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AN INTERPOLATION PROBLEM FOR MULTIVARIATE STATIONARY SEQUENCES

LUTZ KLOTZ

Let \mathbf{X} and \mathbf{Y} be stationarily cross-correlated multivariate stationary sequences. Assume that all values of \mathbf{Y} and all but one values of \mathbf{X} are known. We determine the best linear interpolation of the unknown value on the basis of the known values and derive a formula for the interpolation error matrix. Our assertions generalize a result of Budinský [1].

1. INTRODUCTION

In [1] Budinský studied the following problem. Let \mathbf{X} and \mathbf{Y} be two univariate stationarily cross-correlated stationary sequences. Assume that all values of \mathbf{Y} and all but one values of \mathbf{X} are known. Find the linear interpolation error of the unknown value of \mathbf{X} on the basis of all known values. In the present paper we generalize Budinský's result to multivariate sequences \mathbf{X} and \mathbf{Y} . The main tool of our investigations is the Hellinger-spectral domain of a stationary sequence. H. Salehi first used Hellinger integrals in the interpolation of multivariate stationary sequences, see [6] and [7]. His method was developed and completed by Makagon and Weron, cf. [2, 3], and [8]. Some results of these authors, on which we heavily lean, are summarized in Section 2. Section 3 is devoted to the solution of the interpolation problem mentioned above. We obtain a formula for the interpolation error matrix as well as a recipe for determining the best linear interpolation of the unknown value. Since our formulas are rather difficult to apply in the general situation, in Section 4 we study some special cases and, using some facts on the Moore–Penrose inverse of a non-negative Hermitian block matrix, derive more tractable formulas for the interpolation error matrix.

2. PRELIMINARIES AND NOTATIONS

Let \mathcal{N} , \mathcal{Z} , and \mathcal{C} be the sets of positive integers, integers, and complex numbers, resp. For $r \in \mathcal{N}$, the symbol \mathcal{M}_r , stands for the space of $r \times r$ -matrices with complex entries. If $A \in \mathcal{M}_r$, then A^* , $\mathcal{R}(A)$, $\text{Ker } A$, and $\rho(A)$ denote its adjoint, range, kernel, and rank, resp. Furthermore, A^+ is the Moore–Penrose inverse of A , cf. formulas (1.2) in [4]. If A is regular, its inverse A^{-1} coincides with A^+ .

The symbol I stands for a unit matrix, where its size should become clear from the context.

Let \mathcal{H} be a Hilbert space over \mathbb{C} and \mathcal{H}^r the Cartesian product of r copies of \mathcal{H} . We will consider \mathcal{H}^r as a left \mathcal{M}_r -module, i. e., the generic element \mathbf{u} of \mathcal{H}^r is written as a column vector so that for each $A \in \mathcal{M}_r$ the product $A\mathbf{u}$ is defined in a natural way and belongs to \mathcal{H}^r . The zero element of \mathcal{H}^r is denoted by \mathbf{O}_r , whereas the symbol 0 stands for \mathbf{O}_1 as well as for several zero matrices. For two vectors \mathbf{u}, \mathbf{v} of \mathcal{H}^r let $\langle \mathbf{u}, \mathbf{v} \rangle$ be their Grammian matrix. Finally, \mathbf{e}_k denotes the k th unit vector of \mathbb{C}^r , i. e. the vector whose k th entry is 1 and all its other elements are 0, $k \in \{1, \dots, r\}$.

An r -variate stationary sequence is a map $\mathbf{S} : \mathbf{Z} \ni n \rightarrow \mathbf{s}_n \in \mathcal{H}^r$ such that $\langle \mathbf{s}_m, \mathbf{s}_n \rangle$ depends only on $m - n, m, n \in \mathbf{Z}$. By $\widetilde{\mathcal{M}}$ we denote the time domain of \mathbf{S} , i. e. the closed subspace of \mathcal{H}^r spanned by all $\mathbf{s}_n, n \in \mathbf{Z}$, with coefficients from \mathcal{M}_r . Recall that $\widetilde{\mathcal{M}} = \mathcal{M}^r$, where \mathcal{M} is the closed linear subspace of \mathcal{H} , spanned by the entries of all $\mathbf{s}_n, n \in \mathbf{Z}$.

Let us assume that the spectral measure F of \mathbf{S} is absolutely continuous with respect to the Lebesgue measure σ on the σ -algebra \mathcal{B} of Borel sets of $[-\pi, \pi)$. Let f be the spectral density and $L^2(F)$ the spectral domain of \mathbf{S} , i. e. the left Hilbert \mathcal{M}_r -module of (equivalence classes of) \mathcal{B} -measurable \mathcal{M}_r -valued functions Φ such that $\int_{-\pi}^{\pi} \Phi(\lambda) f(\lambda) \Phi(\lambda)^* \sigma(d\lambda) = \int \Phi f \Phi^* d\sigma$ exists.

In the following we will omit the integration variable and the domain of integration $[-\pi, \pi)$ in the notation. Furthermore, relations between \mathcal{B} -measurable functions are to be understood as relations that hold σ -a. e., although we will not emphasize this each time.

Let U be Kolmogorov's isomorphism between the time domain and the spectral domain of \mathbf{S} , i. e., U is an isometric \mathcal{M}_r -linear isomorphism of $\widetilde{\mathcal{M}}$ onto $L^2(F)$ such that

$$U \mathbf{s}_n = e^{in} I, \quad n \in \mathbf{Z}.$$

Let us consider the Hilbert- \mathcal{M}_r -module $H^2(F)$ of (equivalence classes of) \mathcal{B} -measurable \mathcal{M}_r -valued functions M such that $\text{Ker } M \supseteq \text{Ker } f$ and $\int M f^+ M^* d\sigma$ exists. The mapping

$$V : \Phi \rightarrow \Phi f$$

establishes an isometric \mathcal{M}_r -linear isomorphism of $L^2(F)$ onto $H^2(F)$, cf. [6, Theorem 1] and [3, Theorem 3.3 (b)].

It is not hard to see that

$$V^{-1} M = M f^+, \quad M \in H^2(F). \tag{1}$$

In [3, Theorem 3.4 and Lemma 3.7] and [8, Lemma 4.5 (b)] it was proved the following result.

Lemma 1. A vector \mathbf{u} of $\widetilde{\mathcal{M}}$ is orthogonal to all \mathbf{s}_n , $n \in \mathbf{Z} \setminus \{0\}$, if and only if $VU\mathbf{u}$ is equal to a constant \mathcal{M}_r -valued function, where its value A has the following properties:

$$\mathcal{R}(A) \subseteq \mathcal{R}(f) \tag{2}$$

and

$$\int Af^+A^*d\sigma \tag{3}$$

exists. The matrix A can be computed by

$$A = \langle \mathbf{u}, \mathbf{s}_0 \rangle. \tag{4}$$

Conversely, if $A \in \mathcal{M}_r$ has properties (2) and (3), then there exists a vector $\mathbf{u} \in \widetilde{\mathcal{M}}$, which is orthogonal to all \mathbf{s}_n , $n \in \mathbf{Z} \setminus \{0\}$, such that $VU\mathbf{u} = A$ σ -a. e.

3. AN INTERPOLATION PROBLEM

Let $p, q \in \mathbb{N}$ and let \mathbf{X} be a p -variate and \mathbf{Y} a q -variate stationary sequence such that $\mathbf{S} : \mathbf{s}_n := \begin{pmatrix} \mathbf{x}_n \\ \mathbf{y}_n \end{pmatrix}$, $n \in \mathbf{Z}$, is a $(p+q)$ -variate stationary sequence. Let $\widetilde{\mathcal{M}}_0$ be the closed \mathcal{M}_{p+q} -linear hull of all \mathbf{s}_n , $n \in \mathbf{Z} \setminus \{0\}$, and $\begin{pmatrix} \mathbf{O}_p \\ \mathbf{O}_q \end{pmatrix}$. Denote the vector $\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{O}_q \end{pmatrix}$ by \mathbf{x}'_0 . Motivated by a paper of Budinský [1] we study the following interpolation problem:

Find the orthogonal projection $\widehat{\mathbf{x}}_0$ of \mathbf{x}'_0 onto $\widetilde{\mathcal{M}}_0$ and the interpolation error matrix

$$\Delta := \langle \mathbf{x}'_0 - \widehat{\mathbf{x}}_0, \mathbf{x}'_0 - \widehat{\mathbf{x}}_0 \rangle.$$

Since $\widetilde{\mathcal{M}}_0$ is of the form $\widetilde{\mathcal{M}}_0 = \mathcal{M}_0^{p+q}$, where \mathcal{M}_0 is the closed subspace of \mathcal{H} spanned by the entries of all \mathbf{s}_n , $n \in \mathbf{Z} \setminus \{0\}$, and the entries of \mathbf{y}_0 , the problem is equivalent to determining the orthogonal projections of the entries of \mathbf{x}_0 onto \mathcal{M}_0 . However, we find it convenient to study the larger space $\widetilde{\mathcal{M}}_0$ since this allows us to use the isomorphisms U and V .

First note that the singular part of the spectral measure F of \mathbf{S} does not affect on the interpolation error. So we assume that \mathbf{S} has a spectral density f .

Let

$$\widetilde{\mathbf{x}}_0 := \mathbf{x}'_0 - \widehat{\mathbf{x}}_0.$$

In the following we have to consider block partitions $A := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ of matrices A from \mathcal{M}_{p+q} . In all these cases the left upper block A_{11} is assumed to belong to \mathcal{M}_p .

In particular, the block partition $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{pmatrix}$ of f corresponds to the partition of \mathbf{S} into \mathbf{X} and \mathbf{Y} and the interpolation error matrix Δ has the form

$$\Delta = \begin{pmatrix} \Delta_{11} & 0 \\ 0 & 0 \end{pmatrix}, \tag{5}$$

where Δ_{11} is non-negative Hermitian and belongs to \mathcal{M}_p .

Consider the subset

$$L := \left\{ \begin{pmatrix} \mathbf{c} \\ \mathcal{O}_q \end{pmatrix} \in \mathbf{C}^{p+q} : \begin{pmatrix} \mathbf{c} \\ \mathcal{O}_q \end{pmatrix} \in \mathcal{R}(f) \text{ and } \int \begin{pmatrix} \mathbf{c} \\ \mathcal{O}_q \end{pmatrix}^* f^+ \begin{pmatrix} \mathbf{c} \\ \mathcal{O}_q \end{pmatrix} d\sigma \text{ exists} \right\} \text{ of } \mathbf{C}^{p+q}.$$

Since $\begin{pmatrix} \mathbf{c} \\ \mathcal{O}_q \end{pmatrix} \in L$ if and only if $\begin{pmatrix} \mathbf{c} \\ \mathcal{O}_q \end{pmatrix} \in \mathcal{R}(f)$ and the \mathbf{C}^p -valued function $((f^+)_{11})^{\frac{1}{2}} \mathbf{c}$ is square-integrable, the set L is a subspace of \mathbf{C}^{p+q} . Denote by P the orthogonal projection in \mathbf{C}^{p+q} onto L .

Let E be the \mathcal{M}_{p+q} -valued function

$$E(\lambda) := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \lambda \in [-\pi, \pi].$$

Theorem 2. The interpolation error matrix Δ can be calculated by

$$\Delta = \left(\int P f^+ P d\sigma \right)^+. \tag{6}$$

The orthogonal projection of \mathbf{x}'_0 onto $\widetilde{\mathcal{M}}_0$ is equal to

$$\widehat{\mathbf{x}}_0 = U^{-1}(E - \Delta f^+). \tag{7}$$

Proof. Note that $\widetilde{\mathbf{x}}_0$ is of the form $\begin{pmatrix} \mathbf{u} \\ \mathcal{O}_q \end{pmatrix}$ for some $\mathbf{u} \in \mathcal{M}_0^p$, which implies

$$\langle \widetilde{\mathbf{x}}_0, \mathbf{s}_0 \rangle = \langle \widetilde{\mathbf{x}}_0, \widetilde{\mathbf{x}}_0 \rangle = \Delta. \tag{8}$$

Since $\widetilde{\mathbf{x}}_0$ is orthogonal to all $\mathbf{s}_n, n \in \mathbf{Z} \setminus \{0\}$, from Lemma 1 and (8) we obtain that $VU\widetilde{\mathbf{x}}_0$ is a constant function whose value is equal to Δ . Since VU is an isometry of $\widetilde{\mathcal{M}}$ onto $H^2(F)$, it follows

$$\Delta = \langle \widetilde{\mathbf{x}}_0, \widetilde{\mathbf{x}}_0 \rangle = \int \Delta f^+ \Delta d\sigma. \tag{9}$$

Relations (2) and (5) yield $\mathcal{R}(\Delta) \subseteq \mathcal{R}(f)$. Thus $\int \Delta f^+ \Delta d\sigma = \int \Delta P f^+ P \Delta d\sigma = \Delta \int P f^+ P d\sigma \Delta$. Comparing this with (9), we get

$$\Delta = \Delta \int P f^+ P d\sigma \Delta. \tag{10}$$

If we can show that the range of the matrix $B := \int P f^+ P d\sigma$ is included in $\mathcal{R}(\Delta)$, the result immediately follows from (10). But $\mathcal{R}(B) \subseteq \mathcal{R}(P) \subseteq \mathcal{R}(f)$ and the integral $\int B f^+ B d\sigma = B \int P f^+ P d\sigma B$ exists. According to Lemma 1 there exists a vector \mathbf{u} of $\widetilde{\mathcal{M}}$, which is orthogonal to all $\mathbf{s}_n, n \in \mathbf{Z} \setminus \{0\}$, such that $VU\mathbf{u} = B = \langle \mathbf{u}, \mathbf{s}_0 \rangle \sigma$ -a. e. Moreover, since $\int VU\mathbf{u} f^+ VU \begin{pmatrix} \mathcal{O}_p \\ \mathbf{y}_0 \end{pmatrix} d\sigma = \int B f^+ f \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} d\sigma = \int B \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} d\sigma = 0$, the vector \mathbf{u} even belongs to the orthogonal complement of $\widetilde{\mathcal{M}}_0$. This means that it has the form $\mathbf{u} = D\widetilde{\mathbf{x}}_0$, for some $D \in \mathcal{M}_{p+q}$. Then $B = \langle \mathbf{u}, \mathbf{s}_0 \rangle = \langle D\widetilde{\mathbf{x}}_0, \mathbf{s}_0 \rangle = D \langle \widetilde{\mathbf{x}}_0, \widetilde{\mathbf{x}}_0 \rangle = D\Delta$, which implies $\text{Ker } \Delta \subseteq \text{Ker } B$ and, hence, $\mathcal{R}(B) \subseteq \mathcal{R}(\Delta)$. To prove (7) note that $U\widehat{\mathbf{x}}_0 = U\mathbf{x}'_0 - U\widetilde{\mathbf{x}}_0, U\mathbf{x}'_0 = E$, and $VU\widetilde{\mathbf{x}}_0 = \Delta$ σ -a. e., thus $U\widehat{\mathbf{x}}_0 = V^{-1}\Delta = \Delta f^+$ by (1). \square

Corollary 3. The range of Δ is equal to the range of P .

Proof. It was shown in the proof of Theorem 2 that $\mathcal{R}(\Delta) \subseteq \mathcal{R}(P)$. Thus, if P_Δ denotes the orthogonal projector onto $\mathcal{R}(\Delta)$, we get $\int P f^+ P d\sigma = \Delta^+ = P_\Delta \Delta^+ P_\Delta = P_\Delta \int P f^+ P d\sigma P_\Delta = \int P_\Delta P f^+ P P_\Delta d\sigma = \int P_\Delta f^+ P_\Delta d\sigma$. From this equality it is easy to conclude that $\mathcal{R}(P_\Delta) = \mathcal{R}(P)$. \square

4. SPECIAL CASES

Under additional assumptions formula (6) can be brought into a more explicit form. Because of (5) it is enough to give expressions for Δ_{11} .

Corollary 4. If the values of f are regular matrices and

$$\int (f_{11} - f_{12} f_{22}^{-1} f_{12}^*)^{-1} d\sigma \tag{11}$$

exists, then

$$\Delta_{11} = \left(\int (f_{11} - f_{12} f_{22}^{-1} f_{12}^*)^{-1} d\sigma \right)^{-1}. \tag{12}$$

Proof. If the matrix $f(\lambda)$ is regular, then the left upper block of $f(\lambda)^{-1}$ is equal to $(f_{11}(\lambda) - f_{12}(\lambda) f_{22}(\lambda)^{-1} f_{12}(\lambda)^*)^{-1}$ by the well-known Frobenius formula, $\lambda \in [-\pi, \pi)$. Now the result immediately follows from (6). \square

The following corollary generalizes Theorem 1 of [1].

Corollary 5. Let $p = 1$ and the values of f be regular matrices. Then Δ_{11} can be computed by (12), where the right-hand side of (12) is to be interpreted as 0, if the integral (11) does not exist.

Proof. If (11) exists, the result is a special case of Corollary 4. If (11) does not exist, the projection P is equal to 0. \square

In the statement and the proof of our next corollary we make use of the following result on matrices, which can be easily obtained from formula (3.24) in [4]. If $A \in \mathcal{M}_{p+q}$ and A is non-negative Hermitian, then $\rho(A) = \rho(A_{22}) + \rho(A_{11} - A_{12} A_{22}^+ A_{21})$. In particular, $\rho(A) = \rho(A_{22})$ if and only if $A_{11} - A_{12} A_{22}^+ A_{21} = 0$.

Corollary 6. Let $p = 1$. Then $\Delta_{11} = 0$ if one of the following conditions hold:

- (i) $\rho(f) = \rho(f_{22})$ or, equivalently, $f_{11} - f_{12} f_{22}^+ f_{12}^* = 0$ on a set of positive measure σ .
- (ii) $\rho(f) > \rho(f_{22})$ σ -a. e. and the integral

$$\int (f_{11} - f_{12}f_{22}^+f_{12}^*)^{-1}d\sigma \quad (13)$$

does not exist.

If $\rho(f) > \rho(f_{22})$ σ -a. e. and (13) exists, then Δ_{11} is equal to

$$\Delta_{11} = \left(\int (f_{11} - f_{12}f_{22}^+f_{12}^*)^{-1}d\sigma \right)^{-1}. \quad (14)$$

Proof. It is not hard to see that the condition $\rho(f(\lambda)) = \rho(f_{22}(\lambda))$ is equivalent to the fact that e_1 does not belong to $\mathcal{R}(f(\lambda))$, $\lambda \in [-\pi, \pi]$. So, (i) yields $P = 0$ and, hence, $\Delta_{11} = 0$. If $\rho(f(\lambda)) > \rho(f_{22}(\lambda))$, we have $\rho(f_{11}(\lambda) - f_{12}(\lambda)f_{22}(\lambda)^+f_{12}(\lambda)^*) = 1 = \rho(f_{11}(\lambda))$ and therefore $\rho(f(\lambda)) = \rho(f_{11}(\lambda)) + \rho(f_{22}(\lambda))$. Under this condition the left upper block of $f(\lambda)^+$ is equal to $(f_{11}(\lambda) - f_{12}(\lambda)f_{22}(\lambda)^+f_{12}(\lambda)^*)^{-1}$, cf. formula (3.32) in [4]. Thus, from the non-existence of (13) we again conclude $P = 0$ and the existence of (13) yields (14) because of (6). \square

Corollary 7. Let $p = 1$. Then $\Delta_{11} = 0$ if and only if e_1 belongs to $\mathcal{R}(f)$ σ -a. e. and the integral (13) exists.

Proof. In the proof of Corollary 6 it was mentioned that e_1 belongs to $\mathcal{R}(f(\lambda))$ if and only if $\rho(f(\lambda)) > \rho(f_{22}(\lambda))$, $\lambda \in [-\pi, \pi]$. Hence, Corollary 7 is a consequence of Corollary 6. \square

Now let us use our results to derive a minimality condition for r -variate stationary sequences due to Rozanov [5, Theorem 10.2 of Ch. 2].

An r -variate stationary sequence \mathbf{S} is called minimal in the sense of Rozanov if for each $k \in \{1, \dots, r\}$ the k th entry $s_0^{(k)}$ of \mathbf{s}_0 does not belong to the space \mathcal{H}_k spanned by the entries of all \mathbf{s}_n , $n \in \mathbf{Z} \setminus \{0\}$, and the elements $s_0^{(j)}$, $j \neq k$.

Corollary 8. An r -variate stationary process \mathbf{S} is minimal in the sense of Rozanov if and only if the values of f are regular matrices and all functions on the principal diagonal of f^{-1} are integrable.

Proof. From Corollary 7 it follows that $s_0^{(k)}$ does not belong to \mathcal{H}_k if and only if e_k belongs to $\mathcal{R}(f)$ and the k th function on the principal diagonal of f^+ is integrable. But $e_k \in \mathcal{R}(f)$ for all $k \in \{1, \dots, r\}$ if and only if f^{-1} exists. \square

Remark 9. We conclude with the remark that all results of the present paper can be extended to a multivariate stationary process on a discrete Abelian group in an obvious way.

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REFERENCES

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- [1] P. Budinský: Improvement of interpolation under additional information. In: Proceedings of the 4th Prague Symposium on Asymptotic Statistics (P. Mandl and M. Hušková, eds.), Charles University, Prague 1989, pp. 159–167.
 - [2] A. Makagon: Interpolation error operator for Hilbert space valued stationary stochastic processes. *Probab. Math. Statist.* 4 (1984), 57–65.
 - [3] A. Makagon and A. Weron: q -variate minimal stationary processes. *Studia Math.* 59 (1976), 41–52.
 - [4] R. M. Pringle and A. A. Rayner: *Generalized Inverse Matrices with Applications to Statistics*. Griffin, London 1971.
 - [5] Yu. A. Rozanov: *Stationary Random Processes* (in Russian). Fizmatgiz, Moscow 1963.
 - [6] H. Salehi: The Hellinger square-integrability of matrix-valued measures with respect to a non-negative hermitian measure. *Ark. Mat.* 7 (1967), 299–303.
 - [7] H. Salehi: Application of the Hellinger integrals to q -variate stationary stochastic processes. *Ark. Mat.* 7 (1967), 305–311.
 - [8] A. Weron: On characterizations of interpolable and minimal stationary processes. *Studia Math.* 49 (1974), 165–183.

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