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OPTIMAL STARTING TIMES FOR A CLASS OF SINGLE MACHINE SCHEDULING PROBLEMS WITH EARLINESS AND TARDINESS PENALTIES

Milan Vlach[1] and Karel Zimmermann[2]

A single machine problem with n jobs is considered. The jobs are processed in a given order on a continuously available machine. At any time the machine can handle at most one job, and no preemption is permitted. Every job requires a positive processing time. If a job is not processed within a specified time interval, then a positive penalty is incurred, which is a function of starting time of the job in question. An algorithm is proposed for determining starting times that minimize the maximum penalty subject to the condition that every job must be started within a given time interval. Some extensions to more general objective functions are also presented.

1. INTRODUCTION

We consider a single machine problem with n jobs $1, 2, \dots, n$. The jobs are to be processed in a given order on a continuously available machine. Without any loss of generality we assume that the prescribed order is the natural order $\langle 1, 2, \dots, n \rangle$. At any time, the machine can handle at most one job. No preemption is permitted. Each job j requires a positive processing time p_j. In addition, two time intervals $[\alpha_j, \overline{\beta_j}]$ and $[a_j, b_j]$ are associated with each job j. If job j is not processed within the interval $[a_j, b_j]$, then a positive penalty

$$\max\{a_j - S_j, S_j + p_j - b_j\}$$

is incurred where S_j denotes the starting time of job j. The interval $[\alpha_j, \overline{\beta_j}]$ gives a time window within which job j must be processed, i. e. α_j is the release time and $\overline{\beta_j}$ is the deadline of job j. The objective is to minimize the maximum penalty. The problem can formally be stated as follows:

Minimize the function f given by

$$f(S_1, S_2, \dots, S_n) = \max_{1 \le j \le n} \max\{a_j - S_j, S_j + p_j - b_j, 0\} \tag{1}$$

[1]On leave from Faculty of Mathematics and Physics, Charles University, Prague.

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subject to

$$\alpha_j \le S_j \le \beta_j, \quad j = 1, 2, \ldots, n, \tag{2}$$

where $\beta_j = \overline{\beta_j} - p_j$.

Since the jobs must be processed in the natural order $\langle 1, 2, \ldots, n \rangle$, and since the machine can handle at most one job at a time, the starting times $S_1, S_2 \ldots, S_n$ must satisfy

$$S_j \ge S_{j-1} + p_{j-1}, \quad 2 \le j \le n. \tag{3}$$

Therefore the problem under consideration can be formulated as minimization of the function f given by (1) over the set of all solutions $S = (S_1, S_2, \ldots, S_n)$ of systems (2) and (3). We denote this set by $\boldsymbol{S}$ and call its elements feasible schedules.

The problem considered in this paper belongs to sequencing and scheduling problems with earliness and tardiness penalties, many versions of which are reported e. g. in [1]. The method suggested here is a modification of the threshold method described in [5]. Note that the job processing order is fixed. If this order is not fixed, then even the problem of deciding whether a feasible schedule exists is NP-complete.

2. FEASIBILITY

Obviously, for some data instances, the set of feasible schedules may be empty. If either $\alpha_j = -\infty$ or $\beta_j = \infty$ for each j, then feasible schedules exist. Here we consider the case

$$-\infty < \alpha_j < \beta_j < \infty, \quad 1 \le j \le n.$$

Using constraints (2) and (3), we can determine the earliest and the latest possible starting times α'_j and β'_j of each job j as follows:

$$\alpha'_1 := \alpha_1 \tag{1}$$

$$\alpha'_j := \max\{\alpha'_{j-1} + p_{j-1}, \alpha_j\}, \qquad j = 2, 3, \ldots, n \tag{2}$$

$$\beta'_n := \beta_n \tag{3}$$

$$\beta'_{n-j} := \min\{\beta'_{n-j+1} - p_{n-j}, \beta_{n-j}\}, \quad j = 1, 2, \ldots, n-1. \tag{4}$$

Obviously, a feasible schedule exists if and only if

$$\alpha'_j \le \beta'_j, \quad j = 1, 2, \ldots, n. \tag{5}$$

It is now clear how to decide whether a feasible schedule exists. One just computes α'_j and β'_j from α_j, β_j, p_j using (1)–(4) and verifies validity of (5). It should be pointed out that if (5) holds, then the schedule defined by

$$S_j := \beta'_j, \quad j = 1, 2, \ldots, n \tag{6}$$

is feasible.

3. BACKGROUND FOR A SOLUTION ALGORITHM

For the sake of brevity, we shall often use the standard vector notation, writing e. g. α' instead of $(\alpha'_1, \alpha'_2, \ldots, \alpha'_n)$, and introduce the following notation:

$$\begin{aligned} N &:= \{1, 2, \ldots, n\} \\ F^+(S) &:= \{k \in N | f(S) = S_k + p_k - b_k\} \\ F^-(S) &:= \{k \in N | f(S) = a_k - S_k\} \\ V(S) &:= \{k \in N | S_k = \alpha'_k\} \\ T(S) &:= \textstyle\bigcup_{k \in F^+(S)} T_k(S) \end{aligned}$$

where $T_k(S)$ are defined as follows:

$$T_1(S) := \{1\}$$

$$T_2(S) := \begin{cases} \{2\} & \text{if } S_2 > S_1 + p_1 \\ \{1,2\} & \text{if } S_2 = S_1 + p_1 \end{cases}$$

$$T_3(S) := \begin{cases} \{3\} & \text{if } S_3 > S_2 + p_2 \\ \{2,3\} & \text{if } S_3 = S_2 + p_2 \text{ and } S_2 > S_1 + p_1 \\ \{1,2,3\} & \text{if } S_3 = S_2 + p_2 \text{ and } S_2 = S_1 + p_1 \end{cases}$$

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$$T_n(S) := \begin{cases} \{n\} & \text{if } S_n > S_{n-1} + p_{n-1} \\ \{n-1, n\} & \text{if } S_n = S_{n-1} + p_{n-1} \text{ and } S_{n-1} > S_{n-2} + p_{n-2} \\ \ldots\ldots & \\ \{1, 2, \ldots, n\} & \text{if } S_j = S_{j-1} + p_{j-1} \text{ for } j = 2, \ldots, n. \end{cases}$$

Here $T_k(S)$ denotes the set of indices of all variables which must be decreased, if we want to decrease the variable S_k. If $T_k(S) = \{k, k-1, \ldots, k-l\}$, then such "concatenation" of decreasing variables S_j, $j \in T_k(S)$, is caused by the fact that the inequalities $S_j \geq S_{j-1} + p_{j-1}$ for $j = k, \ldots, k-l+1$ are satisfied as equalities. Therefore job j is started exactly at the moment when job $j-1$ was finished.

The set $T(S)$ contains the indices of all variables that must be decreased, if we want to decrease all variables S_k, $k \in F^+(S)$, i.e. all variables which contribute actively to the value of $f(S)$ and for which $f(S) = S_k + p_k - b_k$. If $f(S) > 0$ and $F^-(S) = \emptyset$ and $S_j > \alpha'_j$ for all $j \in T(S)$, then the decreasing of variables S_j, $j \in T(S)$, is possible and leads to a decrease of the value of f. If $f(S) = 0$, then S is obviously an optimal solution. The corresponding optimization algorithm is based on the following three Lemmas, the proofs of which are given in the Appendix.

Lemma 3.1. If $\bar{S}$ is feasible and $F^-(\bar{S}) \neq \emptyset$, then $f(S) \geq f(\bar{S})$ for every S such that $S \leq \bar{S}$ (i.e. $S_j \leq \bar{S}_j$ for $j = 1, \ldots, n$).

Lemma 3.2. If $\bar{S}$ is a feasible schedule such that $V(\bar{S}) \cap T(\bar{S}) \neq \emptyset$, then $f(S) \geq f(\bar{S})$ for every feasible schedule S such that $S \leq \bar{S}$.

Lemma 3.3. If $\bar{S}$ is a feasible schedule such that simultaneously

$$f(\bar{S}) > 0,\ F^-(\bar{S}) = \emptyset, \quad V(\bar{S}) \cap T(\bar{S}) = \emptyset,$$

then there is a feasible schedule S such that

$$S \leq \bar{S} \quad \text{and} \quad f(S) < f(\bar{S}).$$

As a direct consequence of these results, we obtain the following characterization of optimality:

Theorem 3.1. If a feasible schedule $\bar{S}$ has the property that $f(S) \geq f(\bar{S})$ for every feasible schedule S with $S \not\leq \bar{S}$, then $\bar{S}$ is optimal if and only if

$$f(\bar{S}) = 0 \quad \text{or} \quad F^-(\bar{S}) \neq \emptyset \quad \text{or} \quad V(\bar{S}) \cap T(\bar{S}) \neq \emptyset.$$

4. ALGORITHM

The results of previous sections suggest the following solution procedure. First calculate all ${\alpha'}_j$ and ${\beta'}_j$ and verify whether a feasible schedule exists. If this is the case, that is if ${\alpha'}_j \leq {\beta'}_j$, then take $\bar{S} := \beta'$ as initial approximation. Note that there is no feasible schedule S such that $S \not\leq \bar{S}$ and therefore the assumption of Theorem 3.1 is satisfied for $\bar{S}$. Alternatively, we can take as initial approximation an arbitrary feasible schedule satisfying this assumption. According to Theorem 3.1, if

$$f(\bar{S}) = 0 \quad \text{or} \quad F^-(\bar{S}) \neq \emptyset \quad \text{or} \quad V(\bar{S}) \cap T(\bar{S}) \neq \emptyset$$

holds, then $\bar{S}$ is optimal; if not, then the proof of Lemma 3.3 given in the Appendix provides a hint how to construct a better approximation. It suffices to take a sufficiently small positive ε and define a new approximation $S(\varepsilon)$ by

$$S_j(\varepsilon) = \begin{cases} \bar{S}_j - \varepsilon & \text{for } j \in T(\bar{S}), \\ \bar{S}_j & \text{for } j \notin T(\bar{S}). \end{cases}$$

Obviously, a good strategy is to take ε as large as possible. We shall take the largest ε such that the inequalities (6) – (9) of Appendix remain valid at least as nonstrict inequalities.

In order to determine such an ε, we first note that

$$f(S(\varepsilon)) = \bar{S}_k - \varepsilon + p_k - b_k = f(\bar{S}) - \varepsilon$$

for every $k \in F^+(\bar{S})$ and every sufficiently small positive ε. It follows from (6) of Appendix that we have to restrict the choice of ε by

$$\bar{S}_k - \varepsilon + p_k - b_k \geq \max\{a_k - \bar{S}_k + \varepsilon,\, 0\}$$

for each $k \in F^+(\bar{S})$. This gives

$$\varepsilon \leq \varepsilon_1 := \min\left\{ f(\bar{S}),\, \min_{k \in F^+(\bar{S})} \frac{f(\bar{S}) + \bar{S}_k - a_k}{2} \right\}.$$

Analogously, we obtain from (7) of Appendix that the choice of ε is restricted by

$$\bar{S}_k - \varepsilon + p_k - b_k \geq \max\{a_j - \bar{S}_j,\, \bar{S}_j + p_j - b_j,\, 0\}$$

for all $k \in F^+(\bar{S})$ and $j \notin T(\bar{S})$. Therefore

$$\varepsilon \leq \varepsilon_2 := f(\bar{S}) - \max_{j \notin T(\bar{S})} \max\{a_j - \bar{S}_j,\, \bar{S}_j + p_j - b_j,\, 0\}.$$

From (8) of Appendix we obtain

$$\varepsilon \leq \varepsilon_3 := \min_{j \in T(\bar{S})} (\bar{S}_j - {\alpha'}_j)$$

and from (9) of Appendix we have

$$\varepsilon \leq \varepsilon_4 := \min_j (\bar{S}_j - \bar{S}_{j-1} - p_{j-1})$$

where minimization takes place over all $j \in T(\bar{S})$ such that $j - 1 \notin T(\bar{S})$.

As a next approximation we take the schedule $S(\bar{\varepsilon})$ with

$$\bar{\varepsilon} := \min\{\varepsilon_1,\, \varepsilon_2,\, \varepsilon_3,\, \varepsilon_4\}$$

and repeat the whole procedure with $S(\bar{\varepsilon})$ as a new $\bar{S}$.

It remains to demonstrate that every new approximation satisfies the assumption of Theorem 3.1, and that the process stops with an optimal solution after a finite number of iterations.

Suppose S is a feasible schedule such that $S \not\leq S(\bar{\varepsilon})$. If $S \not\leq \bar{S}$, then $f(S) \geq f(\bar{S})$ by our assumption concerning $\bar{S}$. Therefore, without loss of generality we can assume that $S \leq \bar{S}$ and $S \neq \bar{S}$.

Since $S_j(\bar{\varepsilon}) = S_j$ for every $j \notin T(\bar{S})$, there exists $j_0 \in T(\bar{S})$ such that

$$S_{j_0}(\bar{\varepsilon}) < S_{j_0} < \bar{S}_{j_0}.$$

Therefore $j_0 \in T_k(\bar{S})$ for some $k \in F^+(\bar{S})$. It turns out that

$$S_k(\bar{\varepsilon}) < S_k. \tag{1}$$

This is trivial for $k = j_0$. If $k \neq j_0$, then

$$S_k(\bar{\varepsilon}) = S_{j_0}(\bar{\varepsilon}) + p_{j_0} + \cdots + p_{k-1}.$$

From feasibility of S we have

$$S_{j_0} + p_{j_0} + \cdots + p_{k-1} \leq S_k.$$

Since $S_{j_0}(\bar{\varepsilon}) < S_{j_0}$, we have $S_k(\bar{\varepsilon}) < S_k$. It follows from (1) and the definition of f that

$$S_k(\bar{\varepsilon}) + p_k - b_k < S_k + p_k - b_k \leq f(S).$$

However, $S_k(\bar{\varepsilon}) + p_k - b_k = f(S(\bar{\varepsilon}))$, because $F^+(\bar{S}) \subset F^+(S(\bar{\varepsilon}))$. Therefore $f(S) \geq f(S(\bar{\varepsilon}))$.

It is also easy to show that the process stops with an optimal solution after a finite number of iterations. First we note that at each iteration the objective function value decreases and that

$$F^+(\bar{S}) \subset F^+(S(\bar{\varepsilon})), \quad F^-(\bar{S}) \subset F^-(S(\bar{\varepsilon})),$$
$$V(\bar{S}) \subset V(S(\bar{\varepsilon})), \qquad T(\bar{S}) \subset T(S(\bar{\varepsilon})).$$

If $\bar{\varepsilon} = \varepsilon_1$, then the process stops with optimal $S(\bar{\varepsilon})$ because $f(S(\bar{\varepsilon})) = 0$ or $F^-(S(\bar{\varepsilon})) \neq \emptyset$. If $\bar{\varepsilon} = \varepsilon_2$, then either the process stops with optimal solution or the set $F^+(S(\bar{\varepsilon}))$ becomes larger than $F^+(\bar{S})$. If $\bar{\varepsilon} = \varepsilon_3$, then the set $V(S(\bar{\varepsilon}))$ becomes larger than $V(\bar{S})$. If $\bar{\varepsilon} = \varepsilon_4$, then the set $T(S(\bar{\varepsilon}))$ becomes larger than $T(\bar{S})$. It follows that after at most n iterations the process either stops with an optimal solution or it delivers a nonoptimal schedule $\bar{S}$ such that $T(\bar{S}) = \{1, \ldots, n\}$. In the latter case, we obtain an optimal solution in the next iteration.

Examples.

a) Consider the problem given by the following input data

j	1	2	3	4	5
p_j	2	5	1	6	3
α_j	0	3	0	0	3
β_j	8	12	10	12	20
a_j	2	4	6	5	5
b_j	6	8	10	14	20

Suppose the prescribed order is the natural one, that is $\langle 1, 2, 3, 4, 5 \rangle$. Feasible schedules exist because

j	1	2	3	4	5
α'_j	0	3	8	9	15
β'_j	3	5	10	12	20

As a starting approximation let us take $\bar{S} = \beta'$, that is

$$\bar{S} = (3, 5, 10, 12, 20).$$

The required calculation gives

$$\begin{aligned} f(\bar{S}) &= \max\{0,2,1,4,3\} = 4 \\ F^-(\bar{S}) &= V(\bar{S}) = \emptyset, \quad F^+(\bar{S}) \;=\; T_4(\bar{S}) = T(\bar{S}) = \{4\} \\ \varepsilon_1 &= \min\{4,\ 11/2\} = 4 \\ \varepsilon_2 &= 4 - \max\{0,2,1,3\} = 1 \\ \varepsilon_3 &= 3 \\ \varepsilon_4 &= 12 - 10 - 1 = 1. \end{aligned}$$

Therefore $\bar{\varepsilon} = \min\{4,1,3,1\} = 1$. It follows that a new approximation $S(\bar{\varepsilon})$ will be

$$S(\bar{\varepsilon}) := (3,5,10,11,20).$$

After updating we obtain for the new $\bar{S} := S(\bar{\varepsilon})$:

$$\begin{aligned} f(\bar{S}) &= \max\{0,2,1,3,3\} = 3 \\ F^-(\bar{S}) &= V(S) = \emptyset, \quad F^+(\bar{S}) = \{4,5\} \\ T_4(\bar{S}) &= \{1,2,3,4\}, \quad T_5(\bar{S}) = \{5\}, \quad T(\bar{S}) = \{1,2,3,4,5\} = N \\ \varepsilon_1 &= \min\{3, \min\{9/2,\ 9\}\} = 3 \\ \varepsilon_2 &= +\infty \\ \varepsilon_3 &= \min\{3,2,2,2,5\} = 2 \\ \varepsilon_4 &= +\infty \\ \bar{\varepsilon} &= 2 \\ S(\bar{\varepsilon}) &= (1,3,8,9,18). \end{aligned}$$

For the new $\bar{S} := S(\bar{\varepsilon})$, we obtain

$$\begin{aligned} f(\bar{S}) &= \max\{1,0,0,1,1\} = 1, \\ F^-(\bar{S}) &= \{1\}. \end{aligned}$$

This schedule is optimal according to Theorem 3.1.

b) Consider the problem with the same input data as in the previous case with the exception of p_3 which we set $p_3 := 5$. We obtain

j	1	2	3	4	5
α'_j	0	3	8	13	19
β'_j	3	5	10	12	20

No feasible schedule exists because $\alpha'_4 > \beta'_4$.

5. EXTENSIONS

It can easily be verified that the proposed algorithm can be extended to the case with the objective function

$$\bar{f}(S) := \max_{j \in N} \max\{\varphi_j(a_j - S_j),\ \varphi_j(S_j + p_j - b_j),\ 0\} \tag{1}$$

where all φ_j are strictly increasing continuous real valued functions on $[0,\infty)$ such that $\varphi_j(0)=0$.

Indeed, if $T_j(S)$ and $V(S)$ are defined as previously and if $F^+(S)$, $F^-(S)$ and $T(S)$ are defined by

$$\begin{aligned} F^+(S) &:= \{k\in N | f(S)=\varphi_k(S_k+p_k-b_k)\}\,, \\ F^-(S) &:= \{k\in N | f(S)=\varphi_k(a_k-S_k)\}, \\ T(S) &:= \cup_{k\in F^+(S)} T_k(S), \end{aligned}$$

then Lemmas 3.1 – 3.3 and Theorem 3.1 remain valid.

In order to calculate ε_1, ε_2, ε_3 and ε_4 we first note that the inequality (6) of Appendix must be replaced by

$$\varphi_k(\bar S_k-\varepsilon+p_k-b_k)>\max\{\varphi_k(a_k-\bar S_k+\varepsilon),\,0\}.$$

Therefore the choice of ε is restricted by

$$\begin{aligned} \varphi_k(\bar S_k-\varepsilon+p_k-b_k) &\geq \varphi_k(a_k-\bar S_k+\varepsilon) \\ \varphi_k(\bar S_k-\varepsilon+p_k-b_k) &\geq 0 \end{aligned}$$

for all $k\in F^+(\bar S)$. From the assumption $\varphi_k(0)=0$ and the monotonicity of φ_k, it follows

$$\begin{aligned} \varepsilon &\leq \tfrac{1}{2}(2\bar S_k+p_k-a_k-b_k) \\ \varepsilon &\leq \bar S_k+p_k-b_k \end{aligned}$$

for each $k\in F^+(\bar S)$. Consequently, as ε_1 we can take

$$\varepsilon_1:=\min_{k\in F^+(\bar S)}\min\left\{\bar S_k+p_k-b_k,\,\bar S_k+\tfrac{1}{2}(p_k-a_k-b_k)\right\}.$$

Further we have similarly as in (7) of Appendix for all $k\in F^+(\bar S)$ and $j\notin T(S)$

$$\varphi_k(\bar S_k+p_k-b_k)>\max\{\varphi_j(a_j-\bar S_j),\,\varphi_j(\bar S_j+p_j-b_j),\,0\}$$

so that the choice of ε is restricted (similarly as in the algorithm from Section 4 by

$$\begin{aligned} f(S(\varepsilon)) &= \varphi_k(\bar S_k-\varepsilon+p_k-b_k) \\ &\geq \max\{\varphi_j(a_j-S_j(\varepsilon)),\,\varphi_j(S_j(\varepsilon)+p_j-b_j),\,0\}, \end{aligned}$$

where $k\in F^+(\bar S)$ and $j\notin T(\bar S)$.

Since $j\notin T(\bar S)$, it is $S_j(\varepsilon)=\bar S_j$ and we obtain

$$f(S(\varepsilon))=\varphi_k(S_k-\varepsilon+p_k-b_k)\geq\max\{\varphi_j(a_j-\bar S_j),\,\varphi_j(\bar S_j+p_j-b_j),\,0\}$$

for all $k\in F^+(\bar S)$, $j\notin T(\bar S)$ such that

$$\bar S_k-\varepsilon+p_k-b_k\geq\varphi_k^{-1}(\max\{\varphi_j(a_j-\bar S_j),\,\varphi_j(\bar S_j+p_j-b_j),\,0\}).$$

Therefore ε is further restricted by

$$\varepsilon \le \varepsilon_2 := \min_{k\in F^+(\bar S)} \left(\bar S_k + p_k - b_k - \varphi_k^{-1} \left(\max_{j\in T(\bar S)} \max \left\{ \varphi_j(a_j - \bar S_j),\, \varphi_j(\bar S_j + p_j - b_j) \right\} \right) \right).$$

The remaining restrictions are formally the same as in Section 4 (but with the newly defined $T(\bar S)$), i. e.

$$\varepsilon \le \varepsilon_3 := \min_{j\in T(\bar S)} (\bar S_j - \alpha'_j)$$

and

$$\varepsilon \le \varepsilon_4 := \min(\bar S_j - \bar S_{j-1} - p_{j-1}),$$

where again the minimization takes place over all $j \in T(\bar S)$ such that $j - 1 \notin T(\bar S)$.

As the next approximation, we take the schedule $\bar S(\bar\varepsilon)$ with

$$\bar\varepsilon := \min\{\bar\varepsilon_1, \bar\varepsilon_2, \bar\varepsilon_3, \bar\varepsilon_4\}$$

and repeat the whole procedure with $\bar S(\bar\varepsilon)$ as a new $\bar S$.

The rest of the argument from Section 4 remains unchanged (using the new definitions of $F^+(S)$, $F^-(S)$ and $T(S)$).

The algorithm can be extended also to the objective function

$$\bar{\bar f}(S) := \max_{j\in N} \max\{\psi_j(a_j - S_j),\, \varphi(S_j + p_j - b_j),\, 0\},$$

where φ_j, ψ_j are for all $j \in N$ continuous and increasing with the property $\varphi_j(0) = \psi_j(0) = 0$. In this case, only ε_1 has to be calculated differently and other calculations remain unchanged. It must hold

$$\psi_k(\bar S_k - \varepsilon + p_k - b_k) \ge 0,$$

which implies $\varepsilon \le \bar S_k + p_k - b_k$, and further

$$\varphi_k(\bar S_k - \varepsilon + p_k - b_k) \ge \psi_k(a_k - \bar S_k + \varepsilon)$$

so that

$$\bar S_k - \varepsilon + p_k - b_k \ge \varphi_k^{-1}(\psi_k(a_k - \bar S_k + \varepsilon))$$

or

$$-\varepsilon \ge \varphi_k^{-1}(\psi_k(a_k - \bar S_k + \varepsilon)) - \bar S_k - p_k + b_k.$$

Let us set

$$\chi_k(\varepsilon) := \varphi_k^{-1}(\psi_k(a_k - \bar S_k + \varepsilon)) - \bar S_k - p_k + b_k + \varepsilon.$$

Then $\chi_k(\varepsilon)$ is increasing and continuous with respect to ε and it holds $\chi_k(0) < 0$, $\chi_k(\tilde\varepsilon) > 0$ for sufficiently large $\tilde\varepsilon > 0$. Therefore there exists the unique $\bar\varepsilon_k > 0$ such that $\chi(\bar\varepsilon_k) = 0$ and ε_1 will be chosen as follows:

$$\varepsilon_1 := \min_{k\in F^+(\bar S)} \min\{\bar S_k + p_k - b_k,\, \bar\varepsilon_k\}.$$

6. CONCLUSION

The method proposed in this paper can be characterized briefly as a finite iteration procedure for solving the problem of minimizing penalty function given by (1) subject to the constraints (2) and (3). One starts with a feasible solution $\bar{S}_j = \beta'_j$, $j = 1, 2, \dots, n$ where β'_j is the latest possible time for starting job j. Therefore each variable can only be decreased. Then one tries to reduce the value of $f(\bar{S})$ by decreasing variables S_j with "active" indices in $f(\bar{S})$. If such decreasing is possible without violating the feasibility of the schedule (i. e. without violating inequalities (2) and (3)) and if it leads to a smaller value of f, then one obtains a new better feasible schedule $S(\bar{\varepsilon}) \le \bar{S}$, $S(\bar{\varepsilon}) \ne \bar{S}$ with $f(S(\bar{\varepsilon}) < f(\bar{S})$. Then one sets $\bar{S} := S(\bar{\varepsilon})$ and repeats the calculations with the new $\bar{S}$. If such decreasing of "active" variables is not possible or if it does not lead to a smaller value of f, then $\bar{S}$ is an optimal solution of the problem. The procedure delivers an optimal solution after at most n such iterations, provided a feasible solution exists, which can easily be tested at the beginning. The method is extended to problems with more general objective functions of the form (1).

If there is no feasible schedule, then two natural questions arise. Namely:

i) For a given processing order, what changes in data α_j, β_j, p_j guarantee the existence of a feasible schedule?

ii) For given data α_j, β_j, p_j, what changes in the processing order guarantee the existence of a feasible schedule?

We did not discuss these questions here but hope to give partial answers elsewhere.

APPENDIX

Here we give proofs of Lemmas 3.1, 3.2 and 3.3 respectively.

Proof of Lemma 3.1. By assumption, there is $k \in N$ such that $f(\bar{S}) = a_k - \bar{S}_k$. If $S \le \bar{S}$, then $S_k \le \bar{S}_k$ and therefore $a_k - S_k \ge a_k - \bar{S}_k$. Since by definition of f, $f(S) \ge a_k - S_k$, we have $f(S) \ge f(\bar{S})$. □

Proof of Lemma 3.2. By assumption, there is $j \in N$ such that $j \in V(\bar{S}) \cap T(\bar{S})$. We have $\bar{S}_j = \alpha'_j$ by the definition of $V(\bar{S})$. For $S \le \bar{S}$, we have $\alpha'_j \le S_j \le \bar{S}'_j = \alpha'_j$. Therefore

$$S_j = \bar{S}_j. \tag{1}$$

Since $j \in T(\bar{S})$, there is $k \in F^+(\bar{S})$ such that $j \in T_k(\bar{S})$. It follows

$$f(\bar{S}) = \bar{S}_k + p_k - b_k. \tag{2}$$

From the definition of f, it is obvious that

$$f(S) \ge S_k + p_k - b_k \tag{3}$$

for each S. If $j = k$, then (3) together with (1) and (2) gives the required inequality $f(S) \geq f(\bar{S})$. If $j \neq k$, then $1 \leq j < k$, because $j \in T_k(\bar{S})$. Using (3), we obtain

$$S_k \geq S_j + p_j + \cdots + p_{k-1} \tag{4}$$

for each feasible S. Simultaneously we have

$$\bar{S}_k = \bar{S}_{k-1} + p_{k-1} \ , \ldots , \bar{S}_{j+1} = \bar{S}_j + p_j$$

because $j \in T_k(\bar{S})$. It follows

$$\bar{S}_k = \bar{S}_j + p_j + \cdots + p_{k-1}. \tag{5}$$

Using consecutively (3), (4), (1), (5) and (2) we again obtain $f(S) \geq f(\bar{S})$.

□

Proof of Lemma 3.3. First we note that under our assumption $F^-(\bar{S}) \neq \emptyset$ and that

$$\bar{S}_k + p_k - b_k > \max\{a_k - \bar{S}_k, 0\} \tag{6}$$

for all $k \in F^+(\bar{S})$. Moreover, since $F^+(\bar{S}) \subset T(\bar{S})$, we have

$$\bar{S}_k + p_k - b_k > \max\{a_j - \bar{S}_j, \bar{S}_j + p_j - b_j, 0\} \tag{7}$$

for all $k \in F^+(\bar{S})$ and $j \notin T(S)$. It is also obvious, since $V(\bar{S}) \cap T(\bar{S}) = \emptyset$, that

$$\bar{S}_j > \alpha'_j \tag{8}$$

for all $j \in T(\bar{S})$. It follows from the definition of $T(\bar{S})$ and feasibility of $\bar{S}$ that

$$\bar{S}_j > \bar{S}_{j-1} + p_{j-1} \qquad \text{whenever } j \in T(\bar{S}) \text{ and } j-1 \notin T(\bar{S}) \tag{9}$$

$$\bar{S}_j = \bar{S}_{j-1} + p_{j-1} \qquad \text{whenever } j \in T(\bar{S}) \text{ and } j-1 \in T(\bar{S}) \tag{10}$$

$$\bar{S}_j \geq \bar{S}_{j-1} + p_{j-1} \qquad \text{whenever } j \notin T(\bar{S}) \tag{11}$$

It is obvious that there is a positive ε such that (6) – (11) remain valid. If $\bar{S}$ is replaced by S defined by

$$S_i := \bar{S}_i \qquad \text{for all } i \notin T(\bar{S})$$

$$S_i := \bar{S}_i - \varepsilon \quad \text{for all } i \in T(\bar{S})$$

Since (8) – (11) remain valid and since $S_j = \bar{S}_j \geq \alpha'_j$ for all $j \notin T(\bar{S})$, the schedule S is feasible. Since (6) and (7) remain valid we have (by taking any $k \in F^+(S)$)

$$f(S) = S_k + p_k - b_k = \bar{S}_k - \varepsilon + p_k - b_k < \bar{S}_k + p_k - b_k = f(\bar{S}).$$ □

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Prof. RNDr. Milan Vlach, DrSc., School of Information Science, JAIST, Hokuriku. Japan.

Prof. RNDr. Karel Zimmermann, DrSc., Katedra aplikované matematiky, Matematicko–fyzikální fakulta Univerzity Karlovy (Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University), Malostranské nám. 25, 118 00 Praha 1. Czech Republic.