

## MODIFIED QUASILINEAR FILTERING METHOD FOR ESTIMATION OF PROCESSES IN MULTIDIMENSIONAL NONLINEAR STOCHASTIC SYSTEMS

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The modified quasilinear filtering method has been proposed. This method produces more accurate filter coefficients than the standard quasilinear filtering method. To compute these coefficients, it suffices to know the distribution of the state vector of a stochastic system. It can be determined by using the software for statistical analysis of multidimensional nonlinear stochastic systems. All computations connected with the determination of coefficients of the modified quasilinear filters do not use the results of observations. Therefore they can be computed before the filter design. An example is also presented.

### 1. INTRODUCTION

The methods of optimal filtering of processes determined by nonlinear stochastic differential equations usually yield complicated solutions that are difficult to realize. In practical problems, it is common to use suboptimal filtering methods for estimation of state variables and unknown parameters in stochastic systems. In the case of multidimensional nonlinear systems the practical realization of the extended Kalman filter, as well as similar filters [5], involves considerable computational difficulties. The use of a continuous extended Kalman filter is based on solving  $n(n+3)/2$ ,  $n$  being the dimension of state vector, differential equations for estimation of the state vector directly during the process of observation results. The quasilinear filtering method is therefore used in the case of multidimensional systems.

This method is based on statistical linearization of nonlinear functions of the original system, followed by Kalman filtering of the thus-obtained linearized system. The statistical linearization coefficients are computed under the assumption of normality of the distribution of the state vector. The practical merit of quasilinear filters is the simple realization of these filters in real time, especially in the case of dynamic systems of large dimensions.

The aim of this paper is to develop alternative methods for more accurate computation of the statistical linearization and quasilinear filter coefficients. These methods can improve the accuracy of quasilinear filters.

## 2. STATEMENT OF THE PROBLEM

Let us consider a continuous dynamic system determined by a stochastic differential equation

$$\dot{x} = f(x, t) + G(t) w(t), \quad t \geq t_0. \quad (1)$$

The observed process  $y = y(t)$  is determined by

$$y = h(x, t) + v(t). \quad (2)$$

In equations (1) and (2),  $x \in \mathbb{R}^n$  is the state vector of the system,  $y \in \mathbb{R}^d$  is the vector of observations,  $f(x, t)$  and  $h(x, t)$  are given functions mapping  $\mathbb{R}^n \times \mathbb{R}$  into  $\mathbb{R}^n$  and  $\mathbb{R}^d$  respectively,  $G(t)$  is  $(n \times r)$  matrix,  $\mathbb{R}^n$  is an  $n$ -dimensional Euclidean space,  $w \in \mathbb{R}^r$  and  $v \in \mathbb{R}^d$  are independent normal white noises,  $E[w(t)] = E[v(t)] = 0$ ,  $E[w(t)w(\tau)^T] = Q(t)\delta(t - \tau)$  and  $E[v(t)v(\tau)^T] = R(t)\delta(t - \tau)$ . Initial value  $x_0 = x(t_0)$  is independent of  $v(t)$  and  $w(t)$ ,  $t \geq t_0$ . The matrix  $R(t)$  is uniformly nonsingular in  $t$ .

On the basis of observations  $y_{t_0}^t = \{y(\tau), t_0 \leq \tau \leq t\}$  it is required to estimate the state vector  $x(t)$  by the minimum mean square error criterion. For approximately solving this nonlinear filtering problem, we shall use statistical linearization method.

## 3. THE MODIFIED QUASILINEAR FILTERING METHOD

Let us approximate the nonlinear functions  $f(x, t)$  and  $h(x, t)$  in the equations (1) and (2) by using statistical linearization method [3, 10]. Then we obtain

$$f(x, t) \cong a_0 + a_1(x - m), \quad h(x, t) \cong b_0 + b_1(x - m), \quad m = E[x]. \quad (3)$$

The statistical linearization coefficients  $a_0, a_1, b_0$  and  $b_1$  are determined by minimizing the mean square error. The formulae for these coefficients take the form

$$a_0 = E[f(x, t)], \quad a_1 = E[f(x, t)(x - m)]K^{-1}, \quad (4)$$

$$b_0 = E[h(x, t)], \quad b_1 = E[h(x, t)(x - m)]K^{-1}. \quad (5)$$

Here  $m = E[x]$  and  $K = E[(x - m)(x - m)^T]$  are expectation and covariance matrix of the state vector  $x = x(t)$ , respectively. Substituting the expressions (3) into (1) and (2), we have

$$\dot{\hat{x}} = a_1\hat{x} + a_0 - a_1m + Gw, \quad y = b_1\hat{x} + b_0 - b_1m + v. \quad (6)$$

The system (5) is a model of system (1) and (2). To this model the Kalman filter equations take the form

$$\dot{\hat{x}} = a_1\hat{x} + a_0 - a_1m + Pb_1^T R^{-1}(y - b_1\hat{x} - b_0 + b_1m), \quad (7)$$

$$\dot{P} = a_1P + Pa_1^T - Pb_1^T R^{-1}b_1P + GQG^T, \quad (8)$$

where  $\hat{x} = \hat{x}(t)$  is the estimate of the state vector  $x = x(t)$  and  $P = P(t)$  is the auxiliary  $n \times n$  symmetric matrix. The initial conditions for (6) and (7) are

$\hat{x}_0 = E[x_0]$  and  $P_0 = E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T]$ , respectively. We notice that as different from optimal linear Kalman filter the matrix  $P$  is not the conditional covariance matrix of the filtering error. It is assumed that observability conditions for the nonlinear system (1), (2) and the linearized system (5) are fulfilled [4, 6].

In the standard case the statistical linearization coefficients  $a_0$ ,  $a_1$ ,  $b_0$  and  $b_1$  in (4) are determined approximately under the assumption of normality of the distribution of the state vector  $x(t)$ . In this case the filter (6) and (7) is called the quasilinear filter (QLF) [2, 3, 10].

To increase the accuracy of computation of the coefficients  $a_0$ ,  $a_1$ ,  $b_0$  and  $b_1$  of the QLF (6) and (7), the various approximate methods based on the parametrization of distributions of the state vector are proposed. Approximating the unknown density  $p(x; t)$  or characteristic function  $g(\lambda; t)$  of the state vector  $x(t)$  by some known functions  $p^*(x; \theta)$  and  $g^*(\lambda; \theta)$  depending on a finite-dimensional vector parameter  $\theta$ , we reduce the problem of approximate determination of the distribution to the problem of determination of the vector parameter  $\theta = \theta(t)$  as function of time  $t$ . The vector parameter  $\theta$  may be represented by the set of moments, semi-invariants(cumulants) and quasi- moments [7]. This vector  $\theta$  is defined by ordinary differential equations. The filter (6) and (7) with coefficients  $a_0$ ,  $a_1$ ,  $b_0$  and  $b_1$  determined by using the various methods based on parametrization of distributions such as the method of moments, the method of semi-invariants, the method of quasi-moments and others, we shall call the modified quasilinear filter (MQLF).

Now we pay attention to two circumstances. First, equations determining the statistical linearization coefficients  $a_0$ ,  $a_1$ ,  $b_0$ ,  $b_1$  and the auxiliary matrix  $P$  in (6) and (7) do not contain the results of observations  $y(t)$ , and consequently, may be determined separately (beforehand when the observations are not yet performed). Then the estimate  $\hat{x}$  will be determined by the integration of equation of the MQLF(6) with only the current results of the observations. The equation (6) is sufficiently simple and may be integrated in real time basis during the observation of the studied dynamic system behavior. Secondly, the methods based on parametrization of distributions enable us to determine the true mean square filtering error  $E[(x - \hat{x})^T(x - \hat{x})]$  with any degree of accuracy. The calculations of the accuracy of MQLF do not use the results of observations and prior data are only necessary to fulfill them.

#### 4. PARAMETRIZATION OF DISTRIBUTIONS. METHOD OF MOMENTS FOR CALCULATING THE COEFFICIENTS OF THE MQLF

The coefficients  $a_i$  and  $b_i$  of the MQLF (6) and (7) depend on the expectations in (4). To calculate these expectations it is sufficient to know the one-dimensional distribution (density or characteristic function) of the state vector  $x$  determined by (1). In the general case the density of vector  $x$  is determined by the multi-dimensional Fokker–Planck–Kolmogorov (FPK) equation. The most general methods to find an approximate solution of FPK equation are based on parametrization of distributions, i. e., representation of unknown density in the form of various truncated series (in particular, the Edgeworth series). Among these are the moments method,

the semi-invariants method and the quasi-moments method [7]. These methods allow a practical solution with any degree of accuracy.

To be specific, let us consider the method of moments for calculating the coefficients of the MQLF. In this method the vector parameter  $\theta$  on which depends the function  $p^*(x; \theta)$  approximating the unknown density represents the set of initial or central moments of the state vector  $x \in \mathbb{R}^n$  up to a given order  $N$  :

$$\alpha_{\gamma_1 \dots \gamma_n}(t) = E[x_1(t)^{\gamma_1} \dots x_n(t)^{\gamma_n}], \tag{9}$$

$$\begin{aligned} \mu_{\gamma_1 \dots \gamma_n}(t) &= E[(x_1(t) - m_1)^{\gamma_1} \dots (x_n(t) - m_n)^{\gamma_n}], \\ m_i &= E[x_i(t)], \quad i = 1, \dots, n, \quad \gamma_1 + \dots + \gamma_n = 1, \dots, N, \end{aligned} \tag{10}$$

where  $x_1, \dots, x_n$  and  $m_1, \dots, m_n$  are the components of the state vector  $x$  and its expectation  $m = E[x]$ , respectively.

We obtain from (1) the exact ordinary differential equations for the initial moments  $\alpha_{\gamma_1 \dots \gamma_n}$  [7]

$$\begin{aligned} \dot{\alpha}_{\gamma_1 \dots \gamma_n} &= \sum_{s=1}^n \gamma_s E[f_s(x, t) x_1^{\gamma_1} \dots x_{s-1}^{\gamma_{s-1}} x_s^{\gamma_s-1} x_{s+1}^{\gamma_{s+1}} \dots x_n^{\gamma_n}] \\ &+ \frac{1}{2} \sum_{s=1}^n \gamma_s(\gamma_s - 1) \sigma_{ss} E[x_1^{\gamma_1} \dots x_{s-1}^{\gamma_{s-1}} x_s^{\gamma_s-2} x_{s+1}^{\gamma_{s+1}} \dots x_n^{\gamma_n}] \\ &+ \sum_{h=2}^n \sum_{s=1}^{h-1} \gamma_s \gamma_h \sigma_{sh} E[x_1^{\gamma_1} \dots x_{s-1}^{\gamma_{s-1}} x_s^{\gamma_s-1} x_{s+1}^{\gamma_{s+1}} \dots x_{h-1}^{\gamma_{h-1}} x_h^{\gamma_h-1} x_{h+1}^{\gamma_{h+1}} \dots x_n^{\gamma_n}], \\ &\quad \gamma_1 + \dots + \gamma_n = 1, \dots, N, \end{aligned} \tag{11}$$

where

$$f(x, t) = [f_1(x, t) \dots f_n(x, t)]^T, \quad \sigma(t) = G(t)Q(t)G(t)^T, \quad \sigma = [\sigma_{sh}] \quad (s, h = 1, \dots, n).$$

The method of moments computes the expectations in (10) by replacing the true unknown density  $p(x; t)$  of the state vector  $x(t)$  with some approximate function  $p^*(x; \theta)$ , which is entirely determined by the moments up to the  $N$ th order (Edgeworth series). We have

$$p(x; t) \cong p^*(x; \theta) = N(m, K) \left[ 1 + \sum_{k=3}^N \sum_{\nu_1 + \dots + \nu_n = k} \frac{c_{\nu_1 \dots \nu_n}}{\nu_1! \dots \nu_n!} H_{\nu_1 \dots \nu_n}(x - m) \right], \tag{12}$$

where  $\theta$  is the vector parameter consisting of the moments up to the  $N$ th order,  $N(m, K)$  is normal density with the expectation  $m$  and the covariance matrix  $K$  and  $\{H_\nu(x)\}$  is the sequence of Hermite polynomials which satisfy the biorthogonality condition

$$\int_{\mathbb{R}^n} w(x) H_\nu(x) G_\mu(x) dx = \nu_1! \dots \nu_n! \delta_{\nu\mu}$$

with another sequence of polynomials  $\{G_\nu(x)\}$ . Here  $w(x) = N(0, K)$  is the weight function,  $\nu = [\nu_1 \dots \nu_n]$  and  $\mu = [\mu_1 \dots \mu_n]$  are multi-indexes,  $\delta_{\nu\mu}$  is the Kronecker symbol,  $\delta_{\nu\nu} = 1$  and  $\delta_{\nu\mu} = 0$  ( $\nu \neq \mu$ ). Let  $\bar{w}(x)$  be some density in the  $n$ -dimensional space  $\mathbb{R}^n$ . A system of pairs of polynomials  $\bar{p}_i(x)$  and  $\bar{q}_i(x)$  ( $i = 0, 1, 2, \dots$ ) is called a biorthogonal system with the weight  $\bar{w}(x)$  if

$$\int_{\mathbb{R}^n} \bar{w}(x) \bar{p}_i(x) \bar{q}_j(x) dx = \begin{cases} 0, & i \neq j \\ \text{constant}, & i = j. \end{cases} \tag{13}$$

In special case where  $\bar{p}_i(x) = \bar{q}_i(x)$  ( $i = 0, 1, 2, \dots$ ) the condition (12) takes the form

$$\int_{\mathbb{R}^n} \bar{w}(x) \bar{p}_i(x) \bar{p}_j(x) dx = \begin{cases} 0, & i \neq j \\ \text{constant}, & i = j. \end{cases} \tag{14}$$

In this case the system of polynomials  $\{\bar{p}_i(x)\}$  is orthogonal if it satisfies the condition (13). The sequences of Hermite polynomials  $\{\bar{p}_i(x), \bar{q}_i(x)\} = \{H_i(x), G_i(x)\}$  may serve as examples of biorthogonal systems [7]. Hermite polynomials are often used for an approximate representation of densities.

The coefficients  $c_\nu(t) = c_{\nu_1 \dots \nu_n}(t)$  in (11) are given by

$$c_\nu = \int_{\mathbb{R}^n} p(x; t) G_\nu(x - m) dx = E[G_\nu(x - m)] = G_\nu(\mu), \tag{15}$$

where  $G_\nu(\mu)$  is a linear combination of central moments of the state vector  $x = x(t)$  obtained as a result of replacement in  $G_\nu(x - m)$  of every monomials

$$(x_1 - m_1)^{h_1} \dots (x_n - m_n)^{h_n}$$

by the corresponding central moment  $\mu_{h_1 \dots h_n}$  (9).

Substituting approximate density  $p^*(x; \theta)$  into the expectations in (10) and evaluating that for the coefficients  $c_\nu$  (14) expressed in terms of initial moments, we obtain the differential equations for all initial moments up to the  $N$ th order. These equations are numerically integrated with the initial conditions

$$\alpha_{\gamma_1 \dots \gamma_n}(t_0) = E[x_1(t_0)^{\gamma_1} \dots x_n(t_0)^{\gamma_n}], \quad \gamma_1 + \dots + \gamma_n = 1, \dots, N.$$

In particular case of  $N = 2$  and normal density  $p^*(x; \theta) = N(m, K)$ , (10) represents the equations for the first- and second-order moments  $m$  and  $K$  of the state vector  $x$ . And in this case the equations of the MQLF coincide with those of the QLF. Thus the coefficients of the MQLF (6) and (7) may be determined by the method of moments. Software for determining the distributions of the state vector  $x$  of the continuous and discrete stochastic systems has been developed for various classes of computers. It contains various methods of statistical analysis of the dynamic systems based on parametrization of distributions [1, 8, 9]. This computer software include standard programs for automatic derivation and solution of the differential and difference equations for the distribution parameters (such as moments, semi-invariants and quasi-moments) of the state vector of stochastic systems of arbitrary dimensions with polynomial and some nonpolynomial nonlinearities.

Note that while using a truncated orthogonal expansion small negative values of the density are sometimes obtained. It is natural that any truncated series gives only an approximate value of the function. In using the approximate methods in the form of various truncated series, it is recommended to vary both the number of terms of the series and the form of series. For this purpose the special software is used [1, 8, 9].

We summarize these results in the following theorem.

**Theorem 4.1.** (Modified Quasilinear Filter) Given  $\hat{x}$  and  $P$ , the modified quasilinear filter for the model (1) and (2) is obtained by the following relations:

$$\dot{\hat{x}} = a_1\hat{x} + a_0 - a_1m + Pb_1^T R^{-1}(y - b_1\hat{x} - b_0 + b_1m), \tag{16}$$

$$\dot{P} = a_1P + Pa_1^T - Pb_1^T R^{-1}b_1P + GQG^T, \tag{17}$$

where coefficients  $a_0, a_1, b_0$  and  $b_1$  are determined by using the various methods based on parametrization of distributions. In the case of the method of moments,  $a_0$  and  $a_1$  are obtained by (10).

5. EXAMPLE

Let the dynamic system be described by scalar equation

$$\dot{x} = -x^3 + w, \quad t \geq 0 \tag{18}$$

and the observed process  $y(t)$  be determined by

$$y = x + v. \tag{19}$$

Here  $w = w(t)$  and  $v = v(t)$  are independent normal white noises,  $E[w(t)w(\tau)] = Q\delta(t - \tau)$  and  $E[v(t)v(\tau)] = R\delta(t - \tau)$ , where  $Q$  and  $R$  are known constants. Initial state  $x_0 = x(0)$  is normal random variable with known expectation  $m_0 = E[x_0]$  and variance  $\mathcal{D}_0 = E[(x_0 - m_0)^2]$ . The statistical linearization of the function  $f(x, t) = -x^3$  takes the form (3)

$$-x^3 \cong a_0 + a_1(x - m), \quad m = m(t) = E[x(t)]. \tag{20}$$

The coefficients  $a_0$  and  $a_1$  are determined by the formulae (4)

$$a_0 = -E[x^3], \quad a_1 = -E[x^3(x - m)]/\mathcal{D}, \quad \mathcal{D} = \mathcal{D}(t) = E[(x(t) - m(t))^2]. \tag{21}$$

For finding the estimate of the state variable  $x$ , we shall apply the QLF and MQLF. The equations of the QLF take the form

$$\begin{aligned} \dot{\tilde{x}} &= \bar{a}_1\tilde{x} + \bar{a}_0 - \bar{a}_1m + \bar{P}(y - \tilde{x})/R, \quad \tilde{x}_0 = m_0, \\ \dot{\bar{P}} &= 2\bar{a}_1\bar{P} - \bar{P}^2/R + Q, \quad \bar{P}_0 = \mathcal{D}_0, \end{aligned} \tag{22}$$

where  $\tilde{x} = \tilde{x}(t)$  is the estimate of the state variable  $x = x(t)$  and the statistical linearization coefficients  $\bar{a}_0$  and  $\bar{a}_1$  are evaluated for normal density  $p^*(x; \theta) = N(m, \mathcal{D})$

by using the recursive formula for high-order moments  $\alpha_k = E[x^k]$  of normal distributed random variable,

$$\alpha_k = \alpha_1 \alpha_{k-1} + (k-1) \mathcal{D} \alpha_{k-2}, \quad \alpha_1 = E[x], \quad \mathcal{D} = \alpha_2 - \alpha_1^2, \quad k = 3, 4, \dots \quad (23)$$

We have

$$\begin{aligned} a_0 &\cong \bar{a}_0 = -(\alpha_1 \alpha_2 + 2\mathcal{D} \alpha_1) = 2\alpha_1^3 - 3\alpha_1 \alpha_2, \\ a_1 &\cong \bar{a}_1 = -3(\alpha_1^2 + \mathcal{D}) = -3\alpha_2. \end{aligned}$$

The differential equations for  $\alpha_1$  and  $\alpha_2$  take the form (10)

$$\dot{\alpha}_1 = -\alpha_3, \quad \dot{\alpha}_2 = -2\alpha_4 + Q, \quad (24)$$

where the high-order moments  $\alpha_3$  and  $\alpha_4$  are evaluated for normal density by using the formula (22), i. e.,

$$\alpha_3 = \alpha_1 \alpha_2 + 2\mathcal{D} \alpha_1 = 3\alpha_1 \alpha_2 - 2\alpha_1^3, \quad \alpha_4 = \alpha_1 \alpha_3 + 3\mathcal{D} \alpha_2 = 3\alpha_2^2 - 2\alpha_1^4.$$

The equations of the MQLF take the form

$$\begin{aligned} \dot{\hat{x}} &= a_1 \hat{x} + a_0 - a_1 m + P(y - \hat{x})/R, \quad \hat{x}_0 = m_0, \\ \dot{P} &= 2a_1 P - P^2/R + Q, \quad P_0 = \mathcal{D}_0, \end{aligned} \quad (25)$$

where  $\hat{x} = \hat{x}(t)$  is the estimate of the state variable  $x = x(t)$  and the statistical linearization coefficients  $a_0$  and  $a_1$  are determined by the formulae (4)

$$a_0 = -\alpha_3, \quad a_1 = (-\alpha_4 + \alpha_1 \alpha_3) / (\alpha_2 - \alpha_1^2). \quad (26)$$

For determining the initial moments  $\alpha_k = \alpha_k(t)$ ,  $k = 1, 2, 3, 4$ , we use the method of moments of the fourth order ( $N = 4$ ). According to (10) the differential equations for  $\alpha_k$  take the form

$$\dot{\alpha}_1 = -\alpha_3, \quad \dot{\alpha}_2 = -2\alpha_4 + Q, \quad \dot{\alpha}_3 = -3\alpha_5 + 3Q\alpha_1, \quad \dot{\alpha}_4 = -4\alpha_6 + 6Q\alpha_2. \quad (27)$$

The approximation of the density  $p(x; t)$  of the state variable  $x$  take the form (11)

$$p(x; t) \cong p^*(x; \theta) = N(m, \mathcal{D}) \left[ 1 + \frac{c_3}{3!} H_3(x - m) + \frac{c_4}{4!} H_4(x - m) \right], \quad (28)$$

where Hermite polynomials  $H_\nu(x)$ ,  $G_\nu(x)$  and the coefficients  $c_3$  and  $c_4$  are determined by the formulae

$$\begin{aligned} H_\nu(x) &= (-1)^\nu \exp\left(\frac{x^2}{2\mathcal{D}}\right) \frac{d^\nu}{dx^\nu} \left[ \exp\left(-\frac{x^2}{2\mathcal{D}}\right) \right], \\ G_\nu(x) &= (-1)^\nu \exp\left(\frac{x^2}{2\mathcal{D}}\right) \left\{ \frac{d^\nu}{dy^\nu} \left[ \exp\left(-\frac{\mathcal{D}y^2}{2}\right) \right] \right\}_{y=x/\mathcal{D}}, \\ c_3 &= E[G_3(x - m)], \quad c_4 = E[G_4(x - m)]. \end{aligned}$$

These formulae give the following recursive expressions for Hermite polynomials

$$H_{\nu+1}(x) = \frac{x}{\mathcal{D}}H_{\nu}(x) - \frac{\nu}{\mathcal{D}}H_{\nu-1}(x), \quad G_{\nu+1}(x) = xG_{\nu}(x) - \nu\mathcal{D}G_{\nu-1}(x).$$

In this particular case, we have

$$\begin{aligned} H_0 &= 1, & H_1 &= x/\mathcal{D}, & H_2 &= (x^2 - \mathcal{D})/\mathcal{D}^2, \\ H_3 &= (x^3 - 3\mathcal{D}x)/\mathcal{D}^3, & H_4 &= (x^4 - 6\mathcal{D}x^2 + 3\mathcal{D}^2)/\mathcal{D}^4, \\ G_0 &= 1, & G_1 &= x, & G_2 &= x^2 - \mathcal{D}, \\ G_3 &= x^3 - 3\mathcal{D}x, & G_4 &= x^4 - 6\mathcal{D}x^2 + 3\mathcal{D}^2 \end{aligned} \tag{29}$$

and

$$c_3 = \alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3, \quad c_4 = \alpha_4 - 4\alpha_1\alpha_3 + 12\alpha_1^2\alpha_2 - 3\alpha_2^2 - 6\alpha_1^4. \tag{30}$$

Thus the vector parameter  $\theta$  in (27) represents a 4-dimensional vector

$$\theta = [\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4]^T.$$

To close the equations (26) it is necessary to represent the high-order moments  $\alpha_5$  and  $\alpha_6$  in terms of lower-order moments  $\alpha_k$  ( $k = 1, 2, 3, 4$ ). According to the method of moments and using the approximation of density  $p^*(x; \theta)$  (27), (28) and (29), we obtain

$$\begin{aligned} \alpha_k &= \int_{-\infty}^{\infty} x^k p^*(x; \theta) dx \\ &= \int_{-\infty}^{\infty} x^k N(m, \mathcal{D}) dx + \frac{c_3}{6} \int_{-\infty}^{\infty} x^k H_3(x - m) N(m, \mathcal{D}) dx \\ &\quad + \frac{c_4}{24} \int_{-\infty}^{\infty} x^k H_4(x - m) N(m, \mathcal{D}) dx \quad (\text{at } k = 5, 6). \end{aligned} \tag{31}$$

Denote by

$$I_k = \int_{-\infty}^{\infty} x^k N(m, \mathcal{D}) dx, \quad J_{k,\nu} = \int_{-\infty}^{\infty} x^k H_{\nu}(x - m) N(m, \mathcal{D}) dx. \tag{32}$$

Then the integrals  $I_k$  and  $J_{k,\nu}$  are calculated by the recursive formulae

$$\begin{aligned} I_k &= mI_{k-1} + (k-1)\mathcal{D}I_{k-2}, \quad I_1 = m = \alpha_1, \quad I_2 = \mathcal{D} + m^2 = \alpha_2, \\ J_{k,\nu+1} &= \frac{1}{\mathcal{D}}(J_{k+1,\nu} - mJ_{k,\nu} - \nu J_{k,\nu-1}), \quad J_{k,0} = I_k, \quad J_{k,1} = \frac{1}{\mathcal{D}}(I_{k+1} - mI_k). \end{aligned} \tag{33}$$

Thus the system of ordinary differential equations for initial moments  $\alpha_k$  (26) in which the high-order moments  $\alpha_5$  and  $\alpha_6$  are calculated by the formulae (30), (31) and (32) completely determine the method of moments of the fourth-order.

All the differential equations for the moments (23) and (26) have been numerically integrated by the Euler and Adams–Moulton method, respectively, with a step  $\Delta t = 0.01$ . Initial values are given by  $Q = 0.05$ ,  $R = 0.05$ ,  $m_0 = 4.0$  and  $D_0 = 0.05$ . The results of Monte–Carlo simulation, performed with identical noise sequences for the QLF algorithm (21) and the MQLF algorithm (24), are compared in Figure 1.



**Fig. 1.** Performance of the QLF and MQLF.

Figure 2 shows the true mean square filtering errors

$$P_{qlf} = E(x - \tilde{x})^2, \quad P_{mqf} = E(x - \hat{x})^2$$

for the QLF and MQLF, respectively. The mean square filtering errors  $P_{qlf}$  and  $P_{mqf}$  have been calculated by solving the corresponding statistical analysis problems of the 2-dimensional random processes  $[x \ \tilde{x}]^T$  and  $[x \ \hat{x}]^T$  determined by the stochastic differential equations (17), (18) and (21) and (17), (18) and (24), respectively.

**Fig. 2.** Mean Square Filtering Errors.

The joint density of  $x$  and  $\tilde{x}$  ( $x$  and  $\hat{x}$ ) have been calculated by using the method of moments of the fourth-order. The mean square errors  $P_{qlf}$  and  $P_{mqf}$  estimate the accuracy of filters. As illustrated in Figure 2 the MQLF is better than the QLF. At  $t = 1.0$  we have  $P_{mqf}(1.0) = 0.015$ ,  $P_{qlf}(1.0) = 0.0163$  and relative error  $\epsilon = \epsilon(1.0) = |(P_{mqf} - P_{qlf})/P_{mqf}| \times 100\% = 8.6\%$ . It is a good result especially for high-accuracy control system design (the oscillations of an aircraft in a turbulent air, the gyroscope drift and others). We note that the increasing of accuracy of the MQLF during the filtering is achieved without increasing complexity of computations since the calculations of  $a_i$  and  $b_i$  of the MQLF (6) and (7) don't depend on

measurements and therefore they can be calculated before the filter design. Thus the results of a comparative analysis show that the MQLF are more accurate than the QLF.

## 6. CONCLUSION

The MQLF can be used for a wide range of control problems of large dimensional systems. As a linear Kalman filtering theory, all the computations connected with the determination of a filter gains ( $a_0$ ,  $a_1$  and  $P$ ) are based on a priori data about the model (1) and (2) and do not use the results of observations. Therefore they can be carried out before the MQLF design. The determination of the estimate  $\hat{x}(t)$  from the results of observations reduces to integration of the differential equation (6), which can be done in a real time.

The MQLF also enable us to estimate the state of the system under conditions of uncertainty, when the functions  $f$  and  $h$  in (1) and (2) depend on unknown parameters. For this purpose it is necessary to extend the state vector  $x$  by including in it the unknown parameters as additional components. The MQLF can be used for the discrete and continuous-discrete filtering problems.

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