0^t \beta(\hat{z_s})\hat{y_s}[\hat{x_s} - \vartheta(\hat{z_s})]^+ \,\mathrm{d}s$$
$$\hat{y_t} = \hat{y_0} + \int_0^t \beta(\hat{z_s})\hat{y_s}[\hat{x_s} - \vartheta(\hat{z_s})]^+ - \gamma\hat{y_s} \,\mathrm{d}s$$
$$\hat{z_t} = \int_0^t \gamma\hat{y_s} \,\mathrm{d}s.$$

Therefore $\hat{l}$ is a solution to (R1).

(ii) Functions x and z are monotone and bounded, therefore they have their limits x_∞, z_∞ at infinity. Because $y_t = N - x_t - z_t$ for all $t \in [0, \infty)$, the existence of the limits x_∞ and z_∞ implies the existence of the limit y_∞. Since $z_\infty < \infty$, we get $y_\infty = 0$. Indeed, if $y_\infty > 0$, then there exists a time $T \in [0, \infty)$ and a constant $a > 0$ such that $y_t \geq a$ for all $t > T$. Therefore

$$z_\infty = \int_0^\infty \gamma y_s \,\mathrm{d}s \geq \int_0^T \gamma y_s \,\mathrm{d}s + \int_T^\infty \gamma a \,\mathrm{d}s = \infty.$$

It means that $y_\infty = 0$.

As z_t is a continuous differentiable mapping of $[0, \infty)$ on $[0, z_\infty]$ with positive derivation, it has continuously differentiable inverse z_t^{-1}, we can set $X(z) = x(z_t^{-1})$ and (R2) implies

$$\frac{\mathrm{d}X(z)}{\mathrm{d}z} = \frac{\mathrm{d}x_t/\mathrm{d}t}{\mathrm{d}z_t/\mathrm{d}t} = \frac{-\beta(z_t)Y(z_t)[X(z_t) - \vartheta(z_t)]}{\gamma Y(z_t)}, \quad X(z_0) = x_0,$$

therefore

$$X(z) = \left[x^0 + \int_0^z \frac{\beta(u)}{\gamma}\vartheta(u) \exp\left\{\frac{\int_0^u \beta(s)\,\mathrm{d}s}{\gamma}\right\} \mathrm{d}u\right] \exp\left\{\frac{-\int_0^z \beta(u)\,\mathrm{d}u}{\gamma}\right\}. \tag{3}$$

Finally, let $t \to \infty$ in $z(t) = N - x(t) - y(t)$ to get the equation $z_\infty = N - x_\infty = N - X(z_\infty)$.

(iii) Because $\beta(z_t)$ and x_t are nonincreasing functions of t and ϑ a nondecreasing function of t, it follows that $\beta(z_t)[x_t - \vartheta(z_t)]$ is nonincreasing. Hence $\beta(z_0)[x_0 - \vartheta(z_0)] - \gamma \leq 0$ implies $dy_t \leq 0$, and y_t is nonincreasing. Thus, $\tau_Y = 0$. If $\beta(z_0)[x_0 - \vartheta(z_0)] - \gamma > 0$, then y_t is increasing in neighbourhood of zero and because moreover $y_0 > y_\infty = 0$, we have $0 < \tau_Y < \infty$. Hence continuity and the existence of derivative of y_t imply that $y'_{\tau_Y} = 0$, therefore $\beta(z_{\tau_Y})[x_{\tau_Y} - \vartheta(z_{\tau_Y})] - \gamma = 0$.

Let $\beta(z_{\tau_Y})[x_{\tau_Y} - \vartheta(z_{\tau_Y})] = \gamma$. Denote $T := \inf\{t \geq 0 : \beta(z_t)[x_t - \vartheta(z_t)] = \gamma\}$. From (R1) and $y_{\tau_Y} \geq y_0 > 0$ it follows that x_t is decreasing in T, and so there is no other time t satisfying $\beta(z_t)[x_t - \vartheta(z_t)] = \gamma$. Therefore $\tau_Y = T$ is unique. In the case $\beta(z_0)[x_0 - \vartheta(z_0)] - \gamma < 0$, the uniqueness of τ_Y is obvious. □

Example 2.4. We shall scrutinize the equation $z_\infty = N - X(z_\infty)$ (see Theorem 2.3 (ii)) and assume β to be a constant and $\vartheta(z)$ a general function, later on a linear function.

We apply (3) to get

$$z_\infty = N - \left[x^0 + \int_0^{z_\infty} \frac{\beta}{\gamma}\vartheta(u) \exp\left\{\frac{\int_0^u \beta \,\mathrm{d}s}{\gamma}\right\}\right] \exp\left\{\frac{-\int_0^{z_\infty} \beta \,\mathrm{d}u}{\gamma}\right\}. \tag{5}$$

Denoting $\rho = \beta/\gamma$ then (5) yields

$$z_\infty = N - \mathrm{e}^{-\rho z_\infty}\left[x^0 + \rho \int_0^{z_\infty} \vartheta(u)\mathrm{e}^{\rho u} \,\mathrm{d}u\right]. \tag{6}$$

Choosing a linear vaccination, i. e. $\vartheta(z) = \vartheta_0 + \vartheta_1 z$, where $\vartheta_0 \geq 0$ a $\vartheta_1 \geq 0$, we have

$$C_1 z_\infty = C_2 - C_3 \mathrm{e}^{-\rho z_\infty}, \tag{7}$$

where

$$\begin{aligned} C_1 &= 1 + \vartheta_1, \\ C_2 &= N - \vartheta_0 + \vartheta_1/\rho, \\ C_3 &= x^0 - \vartheta_0 + \vartheta_1/\rho. \end{aligned}$$

The uniqueness of solution to equations (5) depends on the choice of functions $\beta(z)$, $\vartheta(z)$ and initial conditions. If we have more then one solution to the equation, we have to decide, which of them is z_∞.

To illustrate it, go back to the equation (7). What we know is that $C_1 > 0$ and $C_2 > C_3$ hold.

If $C_3 \leq 0$ then the number of vaccinated at $t = 0$ is larger than or equal to the number of susceptibles, hence $\tau_v = 0$ and the assumption of Theorem 2.3 (ii) is not satisfied. In practice, this choice is not a very realistic one, mathematically it leads to $z_\infty = y_0$ by (R3) and, of course, to $y_\infty = 0$.

If $C_3 > 0$ then (7) possess two solutions, but only one positive. It follows that (7) has a unique solution $z_\infty \in [0, N]$.

Example 2.5. Consider again constants β, γ and a linear ϑ in a way that $\tau_v = \infty$ and $\tau_Y \neq 0$. Theorem 2.3 (iii) yields

$$[x_{\tau_Y} - \vartheta(z_{\tau_Y})] = \frac{\gamma}{\beta}. \tag{8}$$

Computing

$$\begin{aligned} X(z) &= \left[x^0 + \int_0^z \frac{\beta}{\gamma}\vartheta(u) \exp\left\{\frac{\int_0^u \beta \,\mathrm{d}s}{\gamma}\right\}\right] \exp\left\{\frac{-\int_0^z \beta \,\mathrm{d}u}{\gamma}\right\} \\ &= \left[x^0 + \frac{\beta}{\gamma}\int_0^z (\vartheta_0 + \vartheta_1 u)\mathrm{e}^{\frac{\beta u}{\gamma}}\right] \mathrm{e}^{-\frac{\beta z}{\gamma}} \\ &= \left(\vartheta_0 - \frac{\vartheta_1}{\rho}\right) + \vartheta_1 z + \left(x^0 + \frac{\vartheta_1}{\rho} - \vartheta_0\right) \mathrm{e}^{-\rho z} \end{aligned} \tag{9}$$

by (3) and substituting $X(z)$ into (8), we arrive at

$$\left(\vartheta_0 - \frac{\vartheta_1}{\rho}\right) + \vartheta_1 z_{\tau_Y} + \left(x^0 + \frac{\vartheta_1}{\rho} - \vartheta_0\right) \mathrm{e}^{-\rho z_{\tau_Y}} - \vartheta_0 - \vartheta_1 z_{\tau_Y} = \frac{1}{\rho}.$$

This implies

$$z_{\tau_Y} = \frac{1}{\rho}\left[\log\left(x^0 + \frac{\vartheta_1}{\rho} - \vartheta_0\right) - \log\left(\frac{1+\vartheta_1}{\rho}\right)\right]. \tag{10}$$

Finally, having on mind that $N = x + y + z$, we get

$$y_{\max} = y_{\tau_Y} = N - X(z_{\tau_Y}) - z_{\tau_Y}, \tag{11}$$

where z_{τ_Y} and $X(z_{\tau_Y})$ are given by (10) and (9), respectively.

3. NUMERICAL RESULTS

This part deals with several problems that arise when one is trying to get some usable results concerning the time of culmination of the epidemics, the largest number of those infected, the influence of vaccination and the comparison of various vaccination strategies.

First, we consider a constant $\beta > 0$ and a linear vaccination, i. e. $\vartheta(z) = \vartheta_0 + \vartheta_1 z$. Having made this choice, we replace the differential equation (R1) by the equation

$$\begin{aligned} x_{n+1} &= x_n - \beta(z_n) y_n \max\{[x_n - \vartheta(z_n)], 0\}\Delta, & x_0 &= x^0 > 0, \\ y_{n+1} &= y_n + (\beta(z_n) y_n \max\{[x_n - \vartheta(z_n)], 0\} - \gamma y_n)\Delta, & y_0 &= y^0 > 0, \\ z_{n+1} &= z_n + \gamma y_n \Delta, & z_0 &= 0, \end{aligned} \tag{R4}$$

where Δ is a difference step.

We solved the equation (R4) with the number of steps 5000 and the difference step $\Delta = 0.016$, because we observed that a choice of smaller step does not change the results significantly. This corresponds to the time interval $(0, 80)$. We decided to use these values, because on this interval, the behavior of the epidemic can be well graphically shown (see Figure 1 and Figure 2). We chose the initial conditions $x^0 = 990$, $y^0 = 10$, what means that at the beginning, 1% of population suffers from the disease, and we observed the behavior of the epidemic with several choices of $\gamma, \beta, \vartheta_0$ and ϑ_1. All computations and graphic results were made by software R.[1]

Figure 1 visualizes the differences in behavior of epidemic for different choices of β with a permanent γ, when no vaccination is applied. Figure 2 shows the differences in behavior of epidemic for different vaccinations.

Although in the first case ($\vartheta_0 = 0$ and $\vartheta_1 = 1$), we have vaccinated 443 individuals by the time $t = 80$, while choosing $\vartheta_0 = 300$ and $\vartheta_1 = 0,2$ we have vaccinated only 363 individual in the same time interval, the evolution of epidemic is less favourable in the former case than in the latter one in the sense that the number of removals

[1] Version R 2.3.1 was used.

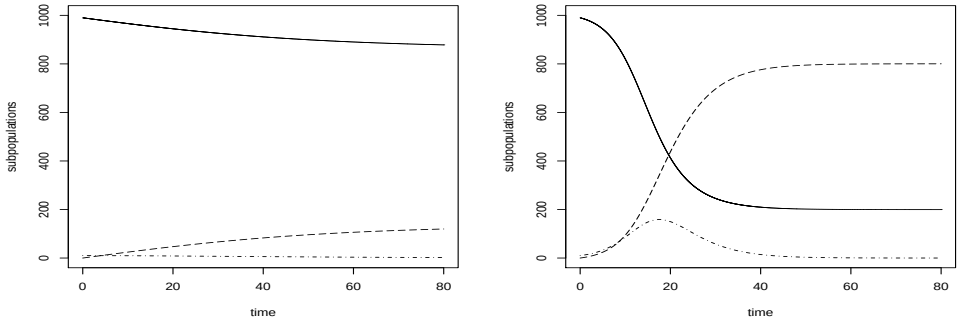

Fig. 1. Behavior of epidemic with $\beta = 0,00025$, $\gamma = 0,25$ (left) and $\beta = 0,0005$, $\gamma = 0,25$ (right). The solid line describes the size of susceptibles, the dot-dashed line the size of infectives and the dashed line the size of removals.

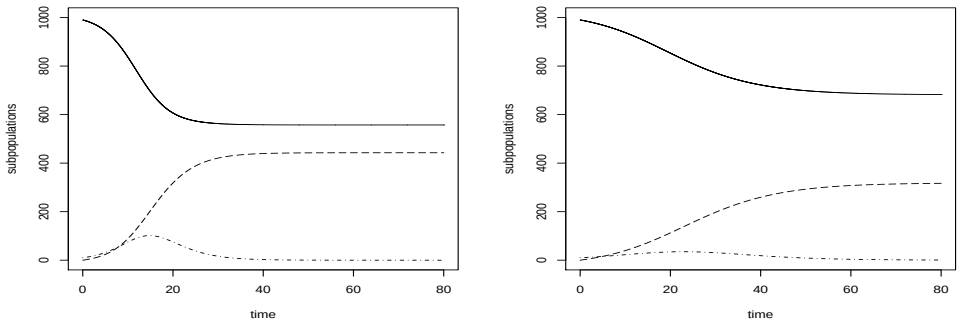

Fig. 2. Behavior of epidemic with $\beta = 0,0005$, $\gamma = 0,25$ and the vaccination either $\vartheta_0 = 0$, $\vartheta_1 = 1$(left) or $\vartheta_0 = 300$, $\vartheta_1 = 0,2$ (right). The solid line describes the size of susceptibles, the dot-dashed line the size of infectives and the dashed line the size of removals.

for the first choice is 443 in comparison with 317 for the second choice. Moreover, the maximal size of infected individuals (35) is also in favor of the latter vaccination compared with the former one (101).

Table 1 summarizes the values obtained by solving equation (R4) for several pairs of the coefficients ϑ_0 and ϑ_1. Here, we approximated x_∞ and z_∞ by x_{12500} and z_{12500}, respectively. The approximation should be a satisfactory one as already the values y_{12500} are observed to be close to zero. To get the results, we produced 12500 steps with difference step $\Delta = 0.016$ (i. e. we observed the time interval $(0, 200)$), choosing β and γ as before, i. e. $\beta = 0.0005$, $\gamma = 0.25$. The initial conditions were

again $x^0 = 990$ and $y^0 = 10$. The table lists the final values of x, y and z, i. e. the numbers of the susceptibles, infectives and removals at time $t = 200$, the total number of vaccinated individuals by the time $t = 200$, the size of maxima of infected individuals and the time of maxima.

Table 1.

ϑ_0	ϑ_1	x_{12500}	z_{12500}	y_{12500}	Vaccinated	max. y	τ_Y
0	0	199.5697	800.4303	5.3 e − 10	0	158.6046	17.552
0	0.2	313.1946	686.8054	5.1 e − 11	137.3611	141.9416	16.848
0	0.5	432.0209	567.9791	3.1 e − 12	283.9895	123.0361	15.984
0	1	557.2330	442.7670	9.8 e − 14	442.7670	101.3385	14.832
0	2	690.4325	309.5675	1.0 e − 15	619.1350	76.0882	13.200
100	0.2	430.9243	569.0757	2.2 e − 09	213.8151	99.6202	18.752
100	0.5	530.3573	469.6427	2.3 e − 10	334.8214	86.0446	17.648
100	1	634.6346	365.3654	1.4 e − 11	465.3654	70.7635	16.192
100	2	744.7960	255.2041	3.1 e − 13	610.4081	53.3868	14.112
300	0.2	681.4626	318.5374	1.2 e − 05	363.7075	34.9557	22.896
300	0.5	736.9506	263.0494	2.8 e − 06	431.5247	30.6612	20.704
300	1	794.3346	205.6654	4.1 e − 07	505.6654	26.0612	17.952
300	2	854.0339	145.9661	2.0 e − 08	591.9322	21.1174	14.304

Numerical results have confirmed our expectations that having determined to provide a fixed number of vaccinations, an epidemic has a better evolution if choosing a more robust pre-vaccination (bigger ϑ_0) because it decreases both the number and the global maximum of the infected individuals. Moreover, comparing 5th and 10th row in Table 1, we can see that for the same running of epidemic (in the mean of remained susceptibles), much less (almost one half) people need to be vaccinated in the case of pre-vaccination.

In Table 2, there are values of z_∞, that we receive as a solution to equation (7) in Example 2.4. We choose again $\beta = 0.0005$, $\gamma = 0.25$, $x^0 = 990$ and $y^0 = 10$ and the vaccination which enters Table 1. We solved the equation by using the divising interval method, we look for a solution in the interval $[0, 1000]$ and we require the error to be less then 0.001.

Table 2.

	$\vartheta_1 = 0$	$\vartheta_1 = 0.2$	$\vartheta_1 = 0.5$	$\vartheta_1 = 1$	$\vartheta_1 = 2$
$\vartheta_0 = 0$	800.2034	686.5820	567.7654	442.5726	309, 4061
$\vartheta_0 = 100$		568.9032	469.4815	365.2218	255.0868
$\vartheta_0 = 300$		318.4655	262.9839	205.6079	145.9184

Comparing the values delivered by Table 1 with those delivered by Table 2, the differences are observed to be less than 0.3.

The values of maxima of infected individuals received by the formula (11) in Example 2.5 are presented by Table 3 choosing β, γ and the initial conditions as above.

Table 3.

	$\vartheta_1 = 0$	$\vartheta_1 = 0.2$	$\vartheta_1 = 0.5$	$\vartheta_1 = 1$	$\vartheta_1 = 2$
$\vartheta_0 = 0$	158.4516	141.7980	122.90494	101.2239	75.9957
$\vartheta_0 = 100$		99.5348	85.9673	70.6963	53.3324
$\vartheta_0 = 300$		34.9380	30.6450	26.0467	21.1049

Comparing Table 1 and Table 3, the differences are seen to be less than 0.2. Hence, we can conclude that (R4) provides approximations close enough to the theoretical values.

ACKNOWLEDGEMENT

This research was supported by the Czech Science Foundation under Grant 201/05/H007 and by the Grant Agency of Charles University under Project 56707-B. The author also thanks to Professor Josef Štěpán for the professional help.

(Received June 27, 2008.)

REFERENCES

[1] H. Amann: Ordinary Differential Equations: An Introduction to Nonlinear Analysis. Walter de Gruyter, Berlin – New York 1990.
[2] N. T. J. Bailey: The Mathematical Theory of Epidemics. Hafner Publishing Company, New York 1957.
[3] D. J. Daley and J. Gani: Epidemic Modelling: An Introduction. Cambridge University Press, Cambridge 1999.
[4] P. Greenwood, L. F. Gordillo, A. S. Marion, and A. Martin-Löf: Bimodal Epidemic Side Distributions for Near-Critical SIR with Vaccination. In preparation.
[5] J. Kalas and Z. Pospíšil: Spojité modely v biologii (Continuous Models in Biology). Masaryk University, Brno 2001.
[6] W. O. Kermack and A. G. McKendrick: A contribution to the mathematical theory of epidemics. Proc. Roy. Soc. London *A 155* (1927), 700–721.
[7] J. Štěpán and D. Hlubinka: Kermack–McKendrick epidemic model revisited. Kybernetika *43* (2007), 395–414.
[8] J. Štěpán: Private communication.
[9] T. Wai-Yuan and W. Hulin: Deterministic and Stochastic Models of AIDS Epidemics and HIV Infections with Intervention. World Scientific, Singapore 2005.

Jakub Staněk, Department of Probability and Mathematical Statistics, Faculty of Mathematics and Physics – Charles University, Sokolovská 83, 186 75 Praha 8. Czech Republic.
e-mail: stanekj@karlin.mff.cuni.cz