KYBERNETIKA — VOLUME 28 (1992), NUMBER 3, PAGES 234 – 238

AN APPROXIMATION OF THE PRESSURE FOR THE TWO-DIMENSIONAL ISING MODEL

Martin Janžura

A sequence of pressure functions corresponding to some one-dimensional models is used to approximate the pressure function of the two-dimensional Ising model. The rate of convergence is derived and the method is demonstrated with a numerical study.

1. INTRODUCTION

The two-dimensional Ising model is the simpliest non-trivial Gibbs random field. Namely, a probability measure μ on the space $\{0,1\}^{Z^2}$ is called to agree with the Ising model if its one-dimensional conditional distributions satisfy the "nearest-neighbor" property and can be expressed in the following way

$$\mu\left(x_t \,|\, x_{Z^2\setminus\{t\}}\right) = \mu\left(x_t \,|\, x_{\partial t}\right) = \Pi_t\left(x_t \,|\, x_{Z^2\setminus\{t\}}\right)$$

for every $t \in Z^d$ and a. e. $x \in \{0,1\}^T[\mu]$, where

$$\Pi_t\left(x_t \,|\, x_{Z^2\setminus\{t\}}\right) = \frac{\exp\left\{-x_t\left(h + J_1\left(x_{t+u} + x_{t-u}\right\} + J_2\left(x_{t+v} + x_{t-v}\right)\right)\right\}}{1 + \exp\left\{-h - J_1\left(x_{t+u} + x_{t-u}\right) - J_2\left(x_{t+v} + x_{t-v}\right)\right\}}$$

are called the local characteristics,

$$\partial t = \left\{s \in Z^2;\ \|t-s\| = 1\right\} = \{u, -u, v, -v\}, \qquad u = (1,0),\ v = (0,1),$$

and h, J_1, J_2 are arbitrary constants.

In general, the system $\{\Pi_t(\cdot|\cdot)\}_{t\in Z^2}$ depending on the triplet (h, J_1, J_2) does not determine the probability measure μ uniquely. The existence, uniqueness, and other properties of the Ising model are closely related to the function called the pressure and defined by the limit

$$\lim_{V\nearrow Z^2} |V|^{-1} \log \sum_{x_V\in\{0,1\}^V} \exp\left\{-h\sum_{t\in V} x_t - J_1 \sum_{t\in V\cap(V-u)} x_t\, x_{t+u} - J_2 \sum_{t\in V\cap(V-v)} x_t\, x_{t+v}\right\} = \\ = \ p\,(h, J_1, J_2)$$

where $V \nearrow Z^2$ means the expansion ensuring $|V|^{-1}\,|V\cap(V-t)| \longrightarrow 1$ for every $t \in Z^2$. (By $|V|$ we denote the cardinality.)

But, with the exception of the famous Onsager's result (cf. [3]), concerning a special case of the problem, no direct way of calculating the pressure p is known. Therefore various approximative methods, using mostly some kind of expansion, are applied. Here, we propose a new approximative method based on an approximation of the pressure of the two-dimensional model by the pressure of some properly chosen one-dimensional models, for which the transfer matrix method is available (cf. [2]).

As will be seen later, the method works quite well in the "high temperature" area (i. e. for "small" parameters h, J_1, J_2) and even in the neighborhood of the critical point it seems to give satisfactory results.

2. BASIC LEMMA

For a fixed positive integer R and a real γ let us consider the two-dimensional model with the state space $\overline{X} = \{0,1\}^R$ and the "nearest-neighbor" local characteristics given by

$$\overline{\Pi}_t^{\gamma}\left(\overline{x}_t \,|\, \overline{x}_{Z^2\setminus\{t\}}\right) = \frac{\exp\left\{-U_\gamma^0(\overline{x}_t) - \sum\limits_{s\in\partial t} U_\gamma^s(\overline{x}_t, \overline{x}_{t+s})\right\}}{\sum\limits_{\overline{y}_t\in\overline{X}} \exp\left\{-U_\gamma^0(\overline{y}_t) - \sum\limits_{s\in\partial t} U_\gamma^s(\overline{y}_t, \overline{x}_{t+s})\right\}}$$

for every $\overline{x}_t \in \overline{X},\ \overline{x}_{\partial t} \in \overline{X}^{\partial t}$, where

$$\begin{aligned}
U_\gamma^0(\overline{x}) &= h\cdot\sum_{i=1}^{R}\overline{x}^i + J_1\sum_{i=1}^{R-1}\overline{x}^i\,\overline{x}^{i+1},\\
U_\gamma^u(\overline{x},\overline{y}) &= \gamma\cdot J_1\cdot\overline{x}^R\,\overline{y}^1, \qquad U^{-u}(\overline{x},\overline{y}) = U^u(\overline{y},\overline{x}),\\
U_\gamma^v(\overline{x},\overline{z}) &= J_2\cdot\sum_{i=1}^{R}\overline{x}^i\,\overline{z}^i + (1-\gamma)\,J_1\,\overline{x}^R\,\overline{z}^1, \qquad U^{-v}(\overline{x},\overline{z}) = U^v(\overline{z},\overline{x}),
\end{aligned}$$

for every $\overline{x},\ \overline{y},\ \overline{z} \in \overline{X}$.

Let us denote by $G_I(\gamma)$ the set of translation invariant probability distributions on $\overline{X}^{Z^2}$ with the one-dimensional conditional distributions equal a.s. to the local characteristics $\left\{\overline{\Pi}_t^{\gamma}\right\}_{t\in Z^2}$.

Finally, we denote by

$$\overline{p}_\gamma\,(h,J_1,J_2) =$$

$$= \lim_{V\nearrow Z^2}|V|^{-1}\log\sum_{\overline{x}_V\in\overline{X}^V}\exp\left\{-\sum_{t\in V}U_\gamma^0(\overline{x}_t) - \sum_{t\in V\cap(V-u)}U_\gamma^u(\overline{x}_t,\overline{x}_{t+u}) - \sum_{t\in V\cap(V-v)}U_\gamma^v(\overline{x}_t,\overline{x}_{t+v})\right\}$$

the pressure corresponding to above defined model.

Lemma. Let $\gamma^* \in [0,1]$ be the point at which the function

$$F(\gamma) = \gamma\overline{p}_1\,(h,J_1,J_2) + (1-\gamma)\overline{p}_0\,(h,J_1,J_2) - \overline{p}_\gamma\,(h,J_1,J_2)$$

assumes its maximum. Then there exists

$$\mu^* \in G_I(\gamma^*)$$

such that

$$\overline{p}_1\,(h,J_1,J_2) - \overline{p}_0\,(h,J_1,J_2) = J_1\left[\mu^*\left(\overline{x}_0^R\cdot\overline{x}_u^1 = 1\right) - \mu^*\left(\overline{x}_0^R\cdot\overline{x}_v^1 = 1\right)\right]$$

holds.

Proof. The statement follows immediately from the equivalence between translation invariant Gibbs states and tangent functionals to the convex functional p (cf. [4], Thm. 8.3) and the general subdifferential calculus (cf. e.g. [5], Sec. 5).

3. MAIN RESULT

Now, let us make clear what was the aim of introducing the models with the "aggregated" state space $\overline{X}$ in the preceding section.

Directly from the definitions it is easy to see that

$$\overline{p}_1\,(h, J_1, J_2) = R \cdot p\,(h, J_1, J_2)$$

holds for every triplet (h, J_1, J_2).

Since for $\gamma = 0$ there is no horizontal interaction, i.e. the model consists of mutually independent columns, we may view the model as a one-dimensional one. And, considering all $\overline{x}_t,\ t \in Z$ as the corresponding segments of a sequence $x_Z = \{x_s\}_{s\in Z} \in \{0,1\}^Z$ (we put $x_s = \overline{x}_t^i$ for $s = t \cdot R + i$), we conclude that

$$\overline{p}_0\,(h, J_1, J_2) = R \cdot p^R\,(h, J_1, J_2)\,,$$

where

$$p^R\,(h, J_1, J_2) =$$
$$= \lim_{n\to\infty} |2n+1|^{-1} \log \sum_{x_{[-n,n]}\in\{0,1\}^{[-n,n]}} \exp\left\{-h \sum_{j=-n}^{n} x_j - J_1 \sum_{i=-n}^{n-1} x_j\,x_{j+1} - J_2 \sum_{i=-n}^{n-R} x_j\,x_{j+R}\right\}$$

is the pressure of the one-dimensional model with the state space $\{0,1\}$ and the local characteristics

$$\Pi_t^0\left(x_t \,|\, x_{Z\setminus\{t\}}\right) = \frac{\exp\left\{-h\,x_t - J_1 x_t\,(x_{t+1} + x_{t-1}) - J_2\,(x_{t-R} + y_{t+R})\right\}}{1 + \exp\left\{-h - J_1\,(x_{t+1} + x_{t-1}) - J_2\,(x_{t-R} + x_{t+R})\right\}},$$

for every $t \in Z,\ x_t \in \{0,1\},\ x_{Z\setminus\{t\}} \in \{0,1\}^{Z\setminus\{t\}}$.

Now, we may formulate the main result on the approximation.

Theorem. For every triplet (h, J_1, J_2) it holds

$$\left|p\,(h, J_1, J_2) - p^R\,(h, J_1, J_2)\right| \le (2R)^{-1}\,|J_1|,$$

and therefore

$$p\,(h, J_1, J_2) = \lim_{R\to\infty} p^R\,(h, J_1, J_2)\,.$$

Proof. The statement follows from Lemma and the considerations above if we realize that the probability measures

$$\nu_u(x,y) = \mu^*\left(\overline{x}_0^R = x,\ \overline{x}_u^1 = y\right), \qquad x,\, y \in \{0,1\},$$

and

$$\nu_v(x,y) = \mu^*\left(\overline{x}_0^R = x,\ \overline{x}_v^1 = y\right), \qquad x,\, y \in \{0,1\}$$

have the same marginals, and therefore

$$|\nu_u(1,1) - \nu_v(1,1)| \le \frac{1}{2}.$$

□

Remark. The values of p^R may be calculated with the aid of the transfer matrix (for details see e.g. [2], Section I.2.1). Of course, actually we are able to calculate p^R for rather small R only. But the convergence is, in fact, quite fast, and even $R = 6$ or $R = 7$, especially in high temperature area (i.e. for rather small interactions), give nice results.

4. NUMERICAL STUDY

Now, we try to demonstrate the method with a particular case which has been chosen in order to make possible a comparison of the obtained results with the rigorous Onsager's one.

Therefore, let $J_1 = J_2 = J \geq 0$ and $h = -2J$.

For $R = 4, 5, 6, 7$ and some $J \in [0, 2]$ the values of $p^R(-2J, J, J)$ obtained by the transfer matrix method (cf. [2], Section I.2.1) are given in the table.

	$J = 0$	$J = 0.5$	$J = 1.0$	$J = 1.5$	$J = 2 \cdot \log(1 + \sqrt{2})$	$J = 2$
$R = 4$	0.6931	0.9589	1.2579	1.5916	1.7800	1.9568
$R = 5$	0.6931	0.9590	1.2595	1.6085	1.8213	2.0320
$R = 6$	0.6931	0.9590	1.2590	1.5999	1.7968	1.9841
$R = 7$	0.6931	0.9590	1.2591	1.6051	1.8158	2.0297

Here, for the critical point $J_c = 2 \log \left(1 + \sqrt{2}\right)$ the exact Onsager's solution gives

$$p\left(-2J_c, J_c, J_c\right) = \log \left(1 + \sqrt{2}\right) + \log 2/2 + 2 \cdot G/\pi \doteq 1.8110692$$

($G = 0.915965594$ is the Catalan's constant).

Trying to make differences between the functions p^R for various R's more evident, we deal with their deviations

$$q^R(J) = p^R(-2J, J, J) - \log 2 - J/2$$

from the line $\log 2 + J/2$ (i.e. their common tangent in $J = 0$) in the following figure.

Similarly, we denote $q\left(J_c\right) = p\left(-2J_c, J_c, J_c\right) - \log 2 - J_c/2 \doteq 0.2365$.

5. CONCLUDING REMARK

Approximation of the described type was at first derived in [1] for purpose of application in mathematical statistics. But here a completely different proof is used, which yields a stronger result and deeper insight into the problem.

(Received March 23, 1991.)

REFERENCES

[1] M. Janžura: Estimating interactions in binary lattice data with nearest-neighbor property. Kybernetika *23* (1987), 136–142.

[2] D. H. Mayer: The Ruelle-Araki transfer operator in classical mechanics. (Lecture Notes in Physics 123.) Springer-Verlag, Berlin – Heidelberg – New York 1980.

[3] L. Onsager: Crystal statistics. Part I: A two-dimensional model with an order-disorder transition. Phys. Rev. *65* (1944), 117–149.

[4] C. Preston: Random Fields. (Lecture Notes in Mathematics 534.) Springer-Verlag, Berlin – Heidelberg – New York 1976.

[5] R. Temam and I. Ekeland: Convex Analysis and Variational Problems. North-Holland, Amsterdam 1976.

RNDr. Martin Janžura CSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation – Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha 8. Czechoslovakia.