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LETTER TO THE EDITOR: CONSISTENCY OF LPC + Ch

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In his paper [4] "Algebraic analysis of LPC+Ch calculus", Kybernetika *31* (1995), No. 1, pp. 99–106 Turunen says in Corollary on p. 106:

> "... (Notice that the third last line on page 195 in [2] stating that LPC+Ch Calculus is consistent is not correct.)"

The reference [2] in Turunen's paper is the same as [2] here.

The system LPC+Ch is consistent, which can be seen quite trivially. For pure syntactical logical systems there are three kinds of consistencies which are quite closely related:

(i) *absolute consistency:* not the every formula is a theorem;
(ii) *canonical consistency:* if α is some fixed theorem, then $\neg\alpha$ is not a theorem;
(iii) *consistency with respect to the negation:* there does not exist a formula α such that α and $\neg\alpha$ both are theorems.

There is still a fourth kind of consistency, namely *consistency with respect to interpretation*, and it cannot be used with purely syntactical systems without possible models they can have. (Different kinds of consistency are considered for example in the following books: Margaris, A., *First Order Mathematical Logic*, Blaisdell, 1967; Rogers, R., *Mathematical Logic and Formalized Theories*, North Holland, 1971; Mendelson, E., *Introduction to Mathematical Logic*, Van Nostrand, 1964; Kleene, S.C., *Mathematical Logic*, Wiley, 1967, and Church, A., *Introduction to Mathematical Logic*, Princeton, 1956.)

LPC+Ch is absolutely consistent, because for example a single M-formula p (an element of the set of atoms) is not a theorem of LPC+Ch. Canonical consistency follows from the consistency with respect to the negation, which is the most important kind of consistency, and which follows from the definition stating that a syntactical logical system is consistent iff all the formulas do not belong to the system. (Note that any system consists of its axioms and theorems.)

We prove that LPC+Ch is consistent with respect to the negation using the well-known method of *PC-transform* (see e. g. [1], p. 145). This method is generally used for proving the consistency of intensional logics. For any M-formula α of LPC+Ch we can form its PC-transform α' in the following way:

1. Rewrite α (if necessary) in primitive notation.
2. Eliminate all occurrences of the identity symbol '=' by using its definition in LPC.
3. Delete all modifiers, quantifiers and individual variables.
4. Replace each distinct predicate variable by a distinct propositional variable.

The resulting expression α' will be a proposition of PC. The PC-transform has the following properties:

(i) Every M-formula will have one and only one PC-transform (though two M-formulas may have the same PC-transform).

(ii) If the PC-transform of α is α', then the PC-transform of $\cdot\ \neg\alpha$ will be $\neg\alpha'$.

Proposition. The system LPC+Ch is consistent with respect to the negation.

P r o o f . We will show that the PC-transform of every provable M-formula in the system LPC+Ch, every theorem of LPC+Ch, is a valid formula of PC by showing that the PC-transform of every axiom is a valid formula of PC and the inference rules preserve this property.

This obviously holds for the axioms of PC, since they are themselves valid formulas of PC. It is also easy to see that the PC-transforms of the axioms of LPC are valid formulas of PC.

The PC-transform of Ax Str is $\alpha' \to \alpha'$ which is valid formula of PC. The PC-transform of Ax Id is $\alpha' \leftrightarrow \alpha'$ which is valid formula of PC.

Every theorem of LPC+Ch is either an axiom or an M-formula obtained from one or more axioms by the inference rules MP, GMP and RS.

Let α', $\beta', \dots$ be respectively the PC-transforms of α, $\beta, \dots$. Suppose β is obtained by MP from α and $\alpha \to \beta$. The PC-transforms of α and $\alpha \to \beta$ are respectively α' and $(\alpha \to \beta)'$. But $(\alpha \to \beta)'$ is the same formula as $\alpha' \to \beta'$. Hence β' may be obtained from α' and $(\alpha \to \beta)'$ by MP in PC. But MP also preserves validity in PC.

The PC-transform of GMP reduces to that of MP.

The PC-transforms of α is identical with that of $\mathcal{F}_i(\alpha)$ for any $\mathcal{F}_i \in \mathbf{0}$. Hence if β is obtained from α by RS, and α' is valid, so is β'.

The PC-transforms of every theorem of LPC+Ch is therefore a valid formula of PC. It follows that for every M-formula α of LPC+Ch, α and $\neg\alpha$ are not both theorems; for if they were, α' and $\neg\alpha'$ would both be valid formulas of PC, which we already know to be impossible. Hence LPC+Ch is consistent with respect to the negation. $\square$

We already mentioned above that a number of distinct M-formulas can have the same PC-transform. In fact an M-formula which is not a theorem of LPC+Ch sometimes has the same PC-transform as one which is. This happens e. g. in the case of $\mathcal{F}(\alpha) \to \alpha$ and $\alpha \to \mathcal{F}(\alpha)$, $\mathcal{F} \in \mathbf{0}$. The former is a theorem while the latter is not, but they both have the PC-transform $\alpha' \to \alpha'$. This gives us a further important result, viz, that LPC+Ch is not maximal consistent (as has been noted already in [2]). For the fact that $\alpha \to \mathcal{F}(\alpha)$ has a valid PC-transform shows that it could be added to LPC+Ch without the system being thereby made inconsistent.

According to the consistency with respect to interpretation, LPC+Ch has several models. If we interpret all the substantiating operators $\mathcal{F}_i \in \mathbf{0}$ as necessities of different strength then some models of multimodal T are models of LPC+Ch (see also some few details in [3]).

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[3] J. K. Mattila: Modified modus ponens and modal logic. Kybernetika (Supplement) *28* (1992), 28–32.

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