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RELATIVE STATIONARY PROBABILITIES

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The paper gives a characterization of extremal relative stationary probabilities, i. e. extremal probabilities on the set of words of some fixed length whose marginals are invariant w. r. t. all feasible shifts. It is shown that a probability measure is relative stationary if and only if it is the projection of a stationary probability and that each extremal relative stationary probability is the projection of an ergodic probability with a finite support.

INTRODUCTION

Let A be a finite non-empty alphabet. Denote by W_k the set A^k of all words of length k, i. e. sequences consisting of k letters from A. We consider the σ-algebra $\mathcal{B}_k = (\exp A)^k$ of all subsets of W_k.

From now on, if not stated otherwise, we suppose that there is given an integer $k \geq 2$.

Further we consider the projections $d, h : W_k \to W_{k-1}$ defined, for every $(a_1, \dots, a_k) \in W_k$, by $d(a_1, \dots, a_k) = (a_1, \dots, a_{k-1})$ and $h(a_1, \dots, a_k) = (a_2, \dots, a_k)$. A probability P on the measurable space $(W_k, \mathcal{B}_k)$ is said to be *relative stationary* if $Ph^{-1} = Pd^{-1}$; denote by $I(W_k)$ the set of all relative stationary probabilities on $(W_k, \mathcal{B}_k)$.

Let $W = A^Z$ denote the set of all sequences indexed by the set Z of all integers and $\mathcal{B} = \bigotimes_{k \in Z} \exp A$ be the product σ-algebra on W. Obviously, if there is given a stationary (i. e. shift-invariant) probability on $(W, \mathcal{B})$ then any its projection onto a "connected segment" W_k of length k represents a relative stationary probability. The characterization 1.4 of the extremal relative stationary probabilities and the extension Theorem 2.2, together with the "Choquet-type" representation of 2.3, imply that this statement can be reversed, i. e. a probability on $(W_k, \mathcal{B}_k)$ is relative stationary if and only if it is the projection of a stationary probability on $(W, \mathcal{B})$. Theorem 3.1 describes the relation between the extremal relative stationary probabilities and ergodic probabilities on $(W, \mathcal{B})$ with a finite support.

The problem of finding the extremal relative stationary probabilities can be viewed on as a special case of a generalized version of the transshipment problem, cf. [1, 5], where systems of measures with given differences of marginals are studied. Here the generalization consists in that the considered projections are somewhat more general than the canonical, coordinate ones. On the other hand, only the special case of systems of probability measures with zero differences of marginals is taken into account. The above-cited references [1, 5] as well as related references [2, 6, 7, 8] indicate that an alternative way of proving the results of the first section is possible. Namely, let us call a set $D \subset W_k$ a set of relative stationary uniqueness, shortly an PSU-set, if $P = Q$ whenever P, $Q \in I(W_k)$ and $\operatorname{supp} P \subset D$, $\operatorname{supp} Q \subset D$. It could be proved that the elementary cycles (introduced in the first section) are the same as the RSU-sets in the present set-up. Following the ideas of the cited papers we could further prove that the extremal relative stationary probabilities have RSU-sets as their supports.

1. EXTREMAL PROBABILITIES

For a pair of words u, $v \in W_k$ we say that v is *tied* to u (write $u \Box v$) if $h(u) = d(v)$; in this case $t(u, v) = h(u) = d(v)$ is the *tie* of u, v. A sequence $S = (w_1, \dots, w_j)$ of words from W_k is a *cycle* if $w_1 \Box w_2 \Box \cdots \Box w_j \Box w_1$. The system of all cycles consisting of j words will be denoted $\mathcal{C}_j(W_k)$. For $S \in \mathcal{C}_j(W_k)$, we define its *restriction* $r(S) = (t(w_1, w_2), \dots, t(w_j, w_1))$. The *extension* $e(S)$ of $S \in \mathcal{C}_j(W_k)$ is the (uniquely determined) cycle in $\mathcal{C}_j(W_{k+1})$ for which $r(e(S)) = S$. The *support* $|S|$ of a cycle S is the set of all the words that S consists of. A cycle S is *elementary* if in $r(S)$ no tie appears more than once.

1.1. Lemma.

(i) For any cycle there is a subsequence which is an elementary cycle.

(ii) If S is an elementary cycle, and R is a (nonempty) subcycle of S, then $R = S$.

Proof. (i) Let $S = (w_1, \dots, w_j) \in \mathcal{C}_j(W_k)$ be a cycle. For $m = 1, \dots, j$ we put $T_m = \{d(w_1), \dots, d(w_m)\}$, $M = \{m : \ h(w_m) \in T_m\}$. As S is finite, M is nonempty. Put $n = \min M$, $q = \max\{m : \ d(w_m) = h(w_n)\}$. Then $(w_q, \dots, w_n)$ is an elementary cycle.

(ii) Write $S = (w_1, \dots, w_j)$ and assume that $w_i \in |R|$ for some $i \in \{1, \dots, j\}$. Then $S \cap d^{-1}(h(w_i)) = \{w_{i+1}\}$, hence $w_{i+1} \in |R|$, and repetition of the argument for all $i = 1, \dots, j$ proves the assertion. $\Box$

1.2. Lemma. For $P \in I(W_k)$ the following assertions hold:

(i) If $w \in \operatorname{supp} P$ then there exists $v \in \operatorname{supp} P$ such that $w \Box v$

(ii) There exist $j \geq 1$ and $S \in \mathcal{C}_j(W_k)$ such that $S \subset \operatorname{supp} P$.

Proof. (i) As $Ph^{-1}(h(w)) > 0$ we get, by relative stationarity of P that $Pd^{-1}(h(w)) > 0$, hence $d^{-1}(h(w)) \neq \emptyset$ and any $v \in d^{-1}(h(w))$ is tied to w.

(ii) Take a $w_1 \in \operatorname{supp} P$ and construct, using (i), elements $w_2, w_3, \dots$ in $\operatorname{supp} P$ such that $w_i \Box w_{i+1}$. As $\operatorname{supp} P$ is finite, $w_{j+1} = w_1$ for some j and $S = (w_1, \dots, w_j) \in \mathcal{C}_j(W_k)$. $\Box$

1.3. Lemma. Let P be a probability on $(W_k, \mathcal{B}_k)$, S be an elementary cycle on W_k and $\operatorname{supp} P \subset |S|$. Then $P \in I(W_k)$ if and only if P is uniform on $|S|$.

P r o o f. If P is uniform on $|S|$, then obviously both Ph^{-1} and Pd^{-1} are uniform on $|r(S)|$, hence $P \in I(W_k)$. On the other hand, let P be relative stationary. By 1.2. (ii) there exists a cycle R for which $|R| \subset \operatorname{supp} P$ and by 1.1. (ii) we get $R = S$; consequently $|S| = \operatorname{supp} P$, $\operatorname{supp} Ph^{-1} = \operatorname{supp} Pd^{-1} = r(S)$ and finally the equality $Ph^{-1} = Pd^{-1}$ implies that P is uniform on $|S|$. □

Note that $I(W_k)$ is a compact convex set if we consider an embedding of $I(W_k)$ into Euclidean space.

1.4. Theorem. A probability P on $(W_k, \mathcal{B}_k)$ is an extremal point of $I(W_k)$ if and only if it is uniform on the support of some elementary cycle S on W_k.

P r o o f. Let P be uniform on $|S|$ and $P = \frac{1}{2}(P_1 + P_2)$ where $P_1, P_2 \in I(W_k)$. Then $\operatorname{supp} P_i \subset |S|$ for $i = 1, 2$ and $P_1 = P_2 = P$ according to 1.3. Consequently P is extremal. On the other hand, suppose that $P \in I(W_k)$ is extremal. By 1.2 (ii) and 1.1. (i) there exists an elementary cycle S such that $|S| \subset \operatorname{supp} P$. If it were $|S| \neq \operatorname{supp} P$, P could be expressed as a nontrivial convex combination of two probabilities from $I(W_k)$ (one of them being the uniform one on $|S|$) and P would not be extremal. Hence $S = \operatorname{supp} P$ and according to 1.3 P is uniform on $|S|$. □

For an elementary cycle S on W_k we denote by U_S the uniform probability on $|S|$.

1.5. Corollary. For every $P \in I(W_k)$ there exist elementary cycles $S_1, \ldots, S_m$ on W_k and $\alpha_1, \ldots, \alpha_m \geq 0$, $\sum_{j=1}^m \alpha_1 = 1$ such that

$$P = \sum_{j=1}^{m} \alpha_j \, U_{S_j}.$$

P r o o f. With respect to 1.4, the assertion is the same as that of the Krein–Millman theorem, cf. e.g. [3]. □

2. EXTENSION OF A RELATIVE STATIONARY PROBABILITY

2.1. Lemma. Let S be an elementary cycle on W_k and $P \in I(W_{k+1})$. Then $Ph^{-1} = U_S$ if and only if $P = U_{e(S)}$.

P r o o f. The definition of an elementary cycle immediately yields that $U_S = U_{e(S)}h^{-1}$. For the "only if" part of the proof, suppose that $Ph^{-1} = U_S$ and express $P = \sum_{j=1}^m \alpha_j U_{S_j}$ according to 1.5; it holds $U_S = \left(\sum \alpha_j \, U_{S_j}\right) \, h^{-1} = \sum \alpha_j \left(U_{S_j} h^{-1}\right) = \sum \alpha_j U_{r(S_j)}$. The last equality follows from

$$U_{S_j} h^{-1}(w) = \begin{cases} 1 \,/\, \|S_j\| & \text{for } w \in r(S_j) \\ 0 & \text{otherwise} \end{cases}$$

and implies $r(S_1) = \dots = r(S_m) = S$ because U_S is extremal. $\square$

For an elementary cycle S on W_k and $m = 0, 1, 2, \dots$ we denote $e^m(U_S) = U_{e^m(S)}$ the probability which is uniform on the elementary cycle $e^m(S)$ on W_{k+m}.

2.2. Theorem. Let S be an elementary cycle on W_k. Then the system $e^m(U_S)$, $m = 0, 1, 2, \dots$, determines a projective system of finite-dimensional distributions on $(W, \mathcal{B})$. If we denote $e^\infty(U_S)$ the corresponding projective limit then $e^\infty(U_S)$ is the only stationary probability on $(W, \mathcal{B})$ whose projection onto W_k is U_S.

P r o o f. Projectivity of the considered system is a straightforward consequence of 2.1. The assertion of the theorem follows from the Daniell–Kolmogorov theorem. $\square$

2.3. Corollary. Let P be a relative stationary probability on $(W_k, \mathcal{B}_k)$. Then there exists a stationary probability on $(W, \mathcal{B})$ whose projection onto W_k is P.

P r o o f. According to 1.5 we may write $P = \sum_{j=1}^m \alpha_j U_{S_j}$; $P_\infty = \sum_{j=1}^m \alpha_j e^\infty(U_{S_j})$ is a stationary probability on $(W, \mathcal{B})$ and its projection onto W_k is P, cf. 2.2. $\square$

3. ERGODIC PROBABILITIES WITH FINITE SUPPORT

A shift θ on $W = A^Z$ is defined, for $x = (x_j)_{j=-\infty}^{\infty}$, by $(\theta(x))_j = x_{j-1}$. A sequence $x \in W$ is *periodic* if there exists a positive integer p such that $\theta^p(x) = x$; the *period* of x is the smallest such p. For a periodic x with the period p we define its *orbit* $\mathcal{O}(x) = \{\theta^j(x) : \ j = 0, \dots p-1\}$.

3.1. Theorem. Let S be an elementary cycle on W_k. Then $e^\infty(U_S)$ is an ergodic probability with a finite support; conversely, each ergodic probability with a finite support is of this type.

P r o o f. Clearly $P = e^\infty(U_S)$ is stationary. If P were not ergodic we could write $P = \frac{1}{2}(P_1 + P_2)$ where $P_j \neq P$, $P_j \in I(W)$ for $j = 1, 2$. Denoting by P'_j the projection of P_j onto W_k we would get $P'_j \neq U_S$, $j = 1, 2$, and according to 2.2 $U_S = \frac{1}{2}(P'_1 + P'_2)$ could not be extremal.

On the other hand, let P denote an ergodic probability with a finite support,then P is uniform on $\operatorname{supp} P = \mathcal{O}(x)$ for some periodic x (cf. [7], pp. 80–81). Let p be the period of x and S denote the projection of $\mathcal{O}(x)$ onto W_{p+1}; S is obviously a cycle and if it were not elementary the period of x would be smaller than p. $\square$

4. DISCUSSION OF RESULTS

There is not given an explicit description of the systems of all the elementary cycles on W_k, $k = 2, 3, \dots$, because of an involved combinatorial nature of such considerations. The results of the second and third sections however indicate that, even though such description is not known, the characterization of the extremal relative stationary probabilities can be helpful.

A straightforward generalization of the paper would consist in finding characterizations of extremal probabilities of projections, on finite subsets, of stationary random fields indexed by Z^d for $d = 2, 3, \ldots$. Here the situation is somewhat more complicated (e. g. the relative stationarity is not a characterization of such projections) and some non-trivial extension of the present paper methods would be required.

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REFERENCES

[1] V. Beneš: Extremal and optimal solutions in the transshipment problem. Comment. Math. Univ. Carolinae *33* (1992), 97–112.

[2] R. G. Douglas: On extremal measures and subspace density. Michigan Math. J. *11* (1964), 243–245.

[3] N. Dunford and J. T. Schwartz: Linear Operators. Part I: General Theory. Interscience Publishers, New York 1963.

[4] R. H. Farell: Representation of invariant measures. Illinois J. Math. *6* (1962), 447–467.

[5] J. H. B. Kemperman: On the role of duality in the theory of moments. In: Proc. Semi-Infinite Programming and Applications 1981 (Lecture Notes in Economics and Math. Systems 215), Springer–Verlag, Berlin – Heidelberg – New York 1983, pp. 63–92.

[6] D. Linhartová: Extremal solutions of a general marginal problems. Comment. Math. Univ. Carolinae *32* (1991), 743–748.

[7] R. Phelps: Lectures on Choquet's Theorem. D. van Nostrand, Princeton, N. J. 1966.

[8] J. Štěpán: Simplicial measures and sets of uniqueness in marginal problem. Statistics and Decisions (to appear).

[9] H. von Weizsäcker and G. Winkler: Integral representation in the set of solutions of a generalized moment problem. Math. Ann. *246* (1979), 23–32.

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