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REMARKS ON FUZZY QUANTITIES WITH FINITE SUPPORT[1]

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The general results concerning the algebraic properties of fuzzy quantities presented, e. g., in [2, 4, 6] can be rather completed and specifically interpreted if the fuzzy quantities with finite support are considered. It is possible to show that the algebraic equivalence relations over such quantities part their class into characteristical subclasses.

1. INTRODUCTION

As shown, e. g., in [2] and [4] the fuzzy numbers and generally fuzzy quantities can be arithmetically handled but the operations over them do not fulfil some important algebraic properties. This lack of algebraic perfectness is not purely technical, it reflects the lack of determinism and the structure of uncertainty typical for the fuzziness. The disproportion between the strict determinism of algebra and vagueness of fuzzy quantities can be avoided if we replace the crisp equality in algebraic rules by certain types of equivalence, as shown, e. g., in [2, 3, 4].

Each equivalence means that some type of equivalence classes is considered instead of single elements of the basic set. In our case the classes of equivalent fuzzy quantities can be especially well specified if only the quantities with finite support are considered. It enables us to formulate a few results which could not be (up to now) derived for the general case, and to use them for a starting point to some conclusions on the nature of fuzziness in quantitative data.

In the whole paper we denote by R the set of all real numbers and by $R_0 = R - \{0\}$ the set of non-zero reals.

By *normal fuzzy quantity* we call any fuzzy subset a of R with the membership function $f_a : \; R \rightarrow [0, 1]$ such that

$$\sup(f_a(x) : \; x \in R) = 1. \tag{1}$$

If, moreover, the support of f_a, i. e. the set

$$\{x \in R : \; f_a(x) > 0\} \tag{2}$$

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is finite, we call a the *finite-support normal fuzzy quantity* (in this paper we mean by fuzzy quantity always the finite-support normal fuzzy quantity). The set of all fuzzy quantities will be in this paper denoted by $\mathbb{R}$. It will be useful to denote by $\mathbb{R}_0$ the set of non-zero fuzzy quantities i. e. such

$$\{a \in \mathbb{R} : \ f_a(0) = 0\}. \tag{3}$$

For $a \in \mathbb{R}$ condition (1) turns into

$$\max(f_a(x) : \ x \in R) = 1, \tag{4}$$

which means that $f_a(x) = 1$ for at least one $x \in R$.

It will be useful to specify the following symbols of some sets of fuzzy quantities

$$\mathbb{R}^+ = \{a \in \mathbb{R} : \ f_a(x) = 0 \text{ for } x \leq 0\}\,, \qquad \mathbb{R}^- = \{a \in \mathbb{R} : \ f_a(x) = 0 \text{ for } x \geq 0\}\,, \tag{5}$$

and the fuzzy quantities from the set

$$\mathbb{R}^\star = \mathbb{R}^+ \cup \mathbb{R}^- \subset \mathbb{R}_0 \tag{6}$$

are called *polarized*.

2. ARITHMETICAL OPERATIONS

Fuzzy quantities represent vague numerical data and, consequently, they should be processed by the usual arithmetical operations or by their close analogy.

Due to the referred literature (and to some other works) we define the addition and multiplication of fuzzy quantities, with respect to the finiteness of their supports, in the following way.

Let a, $b \in \mathbb{R}$ be fuzzy quantities with membership functions f_a, f_b, respectively. By $a \oplus b$ we denote the fuzzy quantity with membership function $f_{a\oplus b}$ such that

$$f_{a\oplus b}(x) = \max_{y\in R}(\min(f_a(y),\, f_b(x-y))) = \max_{z\in R}(\min(f_a(x-z),\, f_b(z)))\,, \quad x \in R. \tag{7}$$

If a, $b \in \mathbb{R}_0 \subset \mathbb{R}$ then we denote by $a \odot b$ the fuzzy quantity with membership function $f_{a\odot b}$, where

$$f_{a\odot b}(x) = \max_{y\in R_0}(\min(f_a(y),\, f_b(x/y))) = \max_{z\in R_0}(\min(f_a(x/z),\, f_b(z)))\,, \quad x \in R. \tag{8}$$

Remark 1. The finiteness of the supports of a and b and relations (7), resp. (8), imply that also $a \oplus b$ and $a \odot b$ have finite supports.

Remark 2. If a, $b \in \mathbb{R}_0$ then (8) implies that also $a \odot b \in \mathbb{R}_0$, i. e. $f_{a\odot b}(0) = 0$.

Fuzzy quantities $a \oplus b$ and $a \odot b$ are called the *sum* and the *product* of a and b, respectively.

As shown in [6] the definition of product $a \odot b$ can be extended to fuzzy quantities from $\mathbb{R} \supset \mathbb{R}_0$, where $f_{a\odot b}(x)$ is given by (8) for $x \neq 0$, and

$$f_{a\odot b}(0) = \max\,(f_a(0),\, f_b(0))\,. \tag{9}$$

This extension is coherent with the original definition of $a \odot b$ for $a, b \in \mathbb{R}_0$ and preserves all its useful properties.

If $x \in R$ then we denote by $\langle x \rangle$ the *degenerated* fuzzy quantity for which

$$f_{\langle x \rangle}(x) = 1, \qquad f_{\langle x \rangle}(y) = 0 \qquad \text{if } y \neq x. \tag{10}$$

For a crisp real number $r \in R$ the products of degenerated $\langle r \rangle$ with fuzzy quantity gets a simplified form. For $a \in \mathbb{R}$ and $r \in R$ we denote $r \cdot a = \langle r \rangle \odot a$, where

$$\begin{aligned} f_{r \cdot a}(x) &= f_a(x/r) \qquad \text{if } r \neq 0, \\ &= f_{\langle 0 \rangle}(x) \qquad \text{if } r = 0. \end{aligned} \tag{11}$$

As shown e. g. in the papers referred below, set $\mathbb{R}$ is a commutative monoid concerning operation $\oplus$ i. e.

$$a \oplus b = b \oplus a, \qquad a \oplus (b \oplus c) = (a \oplus b) \oplus c, \qquad a \oplus \langle 0 \rangle = a \tag{12}$$

for $a, b, c \in \mathbb{R}$. Analogously, $\mathbb{R}$ is a commutative monoid concerning operation $\odot$, i. e. for $a, b, c \in \mathbb{R}$

$$a \odot b = b \odot a, \qquad a \odot (b \odot c) = (a \odot b) \odot c, \qquad a \odot \langle 1 \rangle = a. \tag{13}$$

For the crisp product of $r \in R,\ a, b \in \mathbb{R}$

$$r \cdot (a \oplus b) = (r \cdot a) \oplus (r \cdot b). \tag{14}$$

On the other hand, if we denote for $a \in \mathbb{R}$, $b \in \mathbb{R}_0$ the fuzzy quantities $-a \in \mathbb{R}$ and $1/b \in \mathbb{R}_0$ such that

$$f_{-a}(x) = f_a(-x), \qquad f_{1/b}(y) = f_b(1/y), \qquad f_{1/b}(0) = 0, \qquad x \in R,\ y \in R_0 \tag{15}$$

then the remaining group properties

$$a \oplus (-a) = \langle 0 \rangle \qquad \text{and} \qquad b \odot (1/b) = \langle 1 \rangle, \tag{16}$$

as well as the complementary distributivity

$$(r_1 + r_2) \cdot a = r_1 \cdot a \oplus r_2 \cdot a, \qquad r_1, r_2 \in R, \tag{17}$$

are not generally fulfilled. The roots of this lack of pleasant algebraic properties are discussed in [2, 4, 6]. Generally, it is innatural to demand the validity of strict equations (16) between fuzzy quantities and crisp numbers, especially regarding the fact that the operations $\oplus$ and $\odot$ do increase the fuzziness of the operated quantities. Evidently also the repetitive addition becomes to be different from multiplication by the number of repetitions (e. g. $a \oplus a \neq 2 \cdot a$) if indeterminism enters the process, as expressed by (17). The distributivity of $\oplus$ and $\odot$ is preserved in very special cases only, as shown in [1].

3. ALGEBRAIC EQUIVALENCES

The previous paragraph shows even how to avoid some of the discrepancies connected with the arithmetical processing of fuzzy numbers. It is not rational to implement crisp numbers, 0 and 1, into the manipulation with fuzzy phenomena. Instead of it we should include some kind of "fuzzy zero" (in the additive case) or "fuzzy unit" (in the multiplicative case). It was done in [2] and [4] in the following way.

Let $a \in \mathbb{R}$, $x \in R$. We say that a is x-symmetric iff

$$f_a(x+z) = f_a(x-z) \qquad \text{for all } z \in R. \tag{18}$$

By $\mathbb{S}_x$ we denote the sets of all x-symmetric fuzzy quantities. $\mathbb{S}$ denotes the union

$$\mathbb{S} = \bigcup_{x \in R} \mathbb{S}_x. \tag{19}$$

If a, $b \in \mathbb{R}$ then we say that they are *additively equivalent*, and write $a \sim_\oplus b$ iff there exist s_1, $s_2 \in \mathbb{S}_0$ such that

$$a \oplus s_1 = b \oplus s_2. \tag{20}$$

Analogously, if $b \in \mathbb{R}$ and $y \in R_0$ we say that b is y-transversible iff

$$\begin{aligned} f_b(y \cdot z) &= f_b(y/z) && \text{for } z > 0, \\ &= 0 && \text{for } z \leq 0. \end{aligned} \tag{21}$$

By $\mathbb{T}_y$ we denote the set of all y-transversible fuzzy quantities, and $\mathbb{T}$ denotes the union

$$\mathbb{T} = \bigcup_{y \in R_0} \mathbb{T}_y.$$

It is not difficult to verify that any y-transversible fuzzy quantity belongs to the set $\mathbb{R}^+$ iff $y > 0$ and to $\mathbb{R}^-$ iff $y < 0$ (cf. (5)).

We say that a and b from $\mathbb{R}$ are *multiplicatively equivalent*, in symbols $a \sim_\odot b$, iff there exist t_1, $t_2 \in \mathbb{T}_1$ such that

$$a \odot t_1 = b \odot t_2. \tag{22}$$

Let us remember that

$$a \oplus (-a) \sim_\oplus \langle 0 \rangle \qquad \text{for } a \in \mathbb{R} \tag{23}$$

and

$$b \odot (1/b) \sim_\odot \langle 1 \rangle \qquad \text{for } b \in \mathbb{R}^\star. \tag{24}$$

It was shown (cf. [2, 4]) that $\mathbb{R}$ with the operation $\oplus$ is a commutative group up to the equivalence relation $\sim_\oplus$, and $\mathbb{R}^\star = \mathbb{R}^+ \cup \mathbb{R}^-$ with $\odot$ forms a commutative group up to $\sim_\odot$.

Remark 3. If a, $b \in \mathbb{S}_0$ then evidently $a \sim_\oplus b$ and $a \oplus b \in \mathbb{S}_0$.

Lemma 1. If a, b are fuzzy quantities and $a \in \mathbb{S}_x,\ b \in \mathbb{S}_y$, x, $y \in R$, then $a \sim_{\oplus} b$ iff $x = y$.

Proof. If $a \in \mathbb{S}_x,\ b \in \mathbb{S}_y$ then by [3] $a = \langle x \rangle \oplus s_1,\ b = \langle y \rangle \oplus s_2$ for s_1, $s_2 \in \mathbb{S}_0$. The equivalence $a \sim_{\oplus} b$ means that there exist s_1', $s_2' \in \mathbb{S}_0$ such that

$$\langle x \rangle \oplus s_1 \oplus s_1' = \langle y \rangle \oplus s_2 \oplus s_2', \tag{25}$$

where $s_1 \oplus s_1' = s_1'' \in \mathbb{S}_0,\ s_2 \oplus s_2' = s_2'' \in \mathbb{S}_0$. As $-s_2'' = s_2''$

$$\langle x \rangle \oplus s_1'' \oplus (-(\langle y \rangle \oplus s_2'')) = \langle x - y \rangle \oplus (s_1'' \oplus s_2'') = \langle x - y \rangle \oplus s$$

for $s = s_1'' \oplus s_2'' \in \mathbb{S}_0$. Equality (25) implies that $\langle x - y \rangle \oplus s \in \mathbb{S}_0$ which is possible iff $x - y = 0$ (cf. [3]). □

Lemma 2. If a, b are fuzzy quantities and $a \in \mathbb{T}_x,\ b \in \mathbb{T}_y$, x, $y \in R_0$ then $a \sim_{\odot} b$ iff $x = y$.

Proof. The proof is analogous to the previous one where relevant results from [4] are used. □

4. EQUIVALENCE CLASSES

If we limit our considerations to the fuzzy quantities with finite support, some useful conclusions can be derived. Those ones which concern the additive case were in a slightly modified form presented in [2] (e. g., Lemmas 9 and 10 or Theorems 8 and 9).

The proofs of the following statements demand to proceed rather less strict type of fuzzy quantities than the normal ones considered in the other paragraphs. An *arbitrary fuzzy quantity* with finite support will be called any fuzzy quantity a with membership function $f_a : \ R \to [0,1]$ with finite support (2) but generally not fulfilling the normality assumptions (1) and/or (4).

Lemma 3. Let a be arbitrary fuzzy quantity with finite support, let $s \in \mathbb{S}_0$ be 0-symmetrical fuzzy quantity, let $x_0 \in R$ be such that $f_a(x_0) = f_a(-x_0) = 0$, and let a_0 be an arbitrary fuzzy quantity such that

$$\begin{aligned} f_{a_0}(x) &= f_a(x) \qquad \text{for all } x \in R,\ |x| \neq |x_0| \\ f_{a_0}(x_0) &= f_{a_0}(-x_0) \geq \max\left(f_a(x) :\ x \in R\right). \end{aligned}$$

Then for any $x \in R$ $\quad f_a(x) = f_a(-x)$ iff $f_{a_0}(x) = f_{a_0}(-x)$, and $f_{a \oplus s}(x) = f_{a \oplus s}(-x)$ iff $f_{a_0 \oplus s}(x) = f_{a_0 \oplus s}(-x)$.

Proof. The validity of the first equivalence is evident. Let us denote

$$\{x_1, \ldots, x_n\} = \{x \in R :\ f_a(x) > 0\}. \tag{26}$$

Then for any $y \in R$

$$f_{a_0 \oplus s}(y) = \max_{x \in R} (\min(f_{a_0}(x), f_s(y-x))) = \max_{i=0,1\ldots,n} (\min(f_{a_0}(x_i), f_s(y-x_i))) \,, \tag{27}$$

$$\begin{aligned} f_{a_0 \oplus s}(-y) &= \max_{x \in R} (\min(f_{a_0}(x),\ f_s(-y-x))) = \\ &= \max_{x \in R} (\min(f_{a_0}(x), f_s(y+x))) = \max_{x \in R} (\min(f_{a_0}(-x), f_s(y-x))) \,. \end{aligned} \tag{28}$$

Let $f_{a \oplus s}(y) = f_{a \oplus s}(-y)$ for all $y \in R$, and let us consider the $y_0 \in R$ for which

$$f_{a_0 \oplus s}(y_0) = f_{a_0}(x_0) = \min(f_{a_0}(x_0),\ f_s(y_0 - x_0)) \,.$$

Then, as $f_{a_0}(x_0) \geq f_a(x)$ for all $x \in R$, (27) implies

$$\begin{aligned} f_{a_0 \oplus s}(-y_0) &= \max_x (\min(f_{a_0}(-x),\ f_s(y_0 - x))) = \\ &= \min(f_{a_0}(-x_0),\ f_s(y_0 - x_0))) = f_{a_0}(-x_0) = f_{a_0}(x_0) \end{aligned}$$

and consequently $f_{a_0 \oplus s}(y_0) = f_{a_0 \oplus s}(-y_0)$. If $f_{a_0 \oplus s}(y) \neq f_{a_0}(x_0)$ then it is equal to some of the values of $f_{a \oplus s}(y) = f_{a \oplus s}(-y)$. If, on the other hand, $f_{a_0 \oplus s}(y) = f_{a_0 \oplus s}(-y)$ for all $y \in R$ then for any $y \in R$ either $f_{a \oplus s}(y) = f_{a_0 \oplus s}(y) = f_{a_0 \oplus s}(-y) = f_{a \oplus s}(-y)$ or $f_{a \oplus s}(y) = 0 \neq f_{a_0 \oplus s}(y)$ and then also $f_{a \oplus}(-y) = 0$ as follows from the previous steps of the proof. □

Lemma 4. Let $a \in \mathbb{R}$ be a fuzzy quantity, let $s \in \mathbb{S}_0$ be 0-symmetric fuzzy quantities and let $a \oplus s \in \mathbb{S}_0$. Then $a \in \mathbb{S}_0$ as well.

Proof. Let us denote $\{x_1, \ldots, x_n\}$ by (26). As $s \in \mathbb{S}_0$, equalities (27) and (28) hold. Let us choose the $j \in \{1, \ldots, n\}$ for which

$$f_a(x_j) = \max(f_a(x_i) :\ i = 1, \ldots, n) \,, \tag{29}$$

$$|x_j| \geq |x_i| \qquad \text{for all } x_i \text{ such that } f_a(x_i) = f_a(x_j), \tag{30}$$

$$x_j \geq 0. \tag{31}$$

Such j exists, as the finiteness of support guarantees the fulfilling of (29) and (30) for at least one $j \in \{1, \ldots, n\}$. If (31) is not fulfilled for some j respecting (29) and (30) then there exists $x_j \in R$ such that $x_j < 0$ and $f_a(-x_j) < f_a(x_j)$. Let us choose, then, $y_0 \in R$ such that

$$f_s(y_0 - x_j) \geq f_a(x_j) \qquad \text{and} \qquad f_s(y) < f_a(x_j) \qquad \text{for } y > y_0 - x_j.$$

It means that

$$f_{a \oplus s}(y_0) = \min(f_a(x_j),\ f_s(y_0 - x_j)) > \min(f_a(-x_j),\ f_s(y_0 - x_j)) = f_{a \oplus s}(-y_0)$$

which inequality contradicts the symmetry assumption on $a \oplus s$.

Having chosen $j \in \{1, \ldots, n\}$ and $x_j \in R$ which fulfill (29), (30), (31), we choose the $y_0 \in R$ for which

$$\begin{aligned} &f_s(y_0 - x_j) \geq f_a(x_j), \\ &f_s(y) < f_a(x_j) \qquad \text{for } y < y_0 - x_j, \end{aligned}$$

then by (27) and (28)

$$\begin{aligned} f_{a\oplus s}(y_0) &= \max_{i=1,\dots,n}(\min(f_a(x_i),\, f_s(y_0-x_i))) \geq f_a(x_j,) \\ f_{a\oplus s}(-y_0) &= \max_{x\in R}(\min(f_a(-x),\, f_s(y_0-x))) = \\ &= \min(f_a(-x_j),\, f_s(y_0-x_j))) \leq f_a(-x_j) \leq f_a(x_j). \end{aligned}$$

But the 0-symmetry of $a \oplus s$ means that

$$f_a(x_j) \leq f_{a\oplus s}(y_0) = f_{a\oplus s}(-y_0) \leq f_a(-x_j) \leq f_a(x_j)$$

and, consequently, $f_a(-x_j) = f_a(x_j)$.

Let us construct, now, an arbitrary fuzzy quantity with finite support a_1 such that

$$\begin{aligned} f_{a_1}(x) &= f_a(x) \qquad \text{for } x \in R,\ |x| \neq x_j, \\ f_{a_1}(x_j) &= f_{a_1}(-x_j) = 0. \end{aligned}$$

Lemma 3 implies that $f_{a_1}(x) = f_{a_1}(-x)$ for all x iff $f_a(x) = f_a(-x)$ for all x and that $f_{a\oplus s}(x) = f_{a\oplus s}(-x)$ for all $x \in R$ iff $f_{a_1\oplus s}(x) = f_{a_1\oplus s}(-x)$ for all $x \in R$. It means that we may repeat the previous procedure for a_1 and, after a finite number of analogous induction steps, we prove $f_a(x) = f_a(-x)$ for all $x \in R$. □

Remark 4. It follows from the proof of Lemma 4 that an analogous statement is valid for a being arbitrary fuzzy quantity with finite support such that $f_{a\oplus s}(x) = f_{a\oplus s}(-x)$ for all $x \in R$. Then $f_a(x) = f_a(-x)$ for all $x \in R$, as well.

The following statements concerning the additive equivalence can be derived from the previous lemmas.

Theorem 1. If $a, b \in \mathbb{R}$ are fuzzy quantities then $a \sim_\oplus b$ if and only if $a \oplus (-b) \in \mathbb{S}_0$.

P r o o f. The statement can be proved similarly to Theorem 8 in [2], using previous Lemma 4 and definitoric equality (20). □

Theorem 2. If $a \in \mathbb{R}$ is a fuzzy quantity then $a \sim_\oplus \langle 0 \rangle$ if and only if $a \in \mathbb{S}_0$.

P r o o f. The statement follows from Theorem 1 immediately. □

Theorem 3. Subsets $\mathbb{S}_x$, $x \in R$, form equivalence subclasses of the set of fuzzy quantities $\mathbb{R}$, according to $\sim_\oplus$.

Proof. It was shown by Lemma 1 that for $x, y \in R$, $a \in \mathbb{S}_x$, $b \in \mathbb{S}_y$, the relation $a \sim_\oplus b$ can be true iff $x = y$. Let us suppose that there exists $a \in \mathbb{R} - \mathbb{S}$ and $b \in \mathbb{S}$ such that $a \sim_\oplus b$, and let $x \in R$ be such that $b \in \mathbb{S}_x$. Then $b = \langle x \rangle \oplus s$ for some $s \in \mathbb{S}_0$ and there exist $s_1, s_2 \in \mathbb{S}_0$ for which

$$a \oplus s_1 = b \oplus s_2 = \langle x \rangle \oplus s \oplus s_2.$$

It means

$$a \oplus s_1 \oplus \langle -x \rangle = s \oplus s_1 \oplus \langle 0 \rangle = s \oplus s_1,$$

and by Lemma 4 $a \oplus \langle -x \rangle \in \mathbb{S}_0$. By [3], $a \in \mathbb{S}_{-x} \subset \mathbb{S}$ which contradicts the assumption $a \in \mathbb{R} - \mathbb{S}$. □

Remark 5. Evidently, there exist more equivalence classes in $\mathbb{R}$ than the sets $\mathbb{S}_x$, $x \in R$. However, the specific position of symmetry in fuzzy quantities (cf. [5] or [2]) is even more stressed by the previous result.

The transversible fuzzy quantities and operation of multiplication can be proceeded in a similar way. In the following paragraphs we limit the presentation of the procedure to the points which essentially differ from the additive case.

Let us remember that the concept of arbitrary fuzzy quantity with finite support mentioned above keeps unchanged.

Lemma 5. Let a be an arbitrary fuzzy quantity with finite support, let $f_a(x) = 0$ for all $x \le 0$, let $t \in \mathbb{T}_1$ be a 1-transversible fuzzy quantity, let $x_0 > 0$ be such that $f_a(x_0) = f_a(1/x_0) = 0$, and let a_0 be an arbitrary fuzzy quantity such that

$$f_{a_0}(x) = f_a(x) \qquad \text{for all } x \in R,\ x_0 \ne x \ne 1/x_0,$$
$$f_{a_0}(x_0) = f_{a_0}(1/x_0) \ge f_a(x) \qquad \text{for all } x \in R.$$

Then for any $x \in R_0$ $f_a(x) = f_a(x/1)$ iff $f_{a_0}(x) = f_{a_0}(1/x)$, and $f_{a \odot t}(x) = f_{a \odot t}(1/x)$ iff $f_{a_0 \odot t}(x) = f_{a_0 \odot t}(1/x)$.

Proof. If we denote the support set of a by $\{x_1, \dots, x_n\}$, like in (26), then for any $y > 0$, analogously to (27) and (28),

$$f_{a_0 \odot t}(y) = \max_{x>0} (\min(f_{a_0}(x),\ f_t(y/x))) = \max_{i=0,1,\dots,n} (\min(f_{a_0}(x_i,),\ f_t(y/x_i))) , \tag{32}$$

$$\begin{aligned} f_{a_0 \odot t}(1/y) &= \max_{x>0} (\min(f_{a_0}(x),\ f_t(1/y \cdot x))) = \\ &= \max_{x>0} (\min(f_{a_0}(x),\ f_t(y \cdot x))) = \max_{x>0} (\min(f_{a_0}(1/x),\ f_t(y/x))) . \end{aligned} \tag{33}$$

The assumptions of the lemma immediately imply that for $y \le 0$ $f_{a \odot t}(y) = f_{a_0 \odot t}(y) = 0$ as well as $f_{a_0}(x) = f_a(x) = f_t(x) = 0$ for all $x \le 0$.

Further procedure of the proof is analogous to Lemma 3. If we choose $y_0 \in R_+$ such that

$$f_{a_0 \odot t}(y_0) = f_{a_0}(x_0) \ge f_a(x), \qquad x > 0,$$

then by (32) $f_{a_0\odot t}(1/y_0) = f_{a_0}(x_0) = f_{a_0\odot t}(y_0)$.

If $f_{a_0\odot t}(y) \neq f_{a_0}(x_0)$ then the desired equality follows from the properties of $a \odot t$. The first one of the stated equivalences, namely $f_{a_0}(x) = f_{a_0}(1/x)$ iff $f_a(x) = f_a(1/x)$, is evident from the assumptions. □

Lemma 6. Let $a \in \mathbb{R}_0$ be a fuzzy quantity, let $t \in \mathbb{T}_1$ be 1-transversible fuzzy quantity, and let $a \odot t \in \mathbb{T}_1$. Then $a \in \mathbb{T}_1$ as well.

Proof. If there exists $x_0 \leq 0$ such that $f_a(x_0) > 0$ then (8) implies that also $f_{a\odot t}(y_0) > 0$ for some $y_0 \leq 0$, and in such case $a \odot t \notin \mathbb{T}_1$. Consequently $a \in \mathbb{R}^+$. Further steps of the proof are analogous to the proof of Lemma 4. If we accept the notation $\{x_1, \ldots, x_n\}$ for the support (26) then $x_i > 0$ for all $i = 1, \ldots, n$ and we may choose $j \in \{1, \ldots, n\}$ for which

$$f_a(x_j) \geq f_a(x_i) \qquad \text{for all } i = 1, \ldots, n, \tag{34}$$

$$x_j \geq 1, \tag{35}$$

$$\text{if } f_a(x) = f_a(x_j) \text{ for some } x \in R \text{ then either } 1 \leq x \leq x_j \text{ or } 1 \geq x \geq 1/x_j \tag{36}$$

Such x_j in R does exist, as the finiteness of support guarantees the existence of x_j fulfilling (34). If (35) and (36) are not simultaneously true, i. e. if there exist x_j for which (34) holds, and $0 < x_j < 1$, $f_a(x) < f_a(x_j)$ for all $x \geq 1/x_j > 1$ then we may choose $y_0 > 0$ such that by (32), (33)

$$f_t(y_0/x_j) \geq f_a(x_j) \qquad \text{and} \qquad f_t(y) < f_a(x_j) \qquad \text{for } y > y_0/x_j.$$

But then

$$f_{a\odot t}(y_0) = \min\left(f_a(y_j),\, f_t(y_0/x_j)\right) > \min\left(f_a(1/x_j),\, f_s(y_0/x_j)\right) = f_a(1/y_0)$$

which is impossible because of the transversibility of t. To the $j \in \{1, \ldots, n\}$ and $x_j > 0$ fulfilling (34), (35), (36) we find $y_0 > 0$ for which

$$\begin{aligned} & f_t(y_0/x_j) \geq f_a(x_j) \\ & f_t(y) < f_a(x_j) \qquad \text{for } y < y_0/x_j. \end{aligned}$$

According to (32), (33),

$$\begin{aligned} f_{a\odot t}(y_0) &= \max_{i=1,\ldots,n} (\min(f_a(x_i),\, f_t(y_0/x_i)) \geq f_a(x_j), \\ f_{a\odot t}(1/y_0) &= \max_{x>0} (\min(f_a(1/x),\, f_t(y_0/x))) = \\ &= \min(f_a(1/x_j),\, f_t(y_0/x_j)) \leq f_a(1/x_j) \leq f_a(x_j). \end{aligned}$$

The 1-transversibility of $a \odot t$ means that

$$f_a(x_j) \leq f_{a\odot t}(y_0) = f_{a\odot t}(1/y_0) \leq f_a(1/x_j) \leq f_a(x_j)$$

and, consequently, $f_a(1/x_j) = f_a(x_j)$.

Now we may construct an arbitrary fuzzy quantity a_1 with finite support, where

$$f_{a_1}(x) = f_a(x) \qquad \text{for } x_j \neq x \neq 1/x_j,$$
$$f_{a_1}(x_j) = f_{a_1}(1/x_j) = 0,$$

and using Lemma 5 we may proceed for a_1 analogously to a. After a finite number of induction steps the 1-transversibility of a will be proved. □

Remark 6. A statement analogous to Lemma 6 is valid for a being arbitrary fuzzy quantity with finite support such that $f_{a\odot t}(x) = f_{a\odot t}(1/x)$ for all $x > 0$, as follows from its proof. Then $f_a(x) = f_a(1/x)$ for all $x > 0$.

Theorems analogous to those ones derived for additive case can be obtained for the multiplicative ones, as well.

Theorem 4. If a, $b \in \mathbb{R}^\star$ are fuzzy quantities then $a \sim_\odot b$ if and only if $a \odot (1/b) \in \mathbb{T}_1$.

Proof. If $a \sim_\odot b$ for a, $b \in \mathbb{R}^\star$ then there exist t_1, $t_2 \in \mathbb{T}_1$ such that $a \odot t_1 = b \odot t_2$ and then

$$a \odot t_1 \odot (1/b) = b \odot (1/b) \odot t_2 \in \mathbb{T}_1,$$

as $b \odot (1/b) = t \in \mathbb{T}_1$ and $t \odot t_2 \in \mathbb{T}_1$ as well. Lemma 6 implies that $a \odot (1/b) \in \mathbb{T}_1$. Let, on the other hand, $a \odot (1/b) = t_2 \in \mathbb{T}_1$. Then

$$a \odot (1/b) \odot b = b \odot t_2$$

and, putting $(1/b) \odot b = t_1 \in \mathbb{T}_1$, we obtain the equivalence $a \sim_\odot b$. □

Theorem 5. If $a \in \mathbb{R}$ is a fuzzy quantity then $a \sim_\odot \langle 1 \rangle$ if and only if $a \in \mathbb{T}_1$.

Proof. The statement follows from Theorem 4. □

Theorem 6. For any $y \in R_0$ the subsets $\mathbb{T}_y \subset \mathbb{R}^\star$ form equivalence classes according to the equivalence relation $\sim_\odot$.

Proof. For x, $y \in R_0$ and $a \in \mathbb{T}_x$, $b \in \mathbb{T}_y$ the equivalence $a \sim_\odot b$ can be valid iff $x = y$, as shown in Lemma 2. It means that $\mathbb{T}_x$ are equivalence classes in $\mathbb{T}$. It remains to prove that for $a \in \mathbb{R}^\star - \mathbb{T}$, $b \in \mathbb{T}$ the equivalence $a \sim_\odot b$ is impossible. Let $a \sim_\odot b$ be valid for such a and b, and let $b \in \mathbb{T}_x$, $x \in R_0$. Then $b = \langle x \rangle \odot t$ for some $t \in \mathbb{T}_1$ and there exist t_1, $t_2 \in \mathbb{T}_1$ such that

$$a \odot t_1 = b \odot t_2 = \langle x \rangle \odot t \odot t_2.$$

It means that

$$a \odot t_1 \odot \langle 1/x \rangle = \langle 1 \rangle \odot t \odot t_2 = t \odot t_2 \in \mathbb{T}_1$$

and, by Lemma 6, $a \odot \langle 1/x \rangle \in \mathbb{T}_1$. It means by Theorem 4 that $a \sim_\odot \langle 1/x \rangle$, i.e., $a \in \mathbb{T}_{1/x}$ which contradicts the assumption $a \in \mathbb{R}^\star - \mathbb{T}$. □

Remark 7. Evidently, there exist other equivalence classes in $\mathbb{R}^\star$ different from $\mathbb{T}_x,\ x \in R_0$.

As shown above the equivalence structures for normal fuzzy quantities with finite support are rather more lucid than those ones in more general cases. It simplifies the elaboration of fuzzy-contaminated data with finitely many possible values in real applications. In connection with this it could be interesting to compare the theoretical tools derived here and in other related papers with the uncertainty models in system theory [8] and other branches (e. g., [7]).

The variety of equivalence structures for additive and multiplicative case offers also some deeper considerations about the structure of fuzziness and uncertainty included into fuzzy quantities and their description, analogous to those ones presented in [5]. The mutual relations between additive and multiplicative decomposition of fuzzy quantities could give some information about the essential types of fuzzy uncertainties.

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