KYBERNETIKA — VOLUME *30* (1994), NUMBER 6, PAGES 725 – 743

ROBUST CONTINUOUS-TIME TRACKING AND REGULATION FOR MULTIRATE SAMPLED-DATA SYSTEMS

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In this paper, the robust ripple-free tracking and disturbance rejection problem is solved for multirate sampled-data systems whose matrices are assumed to depend on some "physical" parameters. Making use of a hybrid control system structure, including a continuous-time internal model of the exogenous signals and a periodic discrete-time subcompensator, a ripple-free null steady-state error response is obtained in a neighbourhood of the nominal "physical" parameters of the plant, and a ripple-free dead-beat error response at the nominal ones.

1. INTRODUCTION

The problem of the asymptotic tracking and disturbance rejection of a linear multivariable system subject to unmeasurable disturbances was studied by many authors — see, e. g., [2, 3, 5, 7, 8, 9], [19, 20]), and the references therein. In most of these contributions it is required the compensator to maintain stability, asymptotic tracking and output regulation in spite of independent perturbations of the elements of matrices describing the system. In [14, 16, 17] such a problem was solved for uncertainties or perturbations of "physical" parameters affecting the description of the system.

If the problem of the asymptotic tracking is faced for a continuous-time plant making use of a multirate digital control system, the undesirable ripple which may arise between sampling instants may become unacceptable, especially if the sampling rates are small, and should be avoided [4, 10, 21, 24, 25]. For the single-rate case this can be robustly obtained if a continuous-time internal model of reference signals is included in the forward path of the feedback control system [4, 10].

Here a method for deriving such a continuous-time internal model of both reference signals and disturbance functions is presented for a multirate hybrid control system structure, including a periodic discrete-time subcompensator as in [6]. For the case when the only uncertainties about the description of the plant concern the values of some "physical" parameters, such a control system allows the control requirements to be robustly satisfied, at least in a neighbourhood of the nominal physical parameters of the plant to be controlled, and, in particular, a continuous-time null (i. e., ripple-free) steady-state error response to be guaranteed for all the values of the physical parameters in such a neighbourhood of the nominal ones. Making use of the hybrid control system structure here presented, a continuous-time dead-beat (i. e., ripple-free) convergence of the error response is also obtained at the nominal parameters.

2. PRELIMINARIES

Consider the linear time-invariant plant $\mathcal{P}$ described by

$$\dot{x}(t) = A(\beta)\, x(t) + B(\beta)\, u(t) + M(\beta)\, d(t), \tag{1}$$

$$y(t) = C(\beta)\, x(t) + N(\beta)\, d(t), \tag{2}$$

where $t \in \mathbb{R}$ is time, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the control input, $d(t) \in \mathbb{R}^m$, is the unmeasurable disturbance input, $y(t) \in \mathbb{R}^q$ is the output to be controlled — which is assumed to be measurable — and $A(\beta)$, $B(\beta)$, $C(\beta)$, $M(\beta)$, $N(\beta)$, are matrices with real entries depending on a vector β of parameters, which are subject to variations and/or uncertain, $\beta \in \Omega \subseteq \mathbb{R}^h$, and play the role of the "physical" parameters of the plant. The nominal value β_0 of β is assumed to be an interior point of the set Ω. It is assumed that each of the first $\overline{q}$ components $y_1(t), \ldots, y_{\overline{q}}(t)$ of $y(t)$ must track the corresponding component of the reference vector $r(t) \in \mathbb{R}^{\overline{q}}$, $\overline{q} \leq q$. Therefore, the error signal $e(t) \in \mathbb{R}^q$ for $\mathcal{P}$ is defined by

$$e(t) := y(t) - V r(t), \qquad V := [I_{\bar{q}}\ \ 0]^{\mathrm{T}}, \tag{3}$$

where $I_{\bar{q}}$ is the identity matrix of dimension $\bar{q}$.

It is also assumed that the reference signals $r(\cdot)$ to be asymptotically tracked and the disturbance functions $d(\cdot)$ to be asymptotically rejected are the free output responses of the following exosystem $\mathcal{E}$, whose initial state $z(0)$ is unknown:

$$\dot{z}(t) = F\, z(t), \qquad z(t) \in \mathbb{R}^{\ell} \tag{4}$$

$$r(t) = G\, z(t), \qquad d(t) = H\, z(t). \tag{5}$$

Denote by α_i, $i = 1, \ldots, \mu$, the μ distinct eigenvalues of matrix F, and by $k_i \in \mathbb{Z}^+$ the multiplicity of α_i in the minimal polynomial of F, $i = 1, \ldots, \mu$,where $\mathbb{Z}^+$ is the set of positive integers. It is assumed that the α_i, $i = 1, \ldots, \mu$, are all real and non-negative.

It is also stressed that (4), (5) do not imply that $r(t)$ and $d(t)$ are restricted to contain the same modes; e. g., this is not true for $F = \operatorname{diag}\{\alpha_1 I_{\bar{q}}, \alpha_2 I_m\}$, $\ell = \bar{q} + m$, $G = [I_{\bar{q}}\ 0]$, $H = [0\ I_m]$, and $\alpha_1 \neq \alpha_2$.

When a multirate digital control system is used for plant $\mathcal{P}$, it seems to be reasonable to require for the error $e(t)$ at the nominal parameter $\beta = \beta_0$ not only the dead-beat convergence to zero at the sampling times, but the stronger ripple-free dead-beat convergence, i. e.

$$e(t) = 0, \qquad \forall\, t \geq \bar{t}, \quad t \in \mathbb{R} \tag{6}$$

for some $\bar{t} \in \mathbb{R}, \bar{t} > 0$. Since it seems to be unrealistic to maintain such a property despite of parameter perturbations, for β different from β_0 and belonging to a suitable neighbourhood of β_0 this requirement is weakened to the ripple-free convergence to zero, i. e.,

$$\lim_{t \to +\infty} e(t) = 0 \tag{7}$$

(which is much more than the convergence to zero at the sampling times). It is known that a continuous-time internal model of reference signals is needed in the forward path of the feedback control system for such ripple-free requirements to be satisfied (see [10] for the single-rate case). Therefore, if such an internal model is not entirely contained in the plant $\mathcal{P}$ for $\beta = \beta_0$ and for all β in some neighbourhood of β_0, a continuous-time precompensator $\mathcal{K}_C$ to be connected in series with $\mathcal{P}$ should provide the missing part of the internal model. This justifies the use of the control scheme reported in Fig. 1, where $\mathcal{K}_C$ is a continuous-time time-invariant subcompensator described by

Fig. 1. Structure of the hybrid multirate sampled-data control system Σ.

$$\begin{aligned} \dot{w}_C(t) &= Q_C w_C(t) + R_C u_C(t), \qquad u_C(t) \in \mathbb{R}^p, &(8)\\ u(t) &= J_C w_C(t) + U_C u_C(t), &(9) \end{aligned}$$

$\mathcal{K}_D$ is a discrete-time periodic subcompensator described by

$$\begin{aligned} w_D(k+1) &= Q_D(k)\, w_D(k) + R_D(k)\, e_D(k), \qquad e_D(k) \in \mathbb{R}^q, &(10)\\ u_D(k) &= J_D(k)\, w_D(k), \qquad u_D(k) \in \mathbb{R}^p, &(11) \end{aligned}$$

and the block H_i, $i = 1, \ldots, p$, represents a zeroth-order holder, whose hold interval is $N_i T$, $N_i \in \mathbb{Z}^+$, which connects the ith component $u_{C,i}(t)$ of $u_C(t)$ with the ith component $u_{D,i}(k)$ of $u_D(k)$. It is assumed that for each component $y_i(\cdot)$ of $y(\cdot)$, $i = 1, \ldots, q$ (or, respectively, $r_i(\cdot)$ of $r(\cdot)$, $i = 1, \ldots, \bar{q}$), a discrete-time signal $y_{D,i}(\cdot)$ (or, respectively, $r_{D,i}(\cdot)$) is obtained by sampling $y_i(\cdot)$ (or, respectively $r_i(\cdot)$) with sampling period $Z_i T$, $Z_i \in \mathbb{Z}^+$, i. e.,

$$y_{D,i}(jZ_i) = y_i(jZ_iT), \qquad j = 0, 1, 2, \ldots, \tag{12}$$
$$y_{D,i}(k) = 0, \qquad k \neq jZ_i, \ \forall\, j \in \mathbb{Z}^+, \tag{13}$$

(or, respectively,

$$r_{D,i}(jZ_i) = r_i(jZ_iT), \qquad j = 0, 1, 2, \ldots, \tag{14}$$
$$r_{D,i}(k) = 0, \qquad k \neq jZ_i, \ \forall\, j \in \mathbb{Z}^+). \tag{15}$$

Denoting by $y_D(\cdot)$ and $r_D(\cdot)$ the discrete-time vector functions whose components are $y_{D,i}(k)$, $i = 1, \ldots, q$ and $r_{D,i}(k)$, $i = 1, \ldots, \bar{q}$, respectively, $e_D(k)$ in (10) is expressed by

$$e_D(k) := y_D(k) - V r_D(k), \tag{16}$$

and coincides with the multirate sampling of $e(t)$. It is also assumed that the integers N_i, $i = 1, \ldots, p$ and Z_i, $i = 1, \ldots, q$ have 1 as their greatest common divisor, and that all the hold devices and samplers are synchronized at time $t = 0$. The period ω characterizing the periodic matrices $Q_D(\cdot)$, $R_D(\cdot)$, $J_D(\cdot)$ in (10), (11) is chosen equal to the least common multiple of the integers N_i, $i = 1, \ldots, p$ and Z_i, $i = 1, \ldots, q$.

In view of requirement (6), to be considered for $\beta = \beta_0$, it is natural to require also, for $\beta = \beta_0$, a dead-beat convergence of the free state response of the over-all hybrid control system Σ represented in Fig. 1, instead of the mere exponential decay.

Therefore, the following control problem will be studied here.

Problem 1. (*Robust ripple-free tracking and regulation problem*) Find, if any, linear dynamic compensators $\mathcal{K}_D$ and $\mathcal{K}_C$, described by (10), (11) and (8), (9), respectively, with the matrices $Q_D(\cdot)$, $R_D(\cdot)$ and $J_D(\cdot)$ being periodic of period ω (briefly, ω-periodic), such that the following requirements are satisfied by the overall hybrid control system Σ represented in Fig. 1:

(a) at the nominal parameters of the plant $\mathcal{P}$, i. e., for $\beta = \beta_0$, for all the initial states of Σ at the initial time $t = 0$, the free state response of Σ (i. e., the state response of Σ for $z(0) = 0$) is identically zero for all times $t \geq \tilde{t}$, for some $\tilde{t} > 0$;

(b) for all the initial states $z(0)$ of $\mathcal{E}$ and for all the initial states of Σ, relation (6) is satisfied for some $\bar{t} > 0$, $\bar{t} \in \mathbb{R}$, at the nominal parameters of the plant $\mathcal{P}$, i. e., for $\beta = \beta_0$;

(c) there exists a neighbourhood $\Psi \subseteq \Omega$ of β_0 such that, for all $\beta \in \Psi$, Σ is exponentially stable and relation (7) is satisfied for all the initial states $z(0)$ of $\mathcal{E}$ and for all the initial states of Σ.

By the periodicity of all the time-varying subsystems appearing in Fig. 1, a solution of Problem 1 guarantees requirements (a), (b) and (c) to be satisfied for any (nonzero) initial time. Such a problem will be studied under the following technical assumptions.

Assumption 1. There exists a closed neighbourhood $\Psi_a \subseteq \Omega$ of β_0 such that all the entries of $A(\beta)$, $B(\beta)$, $C(\beta)$ are continuous functions of β in Ψ_a.

In order to state the second technical assumption, denote by $\sigma(E)$ the set of the eigenvalues of a square matrix E, and define

$$\Gamma(\beta) := \sigma(A(\beta)) \cup \sigma(F). \tag{17}$$

Assumption 2. For each element γ of $\Gamma(\beta_0)$, none of the values $\gamma + \jmath 2\pi i/\omega T$, $i \neq 0$, $i \in \mathbb{Z}$, is an element of $\Gamma(\beta_0)$, where $\jmath$ is the imaginary unit.

Remark 1. A proper choice of T trivially allows Assumption 2 to be satisfied. In addition, Assumption 2 implies that none of the values $\jmath 2\pi i/T$, $i \neq 0$, $i \in \mathbb{Z}$, is an element of $\Gamma(\beta_0)$.

Before giving conditions for the existence of a solution of the above stated control problem — together with a design procedure of it — call $\mathcal{S}_C$ the series connection of $\mathcal{K}_C$ and $\mathcal{P}$ (see Fig. 1), rewrite its equations (1), (2), (8), (9) in the following more compact form:

$$\dot{x}_C(t) = A_C(\beta)\, x_C(t) + B_C(\beta)\, u_C(t) + M_C(\beta)\, d(t), \;\; x_C(t) := [x^{\mathrm{T}}(t)\; w_C^{\mathrm{T}}(t)]^{\mathrm{T}}, \tag{18}$$

$$y(t) = C_C(\beta)\, x_C(t) + N_C(\beta)\, d(t), \tag{19}$$

and notice that, if Assumption 1 holds, then the elements of matrices $A_C(\beta)$, $B_C(\beta)$ and $C_C(\beta)$ are continuous functions of β in Ψ_a. Then, denote by $\mathcal{S}_D$ the discrete-time state-space model, having $u_D(k)$ as control input and $e_D(k)$ as output, of the multirate sampled-data system obtained by connecting the hold devices H_i, $i = 1, \ldots, p$, the continuous-time system $\mathcal{S}_C$, the q samplers of the scalar components $y_i(t)$, $i = 1, \ldots, q$, of $y(t)$ and the $\bar{q}$ comparators (see Fig. 1). The following lemma can be deduced directly from [22], [23] (see also [18]).

Lemma 1. For each $\beta \in \Omega$, the discrete-time system $\mathcal{S}_D$ is described by equations of the following form:

$$x_D(k+1) = A_D(\beta,k)\, x_D(k) + B_D(\beta,k)\, u_D(k) + M_D(\beta)\, z_D(k), \;\; x_D(k) \in \mathbb{R}^{n_D} \tag{20}$$

$$e_D(k) = C_D(\beta,k)\, x_D(k) + N_D(\beta,k)\, z_D(k) - V r_D(k), \tag{21}$$

where $z_D(k) = z(kT)$ and $r_D(k)$ satisfy the equations:

$$z_D(k+1) = e^{FT} z_D(k), \tag{22}$$

$$r_D(k) = \Theta(k) z_D(k), \tag{23}$$

with

$$\Theta(k) := \mathrm{diag}\{\tau_1(k), \ldots, \tau_{\bar{q}}(k)\}, \tag{24}$$

$$\tau_i(jZ_i) := 1, \quad j = 0, 1, 2, \ldots, \quad i = 1, \ldots, \bar{q} \tag{25}$$

$$\tau_i(k) := 0 \quad k \neq jZ_i, \quad \forall j \in \mathbb{Z}^+, \quad i = 1, \ldots, \bar{q}, \tag{26}$$

the matrices $A_D(\beta,k)$, $B_D(\beta,k)$, $C_D(\beta,k)$ and $N_D(\beta,k)$ are periodic of period ω for each $\beta \in \Omega$, and, if Assumption 1 holds, all the elements of $A_D(\beta,k)$, $B_D(\beta,k)$ and $C_D(\beta,k)$ are continuous functions of β in Ψ_a for all $k \in \mathbb{Z}$.

3. MAIN RESULT

A solution of Problem 1 is given by the following theorem, whose proof provides a design procedure of $\mathcal{K}_C$ and $\mathcal{K}_D$.

Theorem 1. There exist an ω-periodic discrete-time compensator $\mathcal{K}_D$ and a time-invariant continuous-time compensator $\mathcal{K}_C$ which constitute a solution of Problem 1, under Assumptions 1 and 2, if the following conditions are satisfied:

(i) the triplet $(A(\beta_0), B(\beta_0), C(\beta_0))$ is reachable and observable;

(ii) rank $\begin{bmatrix} A(\beta_0)-\alpha_i I_n & B(\beta_0) \\ C(\beta_0) & 0 \end{bmatrix} = n+q, \quad i=1,2,\dots,\mu.$

P r o o f. It will now be shown the existence of μ continuous-time sub-compensators $\mathcal{K}_1$, $\mathcal{K}_2$, ..., $\mathcal{K}_\mu$ of $\mathcal{K}_C$ such that, for each $j=1,\dots,\mu$, the series connection $\mathcal{S}_j$ of $\mathcal{K}_j$, $\mathcal{K}_{j-1}$, ..., $\mathcal{K}_1$ and $\mathcal{P}$, having the input $u_j(t)$ of $\mathcal{K}_j$ as input and $y(t)$ as output (see Fig. 2), and described by:

Fig. 2. Structure of system $\mathcal{S}_C$.

$$\dot{x}_j(t) = A_j(\beta)\,x_j(t) + B_j(\beta)\,u_j(t) + M_j(\beta)\,d(t), \quad x_j(t) \in \mathbb{R}^{n_j},\ u_j(t) \in \mathbb{R}^p, \tag{1}$$

$$y(t) = C_j(\beta)\,x_j(t) + N_j(\beta)\,d(t), \tag{2}$$

satisfies the following conditions:

Condition 1: Assumptions 1 and 2 and conditions (i) and (ii) of the theorem, rewritten for system $\mathcal{S}_j$ and the matrices describing it, hold;

Condition 2: there exists a neighbourhood $\Psi_j \subseteq \Omega$ of β_0 such that, for each $\beta \in \Psi_j$, the pair $(A_j(\beta), C_j(\beta))$ is observable.

The proof of the existence of $\mathcal{K}_1$, ..., $\mathcal{K}_\mu$ with the above stated properties – whose series connection will constitute the over-all compensator $\mathcal{K}_C$ (see Fig. 2) – will be carried out constructively by induction. Then, assume that $\mathcal{S}_{j-1}$ satisfies

Condition 1 written for $\mathcal{S}_{j-1}$ (for $j=1$ this is true if $\mathcal{P}$ is denoted with $\mathcal{S}_0$). Let $\mathcal{K}_j$ be described by

$$\dot{w}_j(t) = Q_j w_j(t) + R_j u_j(t), \quad w_j(t) \in \mathbb{R}^{k_j q},\ u_j(t) \in \mathbb{R}^p,\ u_C(t) = u_\mu(t), \tag{3}$$

$$u_{j-1}(k) = J_j w_j(t) + U_j u_j(t), \quad u_{j-1}(t) \in \mathbb{R}^p, \quad u_0(t) = u(t), \tag{4}$$

where

$$Q_j := \begin{bmatrix} \alpha_j I_q & I_q & 0 & 0 & \cdots & 0 & 0 \\ 0 & \alpha_j I_q & I_q & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha_j I_q & I_q & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_j I_q & I_q \\ 0 & 0 & 0 & 0 & \cdots & 0 & \alpha_j I_q \end{bmatrix}, \qquad R_j := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & I_q \end{bmatrix}, \tag{5}$$

$$J_j := \begin{bmatrix} E_j & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \qquad U_j := \begin{bmatrix} \widetilde{E}_j & 0 \end{bmatrix}, \tag{6}$$

with $E_j \in \mathbb{R}^{p\times q}$ and $\widetilde{E}_j \in \mathbb{R}^{p\times(p-q)}$ being such that

$$\det \begin{bmatrix} A_{j-1}(\beta_0) - \alpha_j I_{n_{j-1}} & B_{j-1}(\beta_0) E_j \\ C_{j-1}(\beta_0) & 0 \end{bmatrix} \neq 0, \tag{7}$$

$$\det \begin{bmatrix} E_j & \widetilde{E}_j \end{bmatrix} \neq 0, \tag{8}$$

whose existence is guaranteed by the assumption that $\mathcal{S}_{j-1}$ satisfies Condition 1 rewritten for $\mathcal{S}_{j-1}$. Thus it is readily seen that (3) – (8) and Condition 1 rewritten for $\mathcal{S}_{j-1}$, imply that Condition 1 holds and, in addition, that Condition 2 too holds.

Define $\Psi_b := \bigcap_{j=1}^{\mu} \Psi_j$. Notice that, following the above procedure for the design of $\mathcal{K}_C$, the Jordan form of Q_j has q Jordan blocks of dimensions k_j corresponding to the eigenvalue α_j. Notice also that $\alpha_1, \alpha_2, \ldots, \alpha_\mu$ are the only eigenvalues of $\mathcal{K}_C$, and that $\mathcal{S}_C = \mathcal{S}_\mu$ is reachable and observable at $\beta = \beta_0$, and satisfies the whole Condition 1 rewritten for $\mathcal{S}_C$.

Therefore, by Corollaries 3.1 and 3.2 in [23] (see also [22]) and Assumption 2, system $\mathcal{S}_D$ is reachable at all times and reconstructible for $\beta = \beta_0$. Then, call Σ_D the feedback connection of $\mathcal{K}_D$ and $\mathcal{S}_D$, as in Fig. 1, and choose $\mathcal{K}_D$ so that, for $\beta = \beta_0$ and for all the initial states $[x_D^{\mathrm{T}}(0) \quad w_D^{\mathrm{T}}(0)]^{\mathrm{T}}$ of Σ_D, the free state response of Σ_D (i.e., the state response of Σ_D for $z_D(0) = 0$) is zero for all integers $k \geq \hat{k}$, for some $\hat{k} \in \mathbb{Z}^+$ (see, e.g., [11] or [12], [13] for algorithms for the design of $\mathcal{K}_D$). This ensures that requirement (a) is satisfied.

Since, by Assumption 1 and Lemma 1, all the entries of $A_D(\beta,k)$, $B_D(\beta,k)$ and $C_D(\beta,k)$ are continuous functions of β in Ψ_a for all $k \in \mathbb{Z}$, such a compensator $\mathcal{K}_D$ guarantees also the exponential stability of Σ_D for all β within some neighbourhood $\Psi_c \subseteq \Omega$ of β_0.

Notice that all the eigenvalues of $A_C(\beta)$ belong to $\Gamma(\beta)$ and that, by the continuity of $A(\beta)$ in Ψ_a, there exists a neighbourhood $\Psi_d \subseteq \Omega$ of β_0 such that Assumption 2 rewritten with β instead of β_0 holds for each $\beta \in \Psi_d$. Since, for $z_D(0) = 0$, $\mathcal{S}_D$ can be seen as the series connection [22], [23] of three subsystems, namely an ω-periodic discrete-time system, the single-rate sampled-data system corresponding to $\mathcal{P}$ with both hold interval and sampling period equal to T, and a non-dynamic ω-periodic system, by Theorem 4 in [6] the exponential stability of Σ_D for all $\beta \in \Psi_c$ implies the exponential stability of the overall hybrid control system Σ for all $\beta \in \Psi_c \cap \Psi_d$ (see also Remark 1), thus ensuring the first part of requirement (c).

Now, denote by $\eta_{S_C}(t, \beta, x_C(0),\ u_C(\cdot),\ d(\cdot))$ the output response $y(t)$ of the continuous-time system $\mathcal{S}_C$ from the initial state $x_C(0)$ to the control function $u_C(\cdot)$ and to the disturbance function $d(\cdot)$ for the actual value of vector β, and denote by $\eta_{S_D}(k, \beta, x_D(0), u_D(\cdot), z_D(0))$ the output response $e_D(k)$ of the discrete-time system $\mathcal{S}_D$, from the initial state $x_D(0)$, to the control function $u_D(\cdot)$, for the initial state $z_D(0) = z(0)$ of the exogenous system $\mathcal{E}$, for the actual value of β. Then, by the application of Lemma 4 of the Appendix with α_j, k_j, q, $\mathcal{S}_j$, $\mathcal{K}_j$, $\mathcal{S}_{j-1}$ and Q_j instead of α, i, $\overline{q}$, $\overline{S}$, $\overline{S}_1$, $\overline{S}_2$ and $\overline{A}_1$, respectively, Condition 2 for $j = 1, 2, \ldots, \mu$, and the above mentioned Jordan structure of Q_j imply that, for each $\beta \in \Psi_b$ and for each $z(0) \in \mathbb{R}^\ell$ in (4), (5), there exists $\overline{x}_C \in \mathbb{R}^{n_\mu}$ (with $n_\mu = n + q\sum_{i=1}^{\mu} k_i$) such that

$$Vr(t) - \eta_{S_C}(t, \beta, \overline{x}_C, 0, d(\cdot)) = 0, \qquad \forall\, t \geq 0, \quad t \in \mathbb{R}. \tag{9}$$

Therefore, taking into account Fig. 1 and how system $\mathcal{S}_D$ is obtained from system $\mathcal{S}_C$, the hold devices and the samplers [22], [23], it is readily seen that:
(α) for each $\beta \in \Psi_b$ and for each $z_D(0) \in \mathbb{R}^\ell$, there exists $\overline{x}_D \in \mathbb{R}^{n_D}$ such that the state of the hold devices in $\overline{x}_D$ is zero and

$$\eta_{S_D}(k, \beta, \overline{x}_D, 0, z_D(0)) = 0, \qquad \forall\, k \geq 0,\ k \in \mathbb{Z}. \tag{10}$$

Since the discrete-time feedback connection Σ_D of $\mathcal{K}_D$ and $\mathcal{S}_D$ is ω-periodic, its response from the initial time $k = 0$ can be obtained through the feedback connection of the time-invariant "associated systems of $\mathcal{K}_D$ and $\mathcal{S}_D$ at time 0" [15], whose input is the "stacked form" of $z_D(\cdot)$ which has a proper rational z-transform. Since such a time-invariant system is exponentially stable for all $\beta \in \Psi_c$ as Σ_D is, then for each $\beta \in \Psi_c$, for each $z(0) = z_D(0) \in \mathbb{R}^\ell$, and for each initial state of the over-all hybrid control system Σ, the corresponding responses of Σ_D in the $[x_D^{\mathrm{T}}(k)\ \ w_D^{\mathrm{T}}(k)]^{\mathrm{T}}$, $e_D(k)$, $u_D(k)$ variables can be uniquely decomposed as the sum of the transient and steady-state responses, which will be denoted henceforth by the superscripts t and ss, respectively; hence, the corresponding $u_C(t)$ and $e(t)$ responses in the hybrid control system Σ can be decomposed as the sum of the corresponding responses $\tilde{u}_C^t(t)$ and $\tilde{u}_C^{ss}$ and, respectively, $\tilde{e}^t(t)$ and $\tilde{e}^{ss}(t)$.

Therefore, for each $\beta \in \Psi_b \cap \Psi_c \cap \Psi_d$, consider for the discrete-time system Σ_D its steady-state response to $z_D(0)$; by Lemma 3 of the Appendix and the above stated property (α), such a unique steady-state response is characterized by $e_D^{ss}(k) = 0$, $u_D^{ss}(k) = 0$, $w_D^{ss}(k) = 0$, for all $k \geq 0$, and, by Lemma 2 of the Appendix, it coincides for $\mathcal{S}_D$ with the full response from some initial state $x_D(0)$ for which a zero state in the hold devices is maintained, thus implying $\tilde{u}_C^{ss}(t) = 0$ for all $t \geq 0$. Therefore, since Assumption 2 holds in Ψ_d, the application to the series connection of $\mathcal{E}$ and $\mathcal{P}$ of a direct extension of Corollary 3.2 in [23] proves that

$$\tilde{e}^{ss}(t) = 0, \qquad \forall\, t \geq 0, \quad t \in \mathbb{R}, \tag{11}$$

for each $\beta \in \Psi_b \cap \Psi_c \cap \Psi_d$ and for all $z_D(0) \in \mathbb{R}^\ell$.

On the other hand, for each $\beta \in \Psi_b \cap \Psi_c \cap \Psi_d$ and for all $z_D(0) \in \mathbb{R}^\ell$, the corresponding transient response of Σ_D coincides with a free response of Σ_D from some initial state (see Lemma 2 of the Appendix). Hence, $\tilde{e}^t(t)$ and $\tilde{u}^t(t)$ are exponentially convergent to zero by the exponential stability of Σ in $\Psi_c \cap \Psi_d$. This, together with (11), since $e(t) = \tilde{e}^t(t) + \tilde{e}^{ss}(t)$, proves that the second part of requirement (c) is satisfied too with $\Psi := \Psi_b \cap \Psi_c \cap \Psi_d$, and, taking further into account that Σ_D and Σ have dead-beat free responses for $\beta = \beta_0$, proves that also requirement (b) is guaranteed. □

Remark 2. The design procedure contained in the proof of Theorem 1 consists of:

(1) choosing $\mathcal{K}_C$ as the series connection of $\mathcal{K}_1, \ldots, \mathcal{K}_\mu$, with $\mathcal{K}_j$, $j = 1, \ldots, \mu$, being described by equations (3) − (8); and

(2) designing a dead-beat feedback ω-periodic discrete-time controller $\mathcal{K}_D$ for the discrete-time ω-periodic system $\mathcal{S}_D$ corresponding to $\mathcal{S}_C$ (see Fig. 1).

Since, by the choice of $\mathcal{K}_C$, $\mathcal{S}_C$ contains an internal model of the exogenous continuous-time signals for all β in some neighbourhood Ψ_b of β_0 (see Lemmas 3 and 4 of the Appendix), then $\mathcal{S}_D$ contains an internal model of the corresponding discrete-time exogenous signals, thus guaranteeing $e_D(k) = 0$ in the steady-state, for each value of $\beta \in \Psi_b \cap \Psi_c$. This, since Assumption 2 is preserved for all $\beta \in \Psi_d$, and $u(t) = 0$ in the steady-state, guarantees a continuous-time null (i. e., ripple-free) error response in the steady state for all $\beta \in \Psi_b \cap \Psi_c \cap \Psi_d$.

Notice that condition (ii) implies that $p \geq q$ (i. e., $\dim u(t) \geq \dim y(t)$). If, in particular, $p = q$, the construction of the continuous-time precompensator $\mathcal{K}_C$ can be simplified, since it can be chosen as a minimal realization of $\phi^{-1}(s)I_p$, where $\phi(s)$ is the minimal polynomial of F. If, on the contrary, $p > q$, such a simpler precompensator $\mathcal{K}_C$ is not compatible with the detectability of $\mathcal{S}_C$ and $\mathcal{S}_D$, whence with the asymptotic stability of Σ, while the use of a continuous-time postcompensator having the transfer matrix $\phi^{-1}(s)I_q$ (see, e. g., Theorem 9−22 in [1]) is prevented by the hybrid control system structure of Figure 1. If $p > q$, however, the precompensator $\mathcal{K}_C$ here proposed still allows to satisfy the robustness requirement (c) in addition to (a) and (b), and has the same structure as a simi lar precompensator proposed in [18] for single-rate sampled-data control systems and for $k_i = 1$, $i = 1, \ldots, \mu$. Notice that the design of $\mathcal{K}_C$ needs merely the nominal description of the plant $\mathcal{P}$, i. e., its description for $\beta = \beta_0$.

Notice also that the case $\alpha_j \notin \mathbb{R}$ for some $j \in \{1, \ldots, \mu\}$ could be similarly taken into account in the design procedure of $\mathcal{K}_C$, under the same conditions.

It seems useful to clarify the application of Theorem 1 and the above mentioned design procedure by a numerical example. Then, consider the linear time-invariant plant $\mathcal{P}$ described by (1), (2) with

$$A(\beta) = \begin{bmatrix} 1 & 0 & 0 \\ \beta_a & 0.5+\beta_b & 0 \\ 1 & 1 & -1+\beta_b \end{bmatrix}, \; B(\beta) = \begin{bmatrix} 1 & 0 & 1.5+\beta_a \\ -1 & 0 & 0 \\ 0 & 1+\beta_b & -1 \end{bmatrix}, \; M(\beta) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{12}$$

$$C(\beta) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad N(\beta) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{13}$$

where the nominal value of $\beta = [\beta_a \;\; \beta_b]'$ is $\beta_0 = [0.5 \;\; 0]'$, and assume that the first component $y_1(t)$ of $y(t)$ must track a scalar reference signal $r(t) \in \mathbb{R}$, and the second component $y_2(t)$ must be regulated to zero under the action of a disturbance $d(t) \in \mathbb{R}$, with the exosystem $\mathcal{E}$ being described by equation (4), (5) with

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} -1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{14}$$

Moreover, suppose $T = 1$, $N_1 = 1$, $N_2 = 2$, $N_3 = 1$, $Z_1 = 1$ and $Z_2 = 2$. Thus, $\omega = 2$, and it is trivial to check that Assumptions 1 and 2 and conditions (i) and (ii) of Theorem 1 hold. Therefore, according to the step (1) in the above remark, the continuous-time time-invariant precompensator $\mathcal{K}_C$ described by (8). (9) is characterized by

$$Q_C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{15}$$

$$J_C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad U_C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{16}$$

Lastly, a 2-periodic discrete-time dead-beat controller $\mathcal{K}_D$ is easily found with the help of the design procedure contained in [12], [13] in form of a dead-beat observer based controller; namely, $\mathcal{K}_D$ is described by equation (10), (11), where

$$Q_D(k) = A_D(\beta_0, k) – V(k)\, C_D(\beta_0, k) – B_D(\beta_0, k)\, K(k), \; R_D(k) = V(k), \; J_D(k) = -K(k) \tag{17}$$

with

$$A_D(\beta_0, 0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.7183 & 0 & 0.7183 & 0 & 2.7183 & 0 & 0 \\ 0 & 0 & 0 & -0.8766 & 0 & -0.4715 & 0 & 1.0696 & 1.6487 & 0 \\ 0 & 0 & 0 & 0.199 & 0.6321 & 0.0477 & 0.3679 & 1.4965 & 0.8539 & 0.3679 \end{bmatrix}$$

$$A_D(\beta_0, 1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.2183 & 0 & 1.7183 & 0 & 0.7183 & 0 & 2.7183 & 0 & 0 \\ 0 & -0.1612 & 0 & -0.8766 & 0 & -0.4715 & 0 & 1.0696 & 1.6487 & 0 \\ 0 & 0.0093 & 0 & 0.199 & 0.6321 & 0.0477 & 0.3679 & 1.4965 & 0.8539 & 0.3679 \end{bmatrix}$$

$$B_D(\beta_0,0)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0.5&0\\0&0&0.5\\0&1&0\\0&0&1\\3.4365&0.2183&0\\0.8417&-0.1612&0\\0.6531&0.0093&0.1321\end{bmatrix}\quad B_D(\beta_0,1)=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\\0&0&0\\0&0&0.5\\0&0&0\\0&0&1\\3.4365&0&0\\0.8417&0&0\\0.6531&0&0.1321\end{bmatrix}$$

$$C_D(\beta_0,0)=\begin{bmatrix}0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1&0&-1\end{bmatrix}$$

$$C_D(\beta_0,1)=\begin{bmatrix}0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix}$$

$$K(0)=\begin{bmatrix}0&0&0&0.1465&-0.0341&0.4990&-0.0132&1.0738&1.3041&-0.0175\\0&0&0&0.25&0&0.75&0&0&0&0\\0&0&0&-0.1187&0.9565&0.1187&1.4831&0.0547&0.4136&-0.0222\end{bmatrix}$$

$$K(1)=\begin{bmatrix}0&0.4945&0&-0.4271&-0.0320&0.3493&-0.0124&1.0711&1.2839&-0.0164\\0&0&0&0&0&0&0&0&0&0\\0&-0.0104&0&-0.0052&0.9832&0.01037&1.4935&0.0210&0.1594&-0.0086\end{bmatrix}$$

$$V(0)=\begin{bmatrix}0&0\\0&0\\0&0\\0&-1.5518\\-2.1320&-0.4686\\0&-0.6633\\-0.5782&-0.0766\\0&-0.9651\\0&3.1487\\-1.5627&1.0812\end{bmatrix}\quad V(1)=\begin{bmatrix}0&0\\0&0\\0&0\\0&1.1334\\0&-4.1200\\0&0.2990\\0&-1.9212\\0&15.8571\\0&6.3670\\0&7.5125\end{bmatrix}$$

This solution has been found making use of an easy MATLAB implementation of the above quoted algorithm. The entries of $K(k)$ and $V(k)$ (as well as those of the other matrices) have been written above with only five digits. With the synthesized solution $\mathcal{K}_C$ and $\mathcal{K}_D$ the requirements (a), (b) and (c) of Problem 1 are satisfied. This is shown by the results of the simulation tests, which are represented by the experimental diagrams reported in Figures 3, 4, 5 and 6 showing the response of Σ for $\beta=\beta_0$, $z(0)=0$ (i.e., the free response of Σ) and $z(0)=[0\ \ -2]'$, as well as its error responses for some perturbation of vector β from its nominal value β_0.

4. CONCLUSIONS

For a given time-invariant plant whose description has a known and continuous dependence on some "physical" parameters, sufficient conditions and a design procedure have been given for obtaining (under Assumption 2) a ripple-free regulation and tracking by means of a multirate sampled-data control system, at least in a neighbourhood of the nominal physical parameters of the plant, and a ripple-free dead-beat error convergence at the nominal ones. "Large" regions $\Psi\in\Omega$ of convergence could be obtained by making use of robust stabilization design procedures for the choice of the discrete-time controller $\mathcal{K}_D$.

Fig. 3. The components of the continuous-time state response $x(t)$ of the plant $\mathcal{P}$ for $\beta = \beta_0$, $z(0) = 0$, $x(0) = [1\ \ -0.5\ \ 2]'$, $w_C(0) = 0$ and $w_D(0) = 0$.

Obviously, the same sufficient conditions and the same design procedure hold for the case when there is no knowledge about the dependence on the physical parameters of the matrices describing the plant $\mathcal{P}$, and independent perturbations of their entries are considered.

However, it is stressed that condition (ii) of Theorem 1 is not necessary in general for the parameter dependence in (1), (2) (see, e. g., the weaker sufficient conditions given in [18] for a similar problem and for the case of single-rate sampled-data control systems and $k_i = 1$, $i = 1, \dots, \mu$).

Fig. 4. The components of the continuous-time error response for $\beta = \beta_0$, $z(0) = [0\ \ -2]'$, $x(0) = [1\ \ -0.5\ \ 2]'$, $w_C(0) = 0$ and $w_D(0) = 0$.

Fig. 5. The components of the continuous-time control input response $u_C(t)$ of $\mathcal{S}_C$ for $\beta = \beta_0$, $z(0) = [0\ \ -2]'$, $x(0) = [1\ \ -0.5\ \ 2]'$, $w_C(0) = 0$ and $w_D(0) = 0$.

Fig. 6. The components of the continuous-time error response for $\beta = [0.5003\ \ -0.0008]'$, $z(0) = [0\ \ -2]'$, $x(0) = [1\ \ -0.5\ \ 2]'$, $w_C(0) = 0$ and $w_D(0) = 0$.

A. APPENDIX

Consider the linear time-invariant system $\overline{\Sigma}$ described by

$$\Delta\overline{x}(t) = \overline{A}_{\overline{\Sigma}}\overline{x}(t) + \overline{B}_{\overline{\Sigma}}\overline{u}(t), \tag{A.1}$$

$$\overline{y}(t) = \overline{C}_{\overline{\Sigma}}\overline{x}(t) + \overline{D}_{\overline{\Sigma}}\overline{u}(t), \tag{A.2}$$

where $t \in T$ is time, Δ denotes either the differentiation operator (if $T = \mathbb{R}$) or the one-step forward shift operator (if $T = \mathbb{Z}$), $\overline{x}(t) \in \mathbb{R}^{\overline{n}}$, $\overline{u}(t) \in \mathbb{R}^{\overline{p}}$, $\overline{y}(t) \in \mathbb{R}^{\overline{q}}$ and

$\overline{A}_{\overline{\Sigma}}, \overline{B}_{\overline{\Sigma}}, \overline{C}_{\overline{\Sigma}}, \overline{D}_{\overline{\Sigma}}$ are constant matrices with real elements. Denote by $\overline{\varphi}_{\overline{\Sigma}}(t,\overline{x}_0,\overline{u}(\cdot))$ and $\overline{\eta}_{\overline{\Sigma}}(t,\overline{x}_0,\overline{u}(\cdot))$ the state and output responses, respectively, at time t of system $\overline{\Sigma}$ to the initial state $\overline{x}(0)=\overline{x}_0$ and to the input function $\overline{u}(\cdot)$, and denote by $\overline{\mathcal{U}}$ the class of input functions $\overline{u}(\cdot)$ having a proper rational Laplace transform with all the poles in the closed right half-plane (if $T=\mathbb{R}$) or a proper rational z-transform with all the poles outside the open disk of unit radius (if $T=\mathbb{Z}$). Lastly, whenever system $\overline{\Sigma}$ is asymptotically stable, assuming that $\overline{u}(\cdot)\in\overline{\mathcal{U}}$, consider the unique decomposition of $\overline{\varphi}_{\overline{\Sigma}}(t,\overline{x}_0,\overline{u}(\cdot))$ [$\overline{\eta}_{\overline{\Sigma}}(t,\overline{x}_0,\overline{u}(\cdot))$] into the sum of the steady-state response $\overline{\varphi}^{ss}_{\overline{\Sigma}}(t,\overline{u}(\cdot))$ [$\overline{\eta}^{ss}_{\overline{\Sigma}}(t,\overline{u}(\cdot))$] and transient response $\overline{\varphi}^{t}_{\overline{\Sigma}}(t,\overline{x}_0,\overline{u}(\cdot))$ [$\overline{\eta}^{t}_{\overline{\Sigma}}(t,\overline{x}_0,\overline{u}(\cdot))$], consisting, respectively, of the modes of $\overline{u}(\cdot)$ and of the modes of system $\overline{\Sigma}$. The following lemma can be proved directly.

Lemma 2. If system $\overline{\Sigma}$ is asymptotically stable, then for each $\overline{u}(\cdot)\in\overline{\mathcal{U}}$ there exists a unique $\overline{x}_1\in\mathbb{R}^{\overline{n}}$ such that

$$\begin{aligned}
\overline{\varphi}_{\overline{\Sigma}}(t,\overline{x}_1,\overline{u}(\cdot)) &= \overline{\varphi}^{ss}_{\overline{\Sigma}}(t,\overline{u}(\cdot)), \quad \forall\, t\geq 0, && \text{(A.3)}\\
\overline{\eta}_{\overline{\Sigma}}(t,\overline{x}_1,\overline{u}(\cdot)) &= \overline{\eta}^{ss}_{\overline{\Sigma}}(t,\overline{u}(\cdot)), \quad \forall\, t\geq 0, && \text{(A.4)}\\
\overline{\varphi}_{\overline{\Sigma}}(t,\overline{x}_0-\overline{x}_1,0) &= \overline{\varphi}^{t}_{\overline{\Sigma}}(t,\overline{x}_0,\overline{u}(\cdot)), \quad \forall\, t\geq 0, \forall\,\overline{x}_0\in\mathbb{R}^{\overline{n}}, && \text{(A.5)}\\
\overline{\eta}_{\overline{\Sigma}}(t,\overline{x}_0-\overline{x}_1,0) &= \overline{\eta}^{t}_{\overline{\Sigma}}(t,\overline{x}_0,\overline{u}(\cdot)), \quad \forall\, t\geq 0, \forall\,\overline{x}_0\in\mathbb{R}^{\overline{n}}. && \text{(A.6)}
\end{aligned}$$

Now, assume that $\overline{u}(t)=\begin{bmatrix}\overline{r}^{\mathrm{T}}(t) & \overline{d}^{\mathrm{T}}(t)\end{bmatrix}^{\mathrm{T}}$, with $\overline{r}(t)\in\mathbb{R}^{\overline{\ell}}$, $\overline{d}(t)\in\mathbb{R}^{\overline{m}}$ and $\overline{\ell}+\overline{m}=\overline{p}$, and that system $\overline{\Sigma}$ has the feedback structure depicted in Figure 7, where the block denoted by $\overline{V}$ is a linear static link represented by the constant matrix $\overline{V}$, $\overline{S}$ is a linear time-invariant system described by

$$\begin{aligned}
\Delta\overline{x}(t) &= \overline{A}_{\overline{S}}\overline{x}(t)+\overline{B}_{\overline{S}}\overline{e}(t)+\overline{M}_{\overline{S}}\overline{d}(t), && \text{(A.7)}\\
\overline{y}(t) &= \overline{C}_{\overline{S}}\overline{x}(t)+\overline{D}_{\overline{S}}\overline{e}(t)+\overline{N}_{\overline{S}}\overline{d}(t), && \text{(A.8)}
\end{aligned}$$

with $(I+\overline{D}_{\overline{S}})$ nonsingular, and $\overline{r}(t)$, $\overline{d}(t)$ and $\overline{e}(t)$ can have the meaning of the reference, disturbance and error vector, respectively. Denote by $\overline{\theta}_{\overline{\Sigma}}(t,\overline{x}_0,\overline{r}(\cdot),\overline{d}(\cdot))$ the $\overline{e}(t)$ variable response at time t of system $\overline{\Sigma}$ to the initial state $\overline{x}(0)=\overline{x}_0$, and to the input functions $\overline{r}(\cdot)$ and $\overline{d}(\cdot)$; denote by $\overline{\eta}_{\overline{S}}(t,\overline{x}_0,\overline{e}(\cdot),\overline{d}(\cdot))$ the output response $\overline{y}(t)$ at time t of system $\overline{S}$ to the initial state $\overline{x}(0)=\overline{x}_0$, and to the input functions $\overline{e}(\cdot)$ and $\overline{d}(\cdot)$; denote by $\overline{\mathcal{R}}$ and $\overline{\mathcal{D}}$ the classes of functions $\overline{r}(\cdot)$ and $\overline{d}(\cdot)$, respectively, having a proper rational Laplace transform with all the poles in the closed right half-plane (if $T=\mathbb{R}$) or a proper rational z-transform with all the poles outside the open disk of unit radius (if $T=\mathbb{Z}$); and, whenever $\overline{\Sigma}$ is asymptotically stable, for $\overline{r}(\cdot)\in\overline{\mathcal{R}}$ and $\overline{d}(\cdot)\in\overline{\mathcal{D}}$ denote by $\overline{\theta}^{ss}_{\overline{\Sigma}}(t,\overline{r}(\cdot),\overline{d}(\cdot))$ the steady-state response of $\overline{\Sigma}$ in the $\overline{e}(t)$ variable.

The following lemma can be easily deduced from Lemma 2 (see also [19]), and states a simple form of the internal model principle.

Lemma 3. For each pair of functions $\overline{d}(\cdot) \in \overline{\mathcal{D}}$ and $\overline{r}(\cdot) \in \overline{\mathcal{R}}$, if the system $\overline{\Sigma}$ represented in Figure 7 is asymptotically stable, then

$$\overline{\theta}^{ss}_{\overline{\Sigma}}(t, \overline{r}(\cdot), \overline{d}(\cdot)) = 0 \tag{A.9}$$

$$\Updownarrow$$

$$\exists \overline{x}_1 \in \mathbb{R}^{\overline{n}} : \overline{\theta}_{\overline{\Sigma}}(t, \overline{x}_1, \overline{r}(\cdot), \overline{d}(\cdot)) = 0, \quad \forall\, t \geq 0, \tag{A.10}$$

$$\overline{\theta}_{\overline{\Sigma}}(t, \overline{x}_1, \overline{r}(\cdot), \overline{d}(\cdot)) = 0, \quad \forall\, t \geq 0, \tag{A.11}$$

$$\Updownarrow$$

$$\overline{V}\,\overline{r}(t) - \overline{\eta}_{\overline{S}}(t, \overline{x}_1, 0, \overline{d}(\cdot)) = 0, \quad \forall\, t \geq 0. \tag{A.12}$$

Fig. 7. The feedback system $\overline{\Sigma}$.

Now, consider system $\overline{S}$ in the block diagram depicted in Fig. 8, where the block denoted by $\overline{V}$ has the same meaning as it has in Fig. 7 and $\overline{S}$ is assumed to be the series connection of two subsystems $\overline{S}_1$ and $\overline{S}_2$, which are described by equations similar to (A.7), (A.8), $\overline{S}_1$ being not affected by disturbance $\overline{d}(t)$ (i. e., the matrices corresponding to $\overline{M}_{\overline{S}}$ and $\overline{N}_{\overline{S}}$ for $\overline{S}_1$ are zero). In addition, denote by $\overline{A}_1$ the matrix corresponding to $\overline{A}_{\overline{S}}$ for $\overline{S}_1$, and, assuming $T = \mathbb{R}$, for some non-negative $\alpha \in \mathbb{R}$ and some $i \in \mathbb{Z}^+$ denote by $\bar{\mathcal{R}}^i_\alpha$ the subclass of $\bar{\mathcal{R}}$ defined by:

$$\bar{\mathcal{R}}^i_\alpha := \{\bar{r}(\cdot) \;:\; \bar{r}(t) = \sum_{j=1}^{i} \delta_j \frac{t^{j-1}}{(j-1)!} e^{\alpha t}, \quad \forall\, t \geq 0, \quad \delta_j \in \mathbb{R}^{\bar{\ell}}\}, \tag{A.13}$$

and denote by $\bar{\mathcal{D}}^i_\alpha$ the subclass of $\bar{\mathcal{D}}$ which is similarly defined. The proof of the following lemma is straightforward (see also [3, 7, 9] for the case of a feedback connection of $\overline{S}$ as in Fig. 7).

Lemma 4. If system $\overline{S}$ in Fig. 8 is observable, and the Jordan form of matrix $\overline{A}_1$ has $\bar{q}$ Jordan blocks of dimensions not lower than i corresponding to the eigenvalue α, then for each $\bar{d}(\cdot) \in \bar{\mathcal{D}}^i_\alpha$ and for each $\overline{r}(\cdot) \in \bar{\mathcal{R}}^i_\alpha$ there exists $\overline{x}_1 \in \mathbb{R}^{\overline{n}}$ such that (A.12) is satisfied.

Fig. 8. An open-loop connection.

(Received December 2, 1993.)

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