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NOTES ON μ AND ℓ_1 ROBUSTNESS TESTS

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An upper bound for the complex structured singular value related to a linear time-invariant system over all frequencies is given. It is in the form of the spectral radius of the $\mathcal{H}_\infty$-norm matrix of SISO input-output channels of the system when uncertainty blocks are SISO. In the case of MIMO uncertainty blocks the upper bound is the ∞-norm of a special non-negative matrix derived from $\mathcal{H}_\infty$-norms of SISO channels of the system. The upper bound is fit into the inequality relation between the results of μ and ℓ_1 robustness tests.

1. INTRODUCTION

The objective of robust control is to achieve stability and good performance requirements in the presence of uncertainty. Robustness analysis tools and robust control design methods have been developed recently dealing with structured uncertainty for the cases when the signals are measured in the

— $\mathcal{L}_2$ or ℓ_2-norm (energy of the signal): μ -analysis [2, 3, 8, 9],

— $\mathcal{L}_\infty$ or ℓ_∞-norm (maximum amplitude): ℓ_1-analysis [1, 2, 5, 6, 7].

This paper derives a global upper bound for the complex structured singular value related to a linear time-invariant system over all frequencies. We consider both cases when the uncertainty blocks are SISO and MIMO. In SISO case the upper bound can be computed directly as the spectral radius of the $\mathcal{H}_\infty$-norm matrix of input-output channels of the system. However, in MIMO case one has to perform a certain optimization procedure on $\mathcal{H}_\infty$-norms of SISO channels and build a special non-negative matrix. Its ∞-norm gives the upper bound of the μ-test result. The result of the mentioned optimization procedure can be expressed in an explicit form. Thus one will have an insight on how the greatest amplifications of the SISO system components, related to $\mathcal{L}_2$ or ℓ_2 signal norms, affect the result of robustness analysis based on the structured singular value. Remember, that the ℓ_1 -analysis results are directly computed from amplifications, i. e. from ℓ_1- norms of the channels related to the uncertainty structure. This allows us to guess how they influence the robustness test results. This paper shows the counterpart of these relations in the case of the μ-analysis. Further we show how our result fits into the inequality relation between the μ and ℓ_1-tests.

Section 2 gives a brief overview of the mentioned robustness analysis methods. Then some notation used in the paper is introduced. In the first part of Section 3 we derive an upper bound for the result of the structured singular value analysis in the case when the uncertainty blocks are SISO. While the SISO relations are very simple and follow directly from the properties of the complex structured singular value, the MIMO ones need a certain optimization procedure as one can see in the rest of Section 3. Finally Section 4 summarizes the relation between the μ and ℓ_1 robustness test results.

Fig. 1. The robust stability and performance problem formulation.

2. ROBUSTNESS ANALYSIS METHODS

2.1. Robust stability and performance

The general problem formulation of robust stability and performance is shown in Figure 1. The signal w denotes the control inputs or disturbances and z denotes the regulated outputs. The map taking w to z is referred as T_{zw} and Δ models the uncertain part of the system. It is assumed that Δ is structured, i. e. it belongs to the following class:

$$\boldsymbol{\Delta}(n) := \{\Delta = \operatorname{diag}(\Delta_1, \dots, \Delta_n) : \|\Delta_i\| < 1\} \tag{1}$$

where Δ_i are $p_i \times p_i$ systems. Let $p = \sum_{i=1}^{n} p_i$. While Δ models the uncertain part of the system, the linear time-invariant G is the known part including the nominal plant, the controller, any input and output weighting functions and any weighting functions on the perturbations. The system G can be partitioned as follows

$$\begin{aligned} z &= G_{11}w + G_{12}u_\Delta \\ y_\Delta &= G_{21}w + G_{22}u_\Delta \\ u_\Delta &= \Delta y_\Delta. \end{aligned} \tag{2}$$

With this partitioning T_{zw} is in the form

$$T_{zw} = G_{11} + G_{12}\Delta\left(I - G_{22}\Delta\right)^{-1} G_{21}. \tag{3}$$

Definition 1. *Robust Stability.* The system achieves robust stability iff the system is internally stable for all admissible perturbations (for all $\Delta \in \boldsymbol{\Delta}(n)$).

Definition 2. *Robust Performance.* The system achieves robust performance iff

1. the system achieves robust stability, and
2. $\sup_{\Delta \in \mathbf{\Delta}(n)} \|T_{zw}\| \leq 1$.

Fig. 2. Stability vs performance robustness.

The robust performance problem can be transformed into a robust stability problem, which has been shown for both 2-norm and ∞-norm cases [5, 8]. Consider the two systems in Figure 2, where System I corresponds to a performance robustness problem, while System II is formed from System I by connecting z and w through a fictious perturbation, Δ_p, satisfying $\|\Delta_p\| < 1$.

Theorem 1. Consider System I having $\mathbf{\Delta}(n)$ as the class of admissible perturbations and System II having $\mathbf{\Delta}(n+1)$ as the class of admissible perturbations. System I achieves robust performance iff System II is robustly stable.

2.2. μ–analysis

Measuring the signals in the 2-norm and assuming structured linear time-invariant stable uncertainty the μ-tests can be used to analyze robust stability and performance. The structured singular value is a matrix function denoted by $\mu_\Delta(\cdot)$ which depends on the underlying structure $\mathbf{\Delta}$ (a prescribed set of block diagonal matrices) [8].

Definition 3. For $M \in \boldsymbol{C}^{n \times n}$, $\mu_\Delta(M)$ is defined by

$$\mu_\Delta(M) := \left[\min\{\bar{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - M\Delta) = 0\}\right]^{-1} \tag{4}$$

unless no $\Delta \in \mathbf{\Delta}$ makes $I - M\Delta$ singular, in which case $\mu_\Delta(M) := 0$ (the symbol $\bar{\sigma}(\cdot)$ denotes the maximum singular value).

The following lower and upper bounds can be given for the structured singular value $\mu_\Delta(M)$ [3, 8]

$$\max_{U \in \boldsymbol{U}} \rho\left(UM\right) \leq \mu_\Delta\left(M\right) \leq \inf_{D \in \boldsymbol{D}} \bar{\sigma}\left(DMD^{-1}\right) \tag{5}$$

where

$$\boldsymbol{U} = \left\{ U = \mathrm{diag}(U_1, \ldots, U_n) \in \boldsymbol{\Delta} : U^H U = I \right\} \tag{6}$$

and

$$\boldsymbol{D} = \left\{ D = \mathrm{diag}(d_1 I, \ldots, d_n I) : d_i \in R_+ \right\}. \tag{7}$$

The lower bound is equality in all cases, but $\rho\,(UM)$ can have multiple local maxima which are not global. Unfortunately, the upper bound is not always equal to $\mu_\Delta\,(M)$. The following robust stability μ-test is used in frequency domain [8, 9]:

Theorem 2. Suppose that G_{11} is stable, then the uncertain system is stable for all $\Delta \in \boldsymbol{\Delta}(n)$ iff

$$\sup_\omega \mu_\Delta\left(G_{22}(j\omega)\right) \leq 1 \tag{8}$$

where G_{22} is the system mapping internal input signals u_Δ to internal output signals y_Δ.

2.3. ℓ_1–analysis

Measuring the signals in the ∞-norm and assuming structured uncertainty which can be non-linear and time-varying the ℓ_1-test can be used to analyze robust stability and performance. The system G_{22} is partitioned corresponding to the Δ block structure

$$G_{22} = \begin{bmatrix} [G_{22}]_{11} & \ldots & [G_{22}]_{1n} \\ \vdots & & \vdots \\ [G_{22}]_{n1} & \ldots & [G_{22}]_{nn} \end{bmatrix} \tag{9}$$

where $[G_{22}]_{ij}$ is $p_i \times p_j$ system. Let the set J be an index set for all possible collections of rows from the row blocks. For each $j = (j_1, \ldots, j_n) \in J$ define the matrix $\left(\hat{h}_{22}\right)_j$ as follows

$$\left(\hat{h}_{22}\right)_j = \begin{bmatrix} \left\| \left([G_{22}]_{11}\right)_{j_1} \right\|_1 & \ldots & \left\| \left([G_{22}]_{1n}\right)_{j_1} \right\|_1 \\ \vdots & & \vdots \\ \left\| \left([G_{22}]_{n1}\right)_{j_n} \right\|_1 & \ldots & \left\| \left([G_{22}]_{nn}\right)_{j_n} \right\|_1 \end{bmatrix} \tag{10}$$

where $\left([G_{22}]_{ik}\right)_{j_p}$ is the j_pth row of the system $[G_{22}]_{ik}$ and $\left\| \left([G_{22}]_{ik}\right)_{j_p} \right\|_1$ is its ℓ_1-norm. Then the following robust stability ℓ_1-test on G_{22} can be used [2]:

Theorem 3. Given an interconnection of a linear time-invariant stable system G_{22} and n norm bounded perturbation blocks, the system is robustly stable iff

$$\rho\left(\left(\hat{h}_{22}\right)_j\right) \leq 1 \tag{11}$$

holds for all $j \in J$.

3. UPPER BOUND FOR THE μ–TEST RESULT

In this section upper bound for the μ-analysis results is derived for two cases when uncertainty blocks are SISO or MIMO linear time-invariant systems. This upper bound can be computed directly as the spectral radius of the non-negative matrix composed of $\mathcal{H}_\infty$-norms of entries of the matrix $G_{22}(j\omega)$ in the case when the uncertainty blocks are SISO. When these blocks are MIMO we have to perform some more computations on the entries of the matrix containing the $\mathcal{H}_\infty$-norms of the entries of $G_{22}(j\omega)$ and build another non-negative matrix. Then the upper bound of the μ-test will be the ∞-norm (maximum of row-sums) of this new matrix. Further we will compare the μ and ℓ_1-test results. The upper bound derived for the μ -analysis result gives us a good opportunity to obtain these relations in a simple way.

First of all we introduce the matrix $\hat{G}_{22}$ which plays key role in the whole paper. Let $[G_{22}(j\omega)]_{ij}$ be the ijth entry of the matrix $G_{22}(j\omega)$, which has $\mathcal{H}_\infty$-norm in the form of

$$\left\| [G_{22}]_{ij} \right\|_\infty = \sup_\omega \left| [G_{22}(j\omega)]_{ij} \right|. \tag{12}$$

With these the matrix $\hat{G}_{22}$ is defined as

$$\hat{G}_{22} = \begin{bmatrix} \|[G_{22}]_{11}\|_\infty & \cdots & \left\|[G_{22}]_{1p}\right\|_\infty \\ \vdots & & \vdots \\ \left\|[G_{22}]_{p1}\right\|_\infty & \cdots & \left\|[G_{22}]_{pp}\right\|_\infty \end{bmatrix}. \tag{13}$$

First we focus on SISO relations which can be derived easily. Then we turn to MIMO relations which are direct generalizations of the SISO ones.

3.1. Uncertainty with SISO blocks

Dahleh and Diaz–Bobillo [2] summarizes important and general results on the relation between μ and ℓ_1-analysis results when the uncertainties are SISO. Their Theorem 7.6.1 [2, p. 172] claims that

$$\sup_\omega \mu_\Delta \left(G_{22}(j\omega)\right) \le \rho\left(\hat{h}_{22}\right). \tag{14}$$

This proposition is extended here by another inequality which makes the relation physically more transparent and simplifies its proof. The extension is the spectral radius of the matrix $\hat{G}_{22}$.

Proposition 1. Given an interconnection of a linear time-invariant stable system G_{22} and n norm bounded SISO perturbation blocks. Then the following relation exists between the μ and ℓ_1-analysis results

$$\sup_\omega \mu_\Delta \left(G_{22}(j\omega)\right) \le \rho\left(\hat{G}_{22}\right) \le \rho\left(\hat{h}_{22}\right) \tag{15}$$

where $\hat{G}_{22}$ is introduced in (13) being the frequency domain analog of the matrix $\hat{h}_{22}$ defined in (10).

Proof. From the theory of the complex structured singular value (5) it follows that

$$\mu_\Delta\left(G_{22}(j\omega)\right) = \max_{U \in \boldsymbol{U}} \rho\left(UG_{22}(j\omega)\right) \tag{16}$$

where

$$\boldsymbol{U} = \left\{U = \operatorname{diag}(U_1, \ldots, U_n) \in \boldsymbol{\Delta} : U^H U = I\right\}. \tag{17}$$

In our case U_i is a scalar on the unit circle. Since

$$\left|\left[UG_{22}(j\omega)\right]_{ij}\right| \leq \left\|\left[G_{22}\right]_{ij}\right\|_\infty \tag{18}$$

and it is known from the theory of non-negative matrices that if $|A| \leq B$ entrywise then $\rho(A) \leq \rho(B)$ [4], one immediately gets that for each frequency

$$\rho\left(UG_{22}(j\omega)\right) \leq \rho\left(\hat{G}_{22}\right). \tag{19}$$

It means that the following inequality holds

$$\sup_\omega \mu_\Delta\left(G_{22}(j\omega)\right) = \sup_\omega \max_{U \in \boldsymbol{U}} \rho\left(UG_{22}(j\omega)\right) \leq \rho\left(\hat{G}_{22}\right). \tag{20}$$

In SISO case the relation between the $\mathcal{H}_\infty$ and ℓ_1-norm is

$$\left\|\left[G_{22}\right]_{ij}\right\|_\infty \leq \left\|\left[G_{22}\right]_{ij}\right\|_1, \tag{21}$$

Again from the theory of the non-negative matrices we obtain

$$\rho\left(\hat{G}_{22}\right) \leq \rho\left(\hat{h}_{22}\right). \tag{22}$$

So we have verified that

$$\sup_\omega \mu_\Delta\left(G_{22}(j\omega)\right) \leq \rho\left(\hat{G}_{22}\right) \leq \rho\left(\hat{h}_{22}\right). \qquad \square$$

From this proof it is apparent that the spectral radius of the matrix $\hat{G}_{22}$ gives a global upper bound for the structured singular value $\mu_\Delta(G_{22}(j\omega))$ over all frequencies and it is a less strong sufficient condition for the robust stability than the one given by the ℓ_1-test.

3.2. Uncertainty with MIMO blocks

Now we generalize the results of the previous section to the case when the uncertainty blocks are linear time-invariant MIMO systems. An upper bound for μ-test can be computed as ∞-norm of a non-negative matrix derived from matrix $\hat{G}_{22}$ by an optimization procedure.

The following lemma states that an upper bound for μ-analysis result can be found by maximizing the spectral radius of a non-negative matrix which is a product of $\hat{G}_{22}$ and an appropriate non-negative matrix V depending on the perturbation structure.

Lemma 1. Given an interconnection of a linear time-invariant stable system G_{22} and n norm bounded MIMO perturbation blocks. Then the following upper bound holds for the result of the μ-analysis

$$\sup_{\omega} \mu_{\Delta}\left(G_{22}(j\omega)\right) \leq \max_{V} \rho\left(V\hat{G}_{22}\right) \tag{23}$$

where V is non-negative and has the same structure as the perturbation Δ, i. e.

$$V \geq 0, \quad V = \operatorname{diag}(V_1, \ldots, V_n)$$

where V_i is a $p_i \times p_i$ matrix and the jth row v^j of the matrix V satisfies the following condition

$$\left\|v^j\right\|_2^2 = \sum_{k=1}^{p} V_{jk}^2 = 1.$$

P r o o f. From the theory of the complex structured singular value (5) it follows that

$$\mu_{\Delta}\left(G_{22}(j\omega)\right) = \max_{U \in \boldsymbol{U}} \rho\left(UG_{22}(j\omega)\right) \tag{24}$$

where

$$\boldsymbol{U} = \left\{U = \operatorname{diag}(U_1, \ldots, U_n) \in \boldsymbol{\Delta} : U^H U = I\right\}. \tag{25}$$

Denote

$$\bar{\omega} = \arg\sup_{\omega} \mu_{\Delta}\left(G_{22}(j\omega)\right) \tag{26}$$

and

$$\bar{U} = \arg\max_{U \in \boldsymbol{U}} \rho\left(UG_{22}(j\bar{\omega})\right). \tag{27}$$

For the product of two matrices $|AB| \leq |A|\,|B|$ holds entrywise. From the theory of non-negative matrices [4] it follows that if $|A| \leq B$ entrywise then $\rho(A) \leq \rho(B)$. Using these relations we obtain

$$\mu_{\Delta}\left(G_{22}(j\bar{\omega})\right) = \rho\left(\bar{U}G_{22}(j\bar{\omega})\right) \leq \rho\left(\left|\bar{U}G_{22}(j\bar{\omega})\right|\right) \leq \rho\left(\left|\bar{U}\right|\left|G_{22}(j\bar{\omega})\right|\right). \tag{28}$$

Since $|G_{22}(j\omega)| \leq \hat{G}_{22}$, we get the inequality

$$\rho\left(\left|\bar{U}\right|\left|G_{22}(j\bar{\omega})\right|\right) \leq \rho\left(\left|\bar{U}\right|\hat{G}_{22}\right) \tag{29}$$

where $\bar{U}$ is a unitary matrix, so the equality $\bar{U}^H\bar{U} = \bar{U}\bar{U}^H = I$ holds. This property means that the ith and jth rows of the matrix $\bar{U}$, i. e. $\bar{u}^i$ and $\bar{u}^j$, satisfy the condition

$$\bar{u}^i\left(\bar{u}^j\right)^H = \sum_{k=1}^{p} \bar{U}_{ik}\bar{U}_{jk}^H = \delta_{ij} \tag{30}$$

where δ_{ij} is the Kronecker symbol and $\bar{U}_{jk}^H$ is the complex conjugate of the matrix entry $\bar{U}_{jk}$. The above derivation leads to definition of the following optimization problem

$$\bar{V} = \arg\max_{V} \rho\left(V\hat{G}_{22}\right) \tag{31}$$

where $V \geq 0$ and $V = \text{diag}(V_1, \ldots, V_n)$. In addition the jth row v^j of the matrices V has to fulfil the condition

$$\left\|v^j\right\|_2^2 = v^j \left(v^j\right)^T = \sum_{k=1}^{p} V_{jk}^2 = 1. \tag{32}$$

Considering the fact that the matrix $\left|\bar{U}\right|$ is only a suboptimal solution of this optimization problem we find that

$$\sup_\omega \mu_\Delta \left(G_{22}(j\omega)\right) \leq \rho\left(\bar{V}\hat{G}_{22}\right) = \max_V \rho\left(V\hat{G}_{22}\right) \qquad \square$$

So the optimization problem to be solved is finding the non-negative matrix V with the same structure as the uncertainty which maximizes the spectral radius of the product $V\hat{G}_{22}$. First we solve the problem without the restriction on the structure of the matrix V.

Lemma 2. Let P be a $m \times m$ non-negative matrix. Define the optimization problem as

$$\bar{Q} = \arg\max_Q \rho\left(QP\right) \tag{33}$$

on the set of non-negative matrices Q satisfying the following restriction on their ith row denoted by q^i

$$\left\|q^i\right\|_2^2 = \sum_{k=1}^{m} Q_{ik}^2 = 1. \tag{34}$$

Let s_i be the ith row-sum of the matrix P

$$s_i = \sum_{k=1}^{m} P_{ik}. \tag{35}$$

Then each row of optimal solution $\bar{Q}$ is the same and they can be computed as

$$\bar{q}^i = \frac{1}{\sqrt{\sum_{k=1}^{m} s_k^2}} \begin{bmatrix} s_1, & \ldots, & s_m \end{bmatrix}, \quad i = 1, \ldots, m. \tag{36}$$

P r o o f . The spectral radius of an $m \times m$ matrix M is overbounded by its ∞-norm, i. e.

$$\rho\left(M\right) \leq \max_{1 \leq i \leq m} \sum_{k=1}^{m} \left|M_{ik}\right| = \left\|M\right\|_\infty . \tag{37}$$

Let M be a non-negative matrix with each of its row being the same. Let's perform a similarity transformation on M using the transformation matrix

$$T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{38}$$

which has inverse in the form of

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{39}$$

Similarity transformations does not change the eigenvalues of a matrix. If we apply T to M the resulting matrix TMT^{-1} has only one row different from 0. Therefore only one of its eigenvalues is different from 0 and it is equal to the row-sum of the matrix M. This row-sum (the ∞-norm of the non-negative M) is the spectral radius. It means that we have to choose such a matrix Q which has equal rows, so QP will also have equal rows, and makes the row-sum maximal. In accordance with these ideas let us define the following cost function of the row-vector q

$$J(q) = qP\,[1,\ldots,1]^T = q_1 s_1 + q_2 s_2 \ldots + q_m s_m \tag{40}$$

with the following constraints

$$q_1^2 + q_2^2 + \ldots q_m^2 = 1, \quad q_i \geq 0,\ i = 1,\ldots,m \tag{41}$$

and maximize $J(q)$. The above optimization problem can be solved using the Lagrange multiplier method. Define the extended cost function in the form of

$$L(q,\lambda) = q_1 s_1 + q_2 s_2 \ldots + q_m s_m + \lambda(q_1^2 + q_2^2 + \ldots q_m^2 - 1). \tag{42}$$

The optimality conditions are

$$\frac{\partial L}{\partial q_i} = s_i + 2\lambda q_i = 0 \tag{43}$$

and

$$\frac{\partial L}{\partial \lambda} = q_1^2 + q_2^2 + \ldots q_m^2 - 1 = 0. \tag{44}$$

Their solutions are

$$q_i = -\frac{s_i}{2\lambda}, \quad \frac{s_1^2}{4\lambda^2} + \frac{s_2^2}{4\lambda^2} + \cdots + \frac{s_m^2}{4\lambda^2} = 1 \tag{45}$$

and thus $\lambda = \pm\frac{1}{2}\sqrt{s_1^2 + s_2^2 + \ldots s_m^2}$. Considering the restrictions $q_i \geq 0$, the optimal $\bar{q}$ has following entries

$$\bar{q}_j = \frac{s_j}{\sqrt{\sum_{k=1}^m s_k^2}}.$$

□

The structure of the matrix $V = \mathrm{diag}(V_1,\ldots,V_n)$ follows from the structure of the matrix $U = \mathrm{diag}(U_1,\ldots,U_n)$ where U_i and V_i are $p_i \times p_i$ matrices and the sum of their size is $p = \sum_{i=1}^n p_i$. From this restriction it follows that V should not have equal rows in order to achieve better estimate of the result of μ-analysis. Let's partition the matrix $\hat{G}_{22}$ in accordance with the structure of V

$$\hat{G}_{22} = \begin{bmatrix} \hat{G}_{22}^1 \\ \vdots \\ \hat{G}_{22}^n \end{bmatrix}. \tag{46}$$

It means that $\hat{G}_{22}^i$ is a $p_i \times p$ matrix. Define the jth row-sum of the ith partition $\hat{G}_{22}^i$ as follows

$$s_j^i = \sum_{k=1}^{p} \left[\hat{G}_{22}^i\right]_{jk} = \sum_{k=1}^{p} \left\| [G_{22}]_{j+\sum_{m=1}^{i-1} p_m, k} \right\|_\infty, \quad i = 1, \ldots, n, \ j = 1, \ldots, p_i. \tag{47}$$

Let's denote sum of the second powers of ith block row sums by

$$S^i = \sum_{j=1}^{p_i} \left(s_j^i\right)^2 = \sum_{j=1}^{p_i} \left(\sum_{k=1}^{p} \left[\hat{G}_{22}^i\right]_{jk} \right)^2 = \sum_{j=1}^{p_i} \left(\sum_{k=1}^{p} \left\| [G_{22}]_{j+\sum_{m=1}^{i-1} p_m, k} \right\|_\infty \right)^2 \tag{48}$$

where $i = 1, \ldots, n$. We will also need the partial row sums of the ith partition according to the uncertainty structure

$$\hat{s}_j^{i,k} = \sum_{l=1}^{p_k} \left[\hat{G}_{22}^i\right]_{j, l+\sum_{m=1}^{k-1} p_m} = \sum_{l=1}^{p_k} \left\| [G_{22}]_{j+\sum_{m=1}^{i-1} p_m, l+\sum_{m=1}^{k-1} p_m} \right\|_\infty \tag{49}$$

where $i,\, k = 1, \ldots, n, \ j = 1, \ldots, p_i$.

Now we are ready for stating the proposition on the upper bound for the result of μ-analysis in the case when the uncertainties are MIMO linear time-invariant systems.

Proposition 2. Using the notations introduced above define the matrix $\tilde{G}_{22}$ as follows

$$\left[\tilde{G}_{22}\right]_{ik} = \frac{1}{\sqrt{S^i}} \sum_{j=1}^{p_i} s_j^i \hat{s}_j^{i,k}, \quad i,\, k = 1, \ldots, n. \tag{50}$$

Then for the supremum of the structured singular value of the matrix $G_{22}(j\omega)$ over all frequencies, i. e. for the result of μ-analysis, the next inequality holds

$$\sup_\omega \mu_\Delta \left(G_{22}(j\omega)\right) \le \max_V \rho\left(V\hat{G}_{22}\right) \le \left\|\tilde{G}_{22}\right\|_\infty \tag{51}$$

where V has the same properties as in Lemma 1.

Proof. Let the matrix $\bar{V}$ has block $\bar{V}_i$ its jth row $\bar{v}^{i,j}$ contains the scaled ith block row-sums as its entries

$$\bar{v}^{i,j} = \frac{1}{\sqrt{S^i}} \left[s_1^i, \ldots, s_{p_i}^i\right], \quad i = 1, \ldots, n, \ j = 1, \ldots, p_i. \tag{52}$$

Matrices have the property that their spectral radius is not greater than their ∞-norm. It means that $\rho\left(\bar{V}\hat{G}_{22}\right) \leq \left\|\bar{V}\hat{G}_{22}\right\|_{\infty}$. It follows from the previous lemma that the choice of the matrix $\bar{V}$ implies the optimality, i. e. ∞-norm of $\bar{V}\hat{G}_{22}$ is the maximal. $\tilde{G}_{22}$ and $\bar{V}\hat{G}_{22}$ have the same row-sums and so their ∞-norms are the same. □

It is important to note that

$$\rho\left(\tilde{G}_{22}\right) = \rho\left(\bar{V}\hat{G}_{22}\right) \leq \max_{V} \rho\left(V\hat{G}_{22}\right). \tag{53}$$

Equality holds for example in the case of SISO perturbation blocks when $\tilde{G}_{22} = \bar{V}\hat{G}_{22} = \hat{G}_{22}$. Now we demostrate it on a simple example.

Example. Let the $\mathcal{H}_{\infty}$-norm matrix be

$$\hat{G}_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 15 & 5 & 6 \end{bmatrix}.$$

Then the optimal $\bar{V}$ for a SISO and a 2×2 uncertainty block ($p_1 = 1$, $p_2 = 2$) is

$$\bar{V} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{5}{\sqrt{5^2+26^2}} & \frac{26}{\sqrt{5^2+26^2}} \\ 0 & \frac{5}{\sqrt{5^2+26^2}} & \frac{26}{\sqrt{5^2+26^2}} \end{bmatrix}$$

The spectral radius of $\tilde{G}_{22}$ is then 11.3686. For the matrix V of the same structure defined as

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{5}{\sqrt{5^2+26^2}} & \frac{26}{\sqrt{5^2+26^2}} \\ 0 & 0.26 & \sqrt{1-0.26^2} \end{bmatrix}.$$

the spectral radius is greater, and equals to 11.3856.

Note: It is known from the theory of the complex structured singular value [8] that

$$\mu_{\Delta}(M) = \max_{U \in \boldsymbol{U}} \rho(UM) = \max_{U \in \boldsymbol{U}} \rho(MU). \tag{54}$$

It means that we can get another upper bound (maybe smaller) for the μ-test result by application of the presented optimization procedure to the columns of the matrix $\hat{G}_{22}$.

4. RELATION BETWEEN μ AND ℓ_1–TEST RESULTS

The relation between the μ and ℓ_1-analysis results can be investigated based on the previous proposition.

Proposition 3. Given an interconnection of a linear time-invariant stable system G_{22} and n norm bounded MIMO perturbation blocks. Then the following relation exists between the μ and ℓ_1-analysis results

$$\sup_{\omega} \mu_{\Delta}\left(G_{22}(j\omega)\right) \leq \left\|\tilde{G}_{22}\right\|_{\infty} = \max_{1\leq i\leq n} \sqrt{S^i} \leq \sqrt{p_{\max}}\, \|G_{22}\|_1 \tag{55}$$

and for each $j=(j_1,\dots,j_n)\in J$ (See Eq. 10)

$$\rho\left(\left(\hat{h}_{22}\right)_j\right) \leq \|G_{22}\|_1 \tag{56}$$

where $p_{\max}$ is the maximum of uncertainty block sizes, i. e. $p_{\max} = \max\limits_{1\leq i\leq n} p_i$.

Proof. The second inequality has been written only for comparison with the first one. From the definition of the matrix $\left(\hat{h}_{22}\right)_j$ and the ℓ_1-norm of the system G_{22} it follows that

$$\rho\left(\left(\hat{h}_{22}\right)_j\right) \leq \|G_{22}\|_1 = \max_{1\leq i\leq p} \sum_{k=1}^{p} \|[G_{22}]_{ik}\|_1 . \tag{57}$$

Note that $\|G_{22}\|_1 \leq 1$ is the small gain condition for the robust stability in $\mathcal{L}_\infty/\ell_\infty$-sense. The following equation is obtained from the definition of the ∞-norm of a matrix and from the defining equations (47) and (48)

$$\left\|\tilde{G}_{22}\right\|_{\infty} = \max_{1\leq i\leq n} \sum_{k=1}^{n} \left(\frac{1}{\sqrt{S^i}} \sum_{j=1}^{p_i} s_j^i \hat{s}_j^{i,k}\right) = \max_{1\leq i\leq n} \frac{1}{\sqrt{S^i}} \sum_{j=1}^{p_i} \left(s_j^i \sum_{k=1}^{n} \hat{s}_j^{i,k}\right). \tag{58}$$

It can be rewritten using the identity $s_j^i = \sum_{k=1}^{n} \hat{s}_j^{i,k}$ to

$$\left\|\tilde{G}_{22}\right\|_{\infty} = \max_{1\leq i\leq n} \frac{1}{\sqrt{S^i}} \sum_{j=1}^{p_i} \left(s_j^i\right)^2 = \max_{1\leq i\leq n} \sqrt{S^i}. \tag{59}$$

Then the following inequality holds

$$\sqrt{S^i} = \sqrt{\sum_{j=1}^{p_i} \left(s_j^i\right)^2} \leq \sqrt{p_i} \max_{1\leq j\leq p_i} s_j^i = \sqrt{p_i} \max_{1\leq j\leq p_i} \sum_{l=1}^{p} \left\|[G_{22}]_{j+\sum_{m=1}^{i-1} p_m, l}\right\|_{\infty} \tag{60}$$

for each $\sqrt{S^i}$, where $1 \leq i \leq n$. Let us denote the maximum of uncertainty block sizes by $p_{\max}$, i. e. $p_{\max} = \max\limits_{1\leq i\leq n} p_i$. Then we obtain the inequality

$$\sqrt{S^i} \leq \sqrt{p_{\max}} \max_{1\leq j\leq p_i} \sum_{l=1}^{p} \left\|[G_{22}]_{j+\sum_{m=1}^{i-1} p_m, l}\right\|_{\infty} \leq \sqrt{p_{\max}} \max_{1\leq k\leq p} \sum_{l=1}^{p} \|[G_{22}]_{kl}\|_{\infty} . \tag{61}$$

Using the relation between the $\mathcal{H}_\infty$ and ℓ_1-norm of a SISO system we get

$$\sqrt{S^i} \le \sqrt{p_{\max}} \max_{1\le k\le p} \sum_{l=1}^{p} \left\| [G_{22}]_{kl} \right\|_\infty \le \sqrt{p_{\max}} \max_{1\le k\le p} \sum_{l=1}^{p} \left\| [G_{22}]_{kl} \right\|_1 . \tag{62}$$

The above inequality holds for an arbitrary index i. So we have verified that

$$\left\| \tilde{G}_{22} \right\|_\infty \le \sqrt{p_{\max}} \left\| G_{22} \right\|_1 \tag{63}$$

and thus

$$\sup_\omega \mu_\Delta \left(G_{22}(j\omega) \right) \le \| \tilde{G}_{22} \|_\infty \le \sqrt{p_{\max}} \left\| G_{22} \right\|_1 . \qquad \square$$

5. CONCLUSION

A global upper bound for the complex structured singular value related to a linear time-invariant MIMO system over all frequencies has been derived in the paper. We have shown that one can form a special non-negative matrix from the $\mathcal{H}_\infty$-norms of input-output channels of the system which has the property that its spectral radius for SISO perturbation blocks and its ∞-norm for MIMO ones is equal to or greater than the complex structured singular value at any frequency. Thus it provides a sufficient condition for the robust stability or performance of the uncertain system in $\mathcal{L}_2/\ell_2$-sense. On the basis of this result one can estimate how the greatest amplifications of the SISO parts of the system affect the result of robustness analysis. Also we have shown how our result fits into the inequality relation between the μ and ℓ_1-analysis.

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REFERENCES

[1] M. A. Dahleh and M. Khammash: Controller design for plants with structured uncertainty. Automatica *29* (1993), 37–56.

[2] M. A. Dahleh and I. J. Diaz–Bobillo: Control of Uncertain Systems. A Linear Programming Approach. Prentice Hall, NJ 1995.

[3] J. C. Doyle: Structured uncertainty in control system design. In: Proc. of 24th Conference on Decision and Control, Ft. Lauderdale FL 1985, pp. 260–265.

[4] M. Fiedler: Special Matrices and Their Application in Numerical Mathematics. SNTL – Nakladatelství technické literatury, Prague 1981.

[5] M. H. Khammash and J. B. Pearson: Performance robustness of discrete–time systems with structured uncertainty. IEEE Trans. Automat. Control *36* (1991), 398–412.

[6] M. H. Khammash and J. B. Pearson: Analysis and design for robust performance with structured uncertainty. Systems Control Lett. *20* (1993), 179–187.

[7] M. H. Khammash: Necessary and sufficient conditions for the robustness of time–varying systems with applications to sampled–data systems. IEEE Trans. Automat. Control *38* (1993), 49–57.

[8] A. Packard and J. C. Doyle: The complex structured singular value. Automatica *29* (1993), 71–109.

[9] A. L. Tits and M. K. H. Fan: On the small–μ theorem. Automatica *31* (1995), 1199–1201.

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