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ESTIMATION AND TESTING OF COINTEGRATION RELATIONSHIPS WITH STRONGLY SEASONAL MONTHLY DATA[1]

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This paper extends the method proposed in [8] for quarterly nonstationary data, considering the estimation and testing for seasonal cointegration relationships when dealing with strongly seasonal monthly data. The testing procedure is based on the maximum-likelihood estimation of the 'error correction mechanism' for the vector of series considered. Finite sample critical values for the cointegration test statistics at every frequency of interest are obtained by Monte Carlo simulations. Finally, tests are applied to Spanish production indexes data.

1. INTRODUCTION

The concept of cointegration defined in [4] allows us to describe the existence of a stationary or *equilibrium* relationship among individually nonstationary time series.

In economic applications, series that are integrated of order one, $I(1)$, are frequently found among which the existence of possible cointegration relationships is analyzed. On the other hand, many economic series exhibit a strong seasonality which can be characterized by the presence of seasonal roots with modulus one. Series of this type show peaks in their spectra at the corresponding seasonal frequencies.

In [6] (HEGY) the standard cointegration technique is extended to include the possibility that the data present unit roots at seasonal frequencies, suggesting the application of an Engle & Granger type two-step testing procedure to the appropriately filtered series. [8] extends the method developed in [7], which tests for the existence of cointegration relationships among different time series (annual data) as well as the number of possible cointegrating vectors. In his extension [8], Lee considers quarterly data and the possible presence of unit roots at seasonal frequencies as well as at zero frequency. His method is based on the maximum-likelihood estimation of the error correction mechanism for the observed vector of series, and cointegration

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tests are set out at each frequency individually with no prior knowledge about the presence of unit roots at the remaining frequencies. These cointegration tests are developed on the basis of the coefficient matrices of the error correction model.

The aim of the present paper is to extend the testing procedure based on the maximum likelihood estimation developed in [8] to monthly processes.

Section 2 of this paper presents a generalization of the integration and cointegration issues for the definitions of seasonal integrated process and seasonal cointegration. The appropriate error correction mechanism for strongly seasonal monthly data is set out in Section 3. Section 4 extends the general framework of the testing strategy for cointegration considering processes that present unit roots at zero frequency and/or at any seasonal frequencies of interest. The likelihood ratio statistics are derived in each case. A particular case of full cointegration, where all cointegrating vectors coincide, is included in subsection 4.4. In Section 5 the finite sample distributions of the statistics are analyzed by simulation. An example illustrates the implementation of the tests in Section 6. Section 7 presents our conclusions.

2. GENERAL DEFINITIONS OF INTEGRATION AND COINTEGRATION

In recent time series literature the concepts of integration and cointegration have been frequently used to describe the permanent behavior of many macroeconomic time series. However, less attention has been devoted to data series of smaller periodicity than a year. These series exhibit seasonal fluctuations that, in many cases, are of a nonstationary nature. The structure of these series can be characterized by the existence of unitary modulus roots at seasonal frequencies corresponding to peaks in the pseudo-spectrum at the same frequencies (seasonal integration). Consequently, it is also interesting to consider the existence of possible common factors between different series, at any of the seasonal frequencies (seasonal cointegration).

This section presents a set of definitions that generalize the ideas of integration and cointegration, presented in [5] and [3], which were originally formalized in [8].

Definition 2.1. Let $S(L)$ be a polynomial in the lag operator[3] that has a root with modulus one at frequency ω — i. e., $S(L) = (1 - e^{i\omega}L)$ — for $\omega \in (-\pi, \pi]$, and also let D(L) be another polynomial collecting all the unit roots, if any, at seasonal frequencies as well as at zero frequency, which are different from ω. A vector $(n \times 1)$ of series x_t with no deterministic component is said to be *integrated of order d* at frequency ω, and denoted as $x_t \sim I_\omega(d)$, if d is the smallest integer for which the representation $S(L)^d D(L) x_t = C(L)\varepsilon_t$ has the following properties:

(i) The spectrum of $C(L)\,\varepsilon_t$ is bounded away from zero and infinity at all frequencies,

(ii) $\{\varepsilon_t\}$ is a sequence of serially uncorrelated random vectors with finite and constant unconditional variance,

(iii) the initial values are zero, for both ε_t and x_t, for $t \leq 0$.

[3]As usual, L denotes the lag operator.

Due to the presence of $D(L)$, this general definition allows a series x_t to be integrated of a different order at each frequency. Hence, the well known definition of integrated process at zero frequency can be achieved as a particular case when $D(L) = 1$ and $\omega = 0$.

Assuming that $x_t \sim I_\omega(1)$ and $D(L) = 1$, the following implications are obtained from this definition:

(i) the variance of x_t tends to infinity as $t \to \infty$;

(ii) innovations have a permanent effect on the seasonal pattern of x_t (x_t has 'long memory');

(iii) the pseudo-spectrum of x_t takes on asymptotically the form $f(\varphi) = A(\varphi - \omega)^{-2}$ for φ near ω, showing an infinite peak at frequency ω;

(iv) x_t is asymptotically uncorrelated with processes which have unit roots at other frequencies[4] .

Example: An example of seasonal integrated processes for monthly data is

$$(1 - L^{12})\, x_t = \varepsilon_t, \tag{1}$$

which has roots with modulus one at every seasonal frequency as well as at zero frequency. The seasonal difference polynomial $(1 - L^{12}) \equiv \Delta_{12}$ can be factorized as

$$\begin{aligned}(1 - L^{12}) \;=\; & (1 - L)\,(1 + L)\,(1 + iL)\,(1 - iL)\,(1 + (\sqrt{3} + i)\,L/2)\,(1 + (\sqrt{3} - i)\,L/2) \\ & (1 - (\sqrt{3} + i)\,L/2)\,(1 - (\sqrt{3} - i)\,L/2)\,(1 + (i\sqrt{3} + 1)\,L/2) \\ & (1 - (i\sqrt{3} - 1)\,L/2)\,(1 - (i\sqrt{3} + 1)\,L/2)\,(1 + (i\sqrt{3} - 1)\,L/2). \end{aligned} \tag{2}$$

The unit roots of this polynomial are:

$$\begin{array}{lll} \theta_1 = 1; & \theta_2 = -1; & \theta_3 = +i; \\ \theta_4 = -i; & \theta_5 = -\frac{1}{2}(1 + i\sqrt{3}); & \theta_6 = -\frac{1}{2}(1 - i\sqrt{3}); \\ \theta_7 = \frac{1}{2}(1 + i\sqrt{3}); & \theta_8 = \frac{1}{2}(1 - i\sqrt{3}); & \theta_9 = -\frac{1}{2}(\sqrt{3} + i); \\ \theta_{10} = -\frac{1}{2}(\sqrt{3} - i); & \theta_{11} = \frac{1}{2}(\sqrt{3} + i); & \theta_{12} = \frac{1}{2}(\sqrt{3} - i). \end{array}$$

The frequency associated with a particular root is the value of ω in $Re^{\omega i}$ — the polar representation of the root. A root is seasonal if $\omega = 2\pi j/S,\ j = 1, \dots, S - 1$, where S is the number of observations per year (assuming S to be even). When $S = 12$, the seasonal frequencies associated with the seasonal (unit) roots are $\omega = \pi$, $\pm\pi/2$, $\pm 2\pi/3$, $\pm\pi/3$, $\pm 5\pi/6$ and $\pm\pi/6$; corresponding to $6, 3, 9, 8, 4, 2, 10, 7, 5,$ and 11 cycles per year, respectively. Summarizing, Definition 2.1 can be used to

[4] The conditions under which the correlation coefficients approach zero as $T \to \infty$ are given in detail in [6].

point out that the process x_t is $I_\omega(1)$ at these seasonal frequencies and at $\omega = 0$, i. e., x_t has twelve unit roots.

Analogously, the idea of cointegration presented in Engle & Granger's articles can be generalized to define the concept of seasonal cointegration.

Definition 2.2. Let all components of x_t be integrated of order one at frequency ω, i. e., $x_t \sim I_\omega(1)$. The components of x_t are said to be cointegrated at frequency ω, and denoted as $x_t \sim CI_\omega(1,1)$, if there exists a vector $\alpha(\neq 0)$ so that $z_t = \alpha' x_t \sim I_\omega(0)$.

This definition is not at all restrictive, in the sense that it allows different cointegrating vectors at each of the frequencies where unit roots are present.

However, it could be the case that for a vector of nonstationary series with unit roots at some seasonal frequencies and at zero frequency, a single cointegrating vector could eliminate all the unit roots in the series. This situation is formalized in the following definition of *full cointegration*.

Definition 2.3. Let each component of x_t be integrated of order one at some frequencies, not necessarily at the same frequencies for all components. The components of the vector x_t are said to be fully cointegrated, and are denoted as $x_t \sim CI(1,1)$, if there exists a vector $\alpha(\neq 0)$ so that $z_t = \alpha' x_t$ is stationary.

These definitions contain concepts that are quite similar to those derived from the idea of cointegration established in [3]. Hence, if there is cointegration at seasonal frequency ω each of the series contains the same factor $I_\omega(1)$ and an innovation may have a permanent effect on the seasonal behavior of x_t, but only a temporary effect on the seasonal pattern of $z_t = \alpha' x_t$.

3. ERROR CORRECTION MODEL FOR A STRONGLY SEASONAL PROCESS

Based upon the parallelism between cointegrated VAR models and error correction models (ECM) established[5] in [3], a seasonally cointegrated variables system may be represented through either an autoregressive vector (VAR) or, equivalently, using an error correction mechanism.

In this section, we set out the ECM equation that corresponds to a vector of series presenting unit roots at all seasonal frequencies as well as at zero frequency. This equation may be considered as the adaptation for monthly data of the annual and quarterly models presented in [7] and [8], respectively. The model presented sets up a basis on which cointegration tests can be carried out when analyzing the existence of cointegration relationships on autoregressive vectors formed by monthly time series.

[5] Adapted from *Granger's Representation Theorem*.

The data set considered is a monthly sequence of n-dimensional random vectors $\{x_t\}$. We consider a general VAR(p) model. The dynamic of the process is described by the following model

$$x_t = \Phi_1 x_{t-1} + \Phi_2 x_{t-2} + \ldots + \Phi_p x_{t-p} + \varepsilon_t, \quad t = 1, 2, \ldots, T, \tag{3}$$

where $\varepsilon_t \sim \mathrm{NID}_n(0, \Sigma)$ and the $\Phi_1, \ldots, \Phi_p, \Sigma$ are $(n \times n)$ matrices of parameters to be estimated on the basis of T observations.

Since the process x_t is allowed to have unit roots at seasonal frequencies as well as at zero frequency, the determinant of the autoregressive matrix polynomial $\Phi(z) = I - \Phi_1 z - \ldots - \Phi_p z^p$ may have roots on the unit circle at these frequencies. It will be assumed that all the remaining roots of $|\Phi(z)| = 0$ satisfy[6] $|z| > 1$.

Following a procedure parallel to that of the univariant case developed in [2], from equation (3) the ECM representation can be obtained:

$$\begin{aligned} \Delta_{12} x_t \quad = \quad & \Pi_1 y_{1,t-1} + \Pi_2 y_{2,t-1} + \Pi_3 y_{3,t-1} + \cdots + \Pi_{12} y_{12,t-1} \\ & + A_1 \Delta_{12} x_{t-1} + A_2 \Delta_{12} x_{t-2} + \cdots + A_{p-12} \Delta_{12} x_{t-p+12} + \varepsilon_t, \end{aligned} \tag{4}$$

[6]The straight implication is that the nonstationarity of the process will come from unit roots at seasonal and zero frequencies but not from other frequencies.

where

$$\begin{aligned}
y_{1,t} &= Z_1(L)\,x_t \\
&= (1+L+L^2+L^3+L^4+L^5+L^6+L^7+L^8+L^9+L^{10}+L^{11})\,x_t, \\
y_{2,t} &= Z_2(L)\,x_t \\
&= -(1-L+L^2-L^3+L^4-L^5+L^6-L^7+L^8-L^9+L^{10}-L^{11})\,x_t, \\
y_{3,t} &= Z_3(L)\,x_t = -(L-L^3+L^5-L^7+L^9-L^{11})\,x_t, \\
y_{4,t} &= Z_4(L)\,x_t = -(1-L^2+L^4-L^6+L^8-L^{10})\,x_t, \\
y_{5,t} &= Z_5(L)\,x_t \\
&= -\frac{1}{2}(1+L-2L^2+L^3+L^4-2L^5+L^6+L^7-2L^8+L^9+L^{10}-2L^{11})\,x_t, \\
y_{6,t} &= Z_6(L)\,x_t = \frac{\sqrt{3}}{2}(1-L+L^3-L^4+L^6-L^7+L^9-L^{10})\,x_t, \\
y_{7,t} &= Z_7(L)\,x_t \\
&= \frac{1}{2}(1-L-2L^2-L^3+L^4+2L^5+L^6-L^7-2L^8-L^9+L^{10}+2L^{11})\,x_t, \\
y_{8,t} &= Z_8(L)\,x_t = -\frac{\sqrt{3}}{2}(1+L-L^3-L^4+L^6+L^7-L^9-L^{10})\,x_t, \\
y_{9,t} &= Z_9(L)\,x_t \\
&= -\frac{1}{2}(\sqrt{3}-L+L^3-\sqrt{3}L^4+2L^5-\sqrt{3}L^6+L^7-L^9+\sqrt{3}L^{10}-2L^{11})\,x_t, \\
y_{10,t} &= Z_{10}(L)\,x_t \\
&= \frac{1}{2}(1-\sqrt{3}L+2L^2-\sqrt{3}L^3+L^4-L^6+\sqrt{3}L^7-2L^8+\sqrt{3}L^9-L^{10})\,x_t, \\
y_{11,t} &= Z_{11}(L)\,x_t \\
&= \frac{1}{2}(\sqrt{3}+L-L^3-\sqrt{3}L^4-2L^5-\sqrt{3}L^6-L^7+L^9+\sqrt{3}L^{10}+2L^{11})\,x_t, \\
y_{12,t} &= Z_{12}(L)\,x_t \\
&= -\frac{1}{2}(1+\sqrt{3}L+2L^2+\sqrt{3}L^3+L^4-L^6-\sqrt{3}L^7-2L^8-\sqrt{3}L^9-L^{10})\,x_t. \qquad (5)
\end{aligned}$$

The interesting feature in this representation of model (3) is that it makes the set of regressors mutually orthogonal, with each of them collecting the process x_t filtered so that it eliminates, each time, all unit roots except the one associated with one particular frequency[7]. The $Z_k(L)$'s (for $k = 1, \dots, 12$) will be filters performing the function previously described, the Π_k's are $(n \times n)$ coefficient matrices related to the filtered vectors and the A_i's (for $i = 1, \dots, p-12$) are $(n \times n)$ matrices related to the elements included in the regression model to whiten the error ε_t and represent the stationary structure of the model.

The ECM representation (4) will be employed to estimate and test for cointegration relationships between the components of a VAR.

[7]For frequencies associated with complex roots the two filters that leave the two conjugated unit roots must be simultaneously applied.

4. COINTEGRATION TESTS

In this section the testing strategies for the different frequencies of interest will be detailed. It should be pointed out that in the ECM (4) the coefficient matrices $\Pi_1, \Pi_2, \ldots, \Pi_{12}$ convey information concerning the permanent behavior of the series[8]; so if the coefficient matrix Π_k has full rank, then the series do not contain unit roots at the corresponding frequency. If the rank of Π_k is zero, no cointegration relationship will be found at that frequency. However, if there are linear combinations between columns of matrix Π_k, i. e. $0 < \text{rank}(\Pi_k) = r < n$, it can be said that cointegration relationships exist at that frequency. Given that the rank of matrix Π_k is r, it can be shown for a suitable pair of $(n \times r)$ matrices γ_k and α_k, satisfying $\Pi_k = \gamma_k \alpha_k'$, that despite $y_{k,t-1}$ itself being nonstationary, $\alpha_k' y_{k,t-1}$ will be stationary. This would mean that the vector process x_t is cointegrated at the associated frequency whose unit modulus root has not been eliminated in $y_{k,t-1}$.

The proof is straightforward from the ECM (4) if we consider, for instance, $k = 2$. If Π_2 has incomplete rank r, the term $\Pi_2 y_{2,t-1}$ may be rewritten as $(\gamma_2 \alpha_2')\, y_{2,t-1}$, which must be stationary due to the stationarity of the left member in the equality, $(\Delta_{12} x_t)$. The implication described above is obvious substituting $y_{2,t-1}$ for $Z_2(L)\, L x_t$.

$$\gamma_2 Z_2(L) L \alpha_2' x_t \sim I_\pi(0) \iff \alpha_2' x_t \sim I_\pi(0) \iff x_t \text{ cointegrated at } \omega = \pi.$$

The columns of α_k are the cointegrating vectors of the series at that frequency. The space spanned by the columns of α_k, which at the same time coincides with the space spanned by the rows of matrix Π_k, will be called the *seasonal cointegration relationships space* at that frequency.

The natural hypothesis of the cointegration rank test comes from this. Generally, it can be formulated as the hypothesis that *at most* r_k cointegration relationships exist at the corresponding frequency, $\text{rank}(\Pi_k) \le r_k$, against the alternative that $\text{rank}(\Pi_k) > r_k$. The main advantage of this procedure is that several null hypotheses can be tested for each case of interest with no prior knowledge of the existence of cointegration relationships at other frequencies, due to the asymptotic uncorrelatedness between any two series with unit roots at different frequencies.

4.1. Cointegration at zero frequency

To test the existence of *at most* r_1 cointegrating vectors (at least $n - r_1$ unit roots) at zero frequency in the presence of unit roots at some seasonal frequencies, the hypothesis can be formulated as $H_{1_0} : \text{rank}(\Pi_1) \le r_1,\ (r_1 < n)$ vs. $H_{1_a} : \text{rank}(\Pi_1) > r_1$; which can be expressed alternatively as $H_{1_0} : \Pi_1 = \gamma_1 \alpha_1'$. Since neither γ_1 nor α_1 is observable, the test must be based upon estimates of them. Nevertheless, as pointed out in [7] these parameter matrices cannot be estimated, since they form an

[8]Each matrix Π_k informs about the behavior of the series at the frequency whose associated root has not been eliminated in the corresponding $y_{k,t-1}$. Note that for every pair of seasonal frequencies associated with conjugated complex roots the information concerning the permanent behavior of the series is included jointly and inseparably through the corresponding pair of matrices. That is, Π_{k-1} and Π_k $(k = 4, 6, 8, 10, 12)$, in each case.

overparametrization of the model. Therefore the estimates of the spaces spanned by γ_1, $\widehat{sp}(\gamma_1)$, and by α_1, $\widehat{sp}(\alpha_1)$ will be used to test hypothesis H_{1_0}.

Equation (4) will be estimated by maximizing the likelihood function with respect to the parameters $(\Pi_1, \ldots, \Pi_{12})$ and $(\Sigma, A_1, \ldots, A_{p-12})$. Since the way the parameters take part in the likelihood function is independent[9], we can concentrate it sequentially, obtaining an expression depending solely on the parameters of interest Π_1 of the testing hypothesis. The estimates of Π_1 are substituted, recursively, in the corresponding expressions to obtain the estimates of the remaining parameters.

Firstly, for fixed values of $\Pi_1, \ldots, \Pi_{12}$ the maximum-likelihood estimates of $A_1, \ldots$ $\ldots, A_{p-12}$ can be obtained by an OLS regression of $\left(\Delta_{12}x_t - \sum_{k=1}^{12} \Pi_k y_{k,t-1}\right)$ on the lagged seasonal differences $\Delta_{12}x_{t-1}, \ldots, \Delta_{12}x_{t-p+12}$. Alternatively, we can obtain the OLS residuals R_t by first regressing $\Delta_{12}x_t$ on the lagged seasonal differences giving the residuals R_{0t}, then regressing each $y_{k,t-1}$ (k=1,...,12) on the lagged seasonal differences giving the residuals R_{kt} for $k = 1, 2, \ldots, 12$, and finally forming $R_t = R_{0t} - \sum_{k=1}^{12} \Pi_k R_{kt}$.

Then, the ML estimates of Π_k's can be achieved by the following OLS regression

$$R_{0t} = \Pi_1 R_{1t} + \Pi_2 R_{2t} + \ldots + \Pi_{11} R_{11t} + \Pi_{12} R_{12t} + \epsilon_t. \tag{6}$$

Since the parameter matrices $\Pi_2, \ldots, \Pi_{12}$ in equation (6) are independent, the likelihood function can be concentrated on Π_1. Thus regressing R_{0t} and R_{1t} on $(R_{2t}, R_{3t}, \ldots, R_{12t})$, we obtain the residuals U_{0t} and U_{1t}, respectively; thus finally forming $U_t = U_{0t} - \Pi_1 U_{1t}$. The concentrated likelihood function can be rewritten as

$$L(\Pi_1, \Sigma) \propto |\Sigma|^{-T/2} \exp\left(-\frac{1}{2}\sum_{t=1}^{T} U_t' \Sigma^{-1} U_t\right). \tag{7}$$

The ML estimation of the parameter matrix Π_1 may be obtained by the OLS regression $U_{0t} = \Pi_1 U_{1t} + \xi_{1t}$. Since our hypothesis imposes the restriction $\Pi_1 = \gamma_1\alpha_1'$, for a fixed value of α_1 the ML estimation for γ_1 and Σ are equivalent to the LS reduced rank estimation in the regression $U_{0t} = \gamma_1(\alpha_1' U_{1t}) + \eta_{1t}$, obtaining

$$\widehat{\gamma}_1(\alpha_1) = D_{01}\alpha_1(\alpha_1' D_{11}\alpha_1)^{-1} \tag{8}$$

$$\widehat{\Sigma}(\alpha_1) = D_{00} - D_{01}\alpha_1(\alpha_1' D_{11}\alpha_1)^{-1}\alpha_1' D_{10}. \tag{9}$$

where $D_{ij} = T^{-1}\sum_{t=1}^{T} U_{it} U_{jt}'$.

Now the likelihood function is proportional to $|\widehat{\Sigma}(\alpha_1)|^{-T/2}$ and hence, its maximization with respect to α_1 is equivalent to minimizing $\left|D_{00} - D_{01}\alpha_1(\alpha_1' D_{11}\alpha_1)^{-1}\right.$ $\left.\alpha_1' D_{10}\right|$ with respect to α_1. Using commonly known results (see [1] or [7]), the expression to be minimized remains

$$\min_{(\alpha_1)} \frac{|\alpha_1' D_{11}\alpha_1 - \alpha_1' D_{10} D_{00}^{-1} D_{01}\alpha_1|}{|\alpha_1' D_{11}\alpha_1|}. \tag{10}$$

[9] Due to the asymptotic uncorrelatedness between any two series with unit roots at different frequencies, $y_{1,t-1}$ and $y_{j,t-1}$, and the fact that both are at once asymptotically uncorrelated with (stationary) lags of $\Delta_{12}x_t$.

Based upon [7] and [8], α_1 can be estimated by choosing the first r_1 eigenvectors of $D_{10}D_{00}^{-1}D_{01}$ with respect to D_{11}, $\widehat{\alpha}_1 = (v_{1,1}, v_{1,2}, \ldots, v_{1,r_1})$. That is, the eigenvectors corresponding to the r_1 largest eigenvalues, $\widehat{\lambda}_{1i},\ i = 1, \ldots, r_1$ solving $|\lambda_1 D_{11} - D_{10}D_{00}^{-1}D_{01}| = 0$.

Without *a priori* information, a variety of possible optimum choices[10] of the matrix α_1 can be obtained from the resulting $\widehat{\alpha}_1$ that solves the eigenvalue problem above. However, we can always infer the cointegration space of vector x_t at that frequency.

The inference about the number of cointegration relationships at the zero frequency can be carried out through the likelihood ratio test statistic or *trace statistic*. This tests the hypothesis $H_{1_0} : \Pi_1 = \gamma_1\alpha_1'$, through[11]

$$-2\ln(Q) = -T\sum_{i=r_1+1}^{n}\ln(1-\widehat{\lambda}_{1,i}) \tag{11}$$

where $\widehat{\lambda}_{1,r_1+1}, \ldots, \widehat{\lambda}_{1,n}$ are the $(n - r_1)$ smallest eigenvalues of $D_{10}D_{00}^{-1}D_{01}$ with respect to D_{11}, corresponding to the $(n - r_1)$ smallest squared partial canonical correlations of U_1 with respect to U_0.

4.2. Cointegration at seasonal frequency π

Let us now set up the test for the hypothesis that there are *at most* r_2 cointegrating vectors at seasonal frequency π. In this case, the matrix that conveys the information about the behavior at seasonal frequency π is the one related to $y_{2,t-1}$. The confronted hypotheses are $H_{2_0} : \text{rank}(\Pi_2) \leq r_2$ $(r_2 < n)$ vs. $H_{2_a} : \text{rank}(\Pi_2) > r_2$.

Analogously to the previous section, the hypothesis that Π_2 does not have full rank can be formulated as the expression: $H_{2_0} : \Pi_2 = \gamma_2\alpha_2'$. The testing procedure is very similar to the one developed above except that the role played by the vectors of residuals R_{1t} and R_{2t} is reversed.

Given the hypothesis of interest, for a fixed value of α_2 the ML estimation for γ_2 and Σ are equivalent to the LS reduced rank estimation in the regression $U_{0t} = \gamma_2(\alpha_2' U_{2t}) + \eta_{2t}$, causing

$$\widehat{\gamma}_2(\alpha_2) = D_{02}\alpha_2(\alpha_2' D_{22}\alpha_2)^{-1}, \tag{12}$$

$$\widehat{\Sigma}(\alpha_2) = D_{00} - D_{02}\alpha_2(\alpha_2' D_{22}\alpha_2)^{-1}\alpha_2' D_{20}. \tag{13}$$

The likelihood function is proportional to $|\widehat{\Sigma}(\alpha_2)|^{-T/2}$. Maximizing it with respect to α_2 is equivalent to minimizing the expression

$$\min_{(\alpha_2)} \frac{|\alpha_2' D_{22}\alpha_2 - \alpha_2' D_{20}D_{00}^{-1}D_{02}\alpha_2|}{|\alpha_2' D_{22}\alpha_2|} \tag{14}$$

[10] Given that $\Pi_1 = \gamma_1\alpha_1'$ is satisfied, for any $(r_1 \times r_1)$ nonsingular matrix P it is true that $\Pi_1 = \gamma_1\alpha_1' = \widehat{\gamma}_1 PP^{-1}\widehat{\alpha}_1'$.

[11] The asymptotic distribution of the test statistic is a function of the distribution of several stochastic matrices, involving integrals of multivariant Wiener processes of dimension $(n - r_1)$.

The *trace statistic* for the hypothesis that there are *at most* r_2 cointegrating vectors — $(n - r_2)$ unit roots — at seasonal frequency π is

$$-2\ln(Q) = -T \sum_{i=r_2+1}^{n} \ln(1 - \widehat{\lambda}_{2,i}) \tag{15}$$

where $\widehat{\lambda}_{2,r_2+1}, \ldots, \widehat{\lambda}_{2,n}$ are the $(n - r_2)$ smallest eigenvalues of $D_{20}D_{00}^{-1}D_{02}$ with respect to D_{22}, which correspond to the $(n - r_2)$ smallest squared partial canonical correlations of U_2 with respect to U_0.

4.3. Cointegration at the remaining seasonal frequencies

The cointegration analysis at each pair of seasonal frequencies $\pm\pi/2$, $\pm 2\pi/3$, $\pm\pi/3$, $\pm 5\pi/6$ and $\pm\pi/6$ — all associated two by two with conjugated complex unit roots — must take into account simultaneously the information provided by two parameter matrices, in each case. This means that the information about permanent behavior of the series at each pair of conjugated frequencies $\pm\omega_\ell$, for $\ell \in \{4, 6, 8, 10, 12\}$ (where it must be noted that $\omega_4 = \pi/2$, $\omega_6 = 2\pi/3$, $\omega_8 = \pi/3$, $\omega_{10} = 5\pi/6$ and $\omega_{12} = \pi/6$) is conveyed jointly and inseparably through the matrices $\Pi_{\ell-1}$ and Π_ℓ. Therefore, we need to look at the two matrices, $\Pi_{\ell-1}$ and Π_ℓ, simultaneously to test the hypothesis of seasonal cointegration at each pair of conjugated frequencies $(\pm\omega_\ell)$.

In a more general context, the testing procedure might imply that we need to consider *polynomial cointegrating vectors* (PCIV), since one vector is sought to eliminate two unit roots from two different filtered vectors, $y_{\ell-1,t-1}$ and $y_{\ell,t-1}$. If PCIV are employed, both the cointegrating vectors and the error correction term coefficients may be different at different lags.

Henceforth, this paper will assume that cointegration, if any, is *contemporaneous*. Under this assumption the testing procedure is simpler than that which results from a general framework such as the one described above.

Using the above generic notation, the test of interest for either of the five pairs of seasonal frequencies can be formulated by means of the joint hypothesis

$$H_{\ell_0} : \{\Pi_{\ell-1} = \gamma_{\ell-1}\alpha'_\ell\} \cap \{\Pi_\ell = \gamma_\ell \alpha'_\ell\}.$$

Obviously, the restriction of contemporaneous cointegration imposes that the cointegrating vectors must coincide at different lags.

As above, we can develop the testing procedure for each case $\ell \in \{4, 6, 8, 10, 12\}$. For fixed values of Π_ℓ and $\Pi_{\ell-1}$, the maximum-likelihood estimation of the remaining matrices Π_k is equivalent to an OLS estimation in the regression of $(R_{0t} - \Pi_{\ell-1}R_{\ell-1,t} - \Pi_\ell R_{\ell t})$ on $(R_{1t}, R_{2t}, \boldsymbol{P}_\ell)$, where $\boldsymbol{P}_\ell = \boldsymbol{P} - \{R_{\ell-1,t}, R_{\ell t}\}$ and $\boldsymbol{P} = \{R_{3t}, R_{4t}, \ldots, R_{12t}\}$.

If we consider the series of residuals from the OLS regressions of R_{0t}, $R_{\ell-1,t}$ and $R_{\ell t}$ on $(R_{1t}, R_{2t}, \boldsymbol{P}_\ell)$ consecutively and denote them as U_{0t}, $U_{\ell-1,t}$ and $U_{\ell t}$ respectively, we can obtain the MLE of the parameter matrix $[\Pi_{\ell-1} \vdots \Pi_\ell]$ from the regression

$$U_{0t} = \Pi_{\ell-1}U_{\ell-1,t} + \Pi_\ell U_{\ell t} + \xi_{\ell t}.$$

Given the restriction under the joint hypothesis of contemporaneous cointegration, H_{ℓ_0}, for a fixed value of α_ℓ,

$$\begin{aligned}\widehat{\Sigma}(\alpha_\ell) &= D_{00} - \begin{pmatrix} D_{0,\ell-1} & D_{0\ell} \end{pmatrix} \begin{pmatrix} \alpha_\ell & 0 \\ 0 & \alpha_\ell \end{pmatrix} \\ &\times \left[\begin{pmatrix} \alpha'_\ell & 0 \\ 0 & \alpha'_\ell \end{pmatrix} \begin{pmatrix} D_{\ell-1,\ell-1} & D_{\ell-1,\ell} \\ D_{\ell,\ell-1} & D_{\ell,\ell} \end{pmatrix} \begin{pmatrix} \alpha_\ell & 0 \\ 0 & \alpha_\ell \end{pmatrix} \right]^{-1} \\ &\times \begin{pmatrix} \alpha'_\ell & 0 \\ 0 & \alpha'_\ell \end{pmatrix} \begin{pmatrix} D_{\ell-1,0} & D_{\ell 0} \end{pmatrix}. \end{aligned} \tag{16}$$

Now the likelihood function is proportional to $\left|\widehat{\Sigma}(\alpha_\ell)\right|^{-T/2}$ and its maximization is equivalent to minimizing the determinant of expression (16) with respect to α_ℓ.

The *trace statistic* for the hypothesis that there are *at most* r_ℓ seasonal cointegrating vectors — $(n - r_\ell)$ unit roots — at each pair of frequencies $\pm\omega_\ell$ is[12]

$$-2\ln(Q) = -T \sum_{i=r_\ell+1}^{n} \ln(1 - \widehat{\lambda}_{\ell-1,i} - \widehat{\lambda}_{\ell,i}) \tag{17}$$

where $\widehat{\lambda}_{\ell-1,r_\ell+1}, \ldots, \widehat{\lambda}_{\ell-1,n}$ are the $(n-r_\ell)$ smallest eigenvalues of $D_{\ell-1,0}D_{00}^{-1}D_{0,\ell-1}$ with respect to $D_{\ell-1,\ell-1}$; and on the other hand, $\widehat{\lambda}_{\ell,r_\ell+1}, \ldots, \widehat{\lambda}_{\ell,n}$ are the $(n - r_\ell)$ smallest eigenvalues of $D_{\ell,0}D_{00}^{-1}D_{0,\ell}$ with respect to $D_{\ell,\ell}$.

In practice, it is useful to consider a simpler performance of the testing procedure with very little effect (see [8]) on the test when cointegration is *contemporaneous*. This simplification is based upon the structure of the error correction model considered and consists of assuming $\gamma_\ell = 0$ $(\Rightarrow \Pi_\ell = 0)$, $\ell \in \{4, 6, 8, 10, 12\}$.

Under this assumption we can restrict our attention to the matrix $\Pi_{\ell-1}$ to test for cointegration relationships. The hypothesis of interest will be formulated as

$$H_{(\ell-1)_0} : \Pi_{\ell-1} = \gamma_{\ell-1}\alpha'_{\ell-1}.$$

Thus, the testing strategy is similar to that for the zero or seasonal frequency π except that the series of residuals $R_{\ell-1,t}$ — in each particular case — takes the role of R_{1t} or R_{2t}, respectively. So, we obtain that the likelihood ratio test statistic for the hypothesis that there are *at most* $r_{\ell-1}$ seasonal cointegrating vectors at seasonal frequencies $\pm\omega_\ell$ is

$$-2\ln(Q) = -T \sum_{i=r_{\ell-1}+1}^{n} \ln(1 - \widehat{\lambda}_{(\ell-1),i}) \tag{18}$$

where $\widehat{\lambda}_{(\ell-1),r_{\ell-1}+1}, \ldots, \widehat{\lambda}_{(\ell-1),n}$ are the $(n-r_{\ell-1})$ smallest eigenvalues of $D_{\ell-1,0}D_{00}^{-1}$ $D_{0,\ell-1}$ with respect to $D_{\ell-1,\ell-1}$.

[12] The asymptotic distribution of the test statistic is a function of the distribution of several stochastic matrices involving integrals of two mutually independent Wiener processes of dimension $(n - r_\ell)$ [A proof for seasonal frequencies $\pm\pi/2$ can be found in [8]].

4.4. Full Cointegration

In some data series, especially economic series, behavior at different frequencies may be similar due to seasonality in the time series — or even the behavior of the trend — having the same source. This will be reflected in the fact that some (though not necessarily all) cointegrating vectors may coincide. That is, a single cointegrating vector, say α_F, ($\alpha_F = \alpha_1 = \alpha_2 = \alpha_\ell$, $\ell \in \{4, 6, 8, 10, 12.\}$) might remove all unit roots in the system at all frequencies. This is defined as *full cointegration* in Definition 2.3. In this case the ECM (4) will be reduced to

$$\Delta_{12} x_t = \Pi_F(L)\, x_{t-1} + A_1\Delta_{12}x_{t-1} + A_2\Delta_{12}x_{t-2} + \cdots + A_{p-12}\Delta_{12}x_{t-p+12} + \varepsilon_t, \quad (19)$$

where the hypothesis of full cointegration implies that $\Pi_F(L) = \gamma_F(L)\alpha'_F$ must be satisfied. Thus a single vector α'_F may eliminate all the unit roots in the system and $\gamma_F(L)$ is a polynomial matrix having potentially eleven lags.

By arguments similar to those in the previous subsection, we will restrict our attention to the case when cointegration relationships, if any, are contemporaneous. So it is assumed that $\gamma_F(L) = \gamma_{F_{11}} L^{11}$ so the ECM can be written as

$$\Delta_{12} x_t = \Pi_{F_{11}} x_{t-12} + A_1\Delta_{12}x_{t-1} + A_2\Delta_{12}x_{t-2} + \cdots + A_{p-12}\Delta_{12}x_{t-p+12} + \varepsilon_t. \quad (20)$$

Then, we can get the series of residuals R_{Ft} from the OLS regression of x_{t-12} on the lagged seasonal difference $\Delta_{12}x_{t-1}, \ldots, \Delta_{12}x_{t-p+12}$ so, with the series of residuals R_{0t}, we can obtain the likelihood ratio test statistic for the hypothesis $H_{F_0} : \Pi_F = \gamma_F \alpha'_F$.

The *trace statistic* to test the hypothesis that there are *at most* r_F full cointegrating vectors is[13]

$$-2\ln(Q) = -T \sum_{i=r_{F+1}}^{n} \ln(1 - \widehat{\lambda}_{F,i}) \quad (21)$$

where $\widehat{\lambda}_{F,r_F+1}, \ldots, \widehat{\lambda}_{F,n}$ are the $(n - r_F)$ smallest eigenvalues[14] of $S_{F0}S_{00}^{-1}S_{0F}$ with respect to S_{FF}.

5. FINITE SAMPLE CRITICAL VALUES. MONTE CARLO SIMULATIONS

The aim of this section is to analyze the finite sample behavior of trace statistic distributions. This will be achieved by Monte Carlo simulations under the null hypothesis considered — several values of r_k — at any frequencies of interest individually as well as in the full cointegration case.

We then study the power of the test statistics by generating the distribution under several alternative hypotheses.

[13]The asymptotic distribution of the test statistic is a function of twelve standard mutually independent Wiener processes.

[14]Remark: The matrices S_{0F} and S_{FF} correspond to the product matrix of the residuals R_0 and R_F, that is, $S_{0F} = T^{-1}\sum_{t=1}^{T} R_{0t}R'_{Ft}$ and $S_{FF} = T^{-1}\sum_{t=1}^{T} R_{Ft}R'_{Ft}$.

The simulation experiments are designed for a sequence of random $(n \times 1)$vectors $\{x_t\}$ generated by

$$x_t = x_{t-12} + \varepsilon_t, \qquad t = -99, \ldots, 0, 1, \ldots, T \tag{22}$$

where each ε_t is a $(n \times 1)$ vector from a sequence of vectors $\mathrm{NID}(0, I_n)$.

Given the way the series are generated in this paper and since we consider that the cointegration relationships, if any, are *contemporaneous*, the ECM (4) will take the following simpler form to carry out the simulation exercise:

$$\Delta_{12} x_t = \Pi_1 y_{1,t-1} + \Pi_2 y_{2,t-1} + \Pi_3 y_{3,t-1} + \Pi_5 y_{5,t-1} + \Pi_7 y_{7,t-1} + \Pi_9 y_{9,t-1} + \Pi_{11} y_{11,t-1} + \varepsilon_t. \tag{23}$$

By construction, the dynamic (22) of the vector sequence $\{x_t\}$ imply that there is no cointegration relationship between the elements of the vector x_t at any frequencies. It is then clear that the true Π_k's are null matrices. Hence the empirical distributions of the test statistics will be obtained, in each case, under the null hypotheses that $r_k = \mathrm{rank}(\Pi_k) = 0$, $k = 1, 2, 3, 5, 7, 9, 11$, respectively. The critical values are reported in the Appendix.

To study the power of the test at any frequency we simulate several alternative hypotheses, where only cointegration relationships that are contemporaneous have been imposed, and confront them with the correspondent null hypotheses. All these cases indicate that the power of the test for cointegration in rejecting the false null hypotheses increases with T, so tests do not seem inconsistent. In Table 1 we report the results for one of the cases.

Table 1. Power of Trace Statistic. True Model: $(n - r) = (2 - 1) = 1$, $z_t = 2x_t + \varepsilon_t$, where $x_t = x_{t-12} + \zeta_t$.

$H_0 : n - r = 2$		Frequencies						
Quantil	T	**0**	π	$\pm\pi/2$	$\pm 2\pi/3$	$\pm\pi/3$	$\pm 5\pi/6$	$\pm\pi/6$
	100	41.4 %	41.8 %	88.2 %	87.1 %	87.8 %	87.1 %	87.9 %
95 %	300	99.8 %	99.9 %	100 %	100 %	100 %	100 %	100 %
	500	100 %	100 %	100 %	100 %	100 %	100 %	100 %
	100	20.3 %	18.7 %	66.1 %	63.2 %	63.3 %	63.1 %	66.1 %
99 %	300	98.1 %	99.3 %	100 %	100 %	100 %	100 %	100 %
	500	100 %	100 %	100 %	100 %	100 %	100 %	100 %

6. EMPIRICAL APPLICATION: PRODUCTION INDICATORS IN THE SPANISH ECONOMY

In this section we estimate the cointegration rank at each frequency in a set of variables related to Spanish production data from different economic sectors. The existence of cointegration relationships at each frequency implies that the series in the system fluctuate around a cyclical component at that frequency.

Our intention is to include in the variable set considered not only production series from the Industrial Production Index (IPI) but also series from other industrial sectors that are not reflected in this index, and series from the services sector. Thus we try to take into account a more representative set of data on Spanish production. The common element of these series is that they all feature a strong seasonal component. The data are collected from the *Boletín de Indicadores Económicos* published monthly by the Spanish Central Bank (*Banco de España*).

BIECO: IPI. Consumption Goods. Base 1990 = 100.

ESMEYCA: IPI. Investment Goods. Metal structures and Boilermaking. Base 1990 = 100.

MATTRA: IPI. Investment Goods. Transport Material (except cars and motorbikes). Base 1990 = 100.

MAQYBEQ: IPI. Investment Goods. Machinery and other Capital Goods. Base 1990 = 100.

BIEINT: IPI. Intermediate Goods. Base 1990 = 100.

ACERO: Domestic Steel Production. In Thousands of Tons.

VENGRA: Multiples Sale Index. Base 1983 = 100.

PERNOCVI: Tourism and Travelling. Nights spent by Travellers in Hotels. In Thousands of People.

We analyze monthly data for the period between January 1975 and March 1995. Figure 1 shows the logarithmic transformations of the series. The series show strong components in their seasonal and trend pattern, so we can expect to find unit roots at seasonal frequencies as well as at zero frequency. Hence, we can test and estimate cointegration relationships in the system at different frequencies.

Fig. 1. Production Series (log).

To carry out the unit roots tests for each individual series, we use the critical values reported in Table A.1 in the Appendix. These tests corroborate the impression reflected by the graphics of the series as to the existence of a unit root at every frequency in each series. Once we have analyzed the individual structure in each series, the cointegration tests for the different frequencies are applied.

Rank and cointegration relationships are estimated in the system formed by the variables BIECO, ESMEYCA, MATTRA, MAQYBEQ, BIEINT, ACERO, VENGRA and PERNOCVI. For the choice of the lag length p in the VAR, the usual model selection methods such as the Akaike Information Criterion (AIC) and Schwarz Criterion (SC) are used and Box-Pierce Q-statistics are also examined to test for uncorrelatedness of residuals. The ECM representation is adjusted for a VAR(22). Table 2 reports the results for the trace statistics to test the number of cointegration relationships at every frequency.

Table 2. Cointegration Tests.

$\boldsymbol{H_0 : n - r(\Pi_k)}$		Trace Statistic at Frequencies						
$n = 8$	$T = 214$	0	π	$\pm\pi/2$	$\pm2\pi/3$	$\pm\pi/3$	$\pm5\pi/6$	$\pm\pi/6$
$n - r = 8$	$r(\Pi_k) = 0$	334.0[a]	195.4[a]	159.4	162.6[a]	151.6	182.5[a]	182.8[a]
$n - r = 7$	$r(\Pi_k) = 1$	229.6[a]	120.8	104.3	103.3	95.0	114.3	101.6
$n - r = 6$	$r(\Pi_k) = 2$	153.4[a]	72.2	60.3	67.3	54.4	61.8	56.6
$n - r = 5$	$r(\Pi_k) = 3$	98.1[a]	41.8	37.7	37.7	26.2	32.4	24.7
$n - r = 4$	$r(\Pi_k) = 4$	59.5[a]	20.3	21.2	17.8	12.4	12.8	4.7
$n - r = 3$	$r(\Pi_k) = 5$	28.4[a]	10.0	8.0	6.7	3.3	4.8	2.0
$n - r = 2$	$r(\Pi_k) = 6$	12.4	4.2	4.0	3.2	1.5	1.8	0.3
$n - r = 1$	$r(\Pi_k) = 7$	2.1	0.0	0.7	1.1	0.0	0.0	0.0

[a]Significant at the 5 % level.

The results show no cointegration at seasonal frequencies $\pm\pi/2$ and $\pm\pi/3$. However, it seems that there is evidence of cointegration relationships at the remaining frequencies. The estimated cointegration ranks are $r_0 = 6$ and $r_\pi = r_{\pm2\pi/3} = r_{\pm5\pi/6} = r_{\pm\pi/6} = 1$.

The optimum estimates of the cointegrating vectors at each frequency can be obtained by the method explained in Section 4. These estimates are:

$$\widehat{\alpha}_1 = (\widehat{v}_{1,1}, \ldots, \widehat{v}_{1,6}) = \begin{bmatrix} 9.27 & 2.83 & -2.39 & 2.68 & 4.62 & 4.00 \\ 2.51 & -3.42 & 2.35 & -2.39 & -0.63 & 1.58 \\ -1.85 & 1.64 & -1.57 & -0.62 & -1.00 & -1.65 \\ 0.17 & -1.28 & -0.25 & 3.79 & 0.64 & 1.42 \\ -10.09 & -4.63 & 6.53 & -4.44 & 0.81 & -4.48 \\ 1.66 & 0.28 & -3.00 & 1.49 & -0.58 & 0.05 \\ -0.61 & 0.52 & -0.51 & -0.62 & -0.58 & -0.32 \\ -0.46 & 1.83 & 0.25 & -0.28 & -1.44 & -0.26 \end{bmatrix}$$

$$\begin{aligned}
\widehat{\alpha}_2 &= (\widehat{v}_{2,1}) = \left[\begin{array}{cccccccc} -11.96 & -2.21 & 1.28 & -3.18 & 23.78 & -1.33 & 9.30 & 7.27 \end{array}\right]' \\
\widehat{\alpha}_5 &= (\widehat{v}_{5,1}) = \left[\begin{array}{cccccccc} -16.96 & 8.98 & 0.48 & 6.78 & -4.16 & -11.63 & -7.50 & 20.06 \end{array}\right]' \\
\widehat{\alpha}_9 &= (\widehat{v}_{9,1}) = \left[\begin{array}{cccccccc} 19.90 & -7.86 & 2.38 & 2.02 & -23.72 & 6.03 & -2.45 & 4.62 \end{array}\right]' \\
\widehat{\alpha}_{11} &= (\widehat{v}_{11,1}) = \left[\begin{array}{cccccccc} 6.40 & -11.79 & 2.67 & -2.89 & 32.31 & -0.97 & 4.18 & 2.93 \end{array}\right]'
\end{aligned}$$

Summarizing, the ECM that describes the dynamic of production is

$$\begin{aligned}
\Delta_{12}x_t &= \gamma_1\alpha_1' y_{1,t-1} + \gamma_2\alpha_2' y_{2,t-1} + \gamma_5\alpha_5' y_{5,t-1} + \gamma_9\alpha_9' y_{9,t-1} + \gamma_{11}\alpha_{11}' y_{11,t-1} + \\
&\quad + A_1\Delta_{12}x_{t-1} + \cdots + A_{10}\Delta_{12}x_{t-10} + \varepsilon_t, \qquad (24)
\end{aligned}$$

where γ_1, α_1 are (8×6) matrices and $\gamma_2, \alpha_2, \gamma_5, \alpha_5, \gamma_9, \alpha_9, \gamma_{11}, \alpha_{11}$ are (8×1) matrices. Estimation of $\gamma_1, \gamma_2, \gamma_5, \gamma_9, \gamma_{11}$ can be carried out as described in Sections 4.1, 4.2 and 4.3.

7. CONCLUDING REMARKS

The main result of this study is the provision of a testing framework for the cointegration ranks in a system of nonstationary monthly processes. Its particularity is that it allows these tests to be implemented at each frequency of interest in the presence of unitary modulus roots at other frequencies. The extension made here — from the research in [8] — provides the empirical distributions of the test statistics for finite samples with different numbers of observations.

The statistical properties of these distributions seem to coincide with those found in previous similar studies, and are not far from those we would have desired at the beginning of our work. In particular, the power experiments show that the testing procedure presented is not inconsistent.

The method is applied to Spanish economic data in the form of monthly production indicators, which illustrates the implementation of the testing procedures. The results enable the error correction mechanism describing the dynamic of the system formed by these production variables to be identified.

We would expect inclusion of a constant term to affect — similarly to previous studies — the distribution of the test statistics for cointegration at zero frequency, and inclusion of seasonal dummies to change the distributions of the test statistics for cointegration at seasonal frequencies.

Finally in practice, the assumption mentioned above, $\Pi_4 = \Pi_6 = \Pi_8 = \Pi_{10} = \Pi_{12} = 0$, hardly affects the distributions of the test statistics from the point of view of contemporaneous cointegration analysis. However, this contemporaneous cointegration analysis is included in a more general context that implies considering polynomial cointegrating vectors (PCIV) which are difficult to find in practice, and a much more complicated testing procedure will be required to study cointegrating relationships.

8. APPENDIX: TABLES OF TRACE STATISTICS

These tables report the critical values of the empirical distributions for the different test statistics presented in Section 4; i.e., for the trace statistics (11), (15), (18) and (21).

The distribution of probabilities for each test statistic has been approached by the distribution of frequencies resulting from calculating repeatedly the trace statistic under the null hypotheses of no cointegration ($r = 0$) for values of $n = 1, 2, 3, 4, 5, 6, 7$ and 8, respectively. The number of observations for the finite sample considered is T = 100, 200, 300 and 500. The number of replications is 20,000 except in the cases $(n - r) = 6, 7, 8$, where 3,000 replications are employed.

Each table reports the critical values for the cointegration tests at each frequency individually and for the particular case of full cointegration. We use the statistics program RATS.

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Table A.1 Quantiles in the distribution of Trace Statistics $H_0 : n - r = 1$.

n−r = 1		*Quantiles*							
Frequency	**T**	**1 %**	**5 %**	**10 %**	**50 %**	**90 %**	**95 %**	**97.5 %**	**99 %**
$\omega = 0$	**100**	0.00	0.00	0.02	0.55	3.20	4.54	5.95	7.86
	200	0.00	0.00	0.02	0.62	3.23	4.52	5.71	7.45
	300	0.00	0.00	0.02	0.61	3.04	4.29	5.46	7.23
	500	0.00	0.00	0.02	0.60	2.99	4.15	5.35	6.91
$\omega = \pi$	**100**	0.00	0.00	0.02	0.54	3.16	4.50	5.82	7.97
	200	0.00	0.00	0.02	0.62	3.21	4.44	5.76	7.52
	300	0.00	0.00	0.02	0.62	3.10	4.40	5.66	7.29
	500	0.00	0.00	0.02	0.62	3.06	4.22	5.40	6.99
$\omega = \pi/2$	**100**	0.00	0.00	0.02	0.57	3.23	4.61	6.00	7.73
	200	0.00	0.00	0.01	0.56	3.22	4.48	5.86	7.69
	300	0.00	0.00	0.01	0.57	3.17	4.44	5.66	7.40
	500	0.00	0.00	0.01	0.58	3.16	4.39	5.54	7.30
$\omega = 2\pi/3$	**100**	0.00	0.00	0.01	0.55	3.20	4.52	5.92	7.88
	200	0.00	0.00	0.01	0.56	3.12	4.39	5.71	7.52
	300	0.00	0.00	0.01	0.56	3.11	4.35	5.60	7.34
	500	0.00	0.00	0.01	0.56	3.11	4.34	5.49	7.18
$\omega = \pi/3$	**100**	0.00	0.00	0.01	0.53	3.20	4.54	5.96	7.64
	200	0.00	0.00	0.01	0.56	3.16	4.44	5.67	7.54
	300	0.00	0.00	0.01	0.55	3.12	4.33	5.63	7.18
	500	0.00	0.00	0.01	0.56	2.99	4.18	5.46	7.10
$\omega = 5\pi/6$	**100**	0.00	0.00	0.01	0.54	3.28	4.68	6.14	8.12
	200	0.00	0.00	0.01	0.57	3.20	4.52	5.86	7.67
	300	0.00	0.00	0.01	0.57	3.14	4.41	5.66	7.10
	500	0.00	0.00	0.01	0.55	3.03	4.23	5.52	7.33
$\omega = \pi/6$	**100**	0.00	0.00	0.02	0.56	3.29	4.68	5.92	7.93
	200	0.00	0.00	0.01	0.56	3.18	4.47	5.87	7.64
	300	0.00	0.00	0.01	0.56	3.14	4.41	5.61	7.32
	500	0.00	0.00	0.01	0.56	3.06	4.28	5.60	7.21
	100	0.00	0.00	0.01	0.51	3.08	4.33	5.70	7.33
Full	**200**	0.00	0.00	0.01	0.47	2.86	4.08	5.33	6.97
Cointe-	**300**	0.00	0.00	0.01	0.45	2.74	3.86	5.06	6.76
gration	**500**	0.00	0.00	0.01	0.42	2.66	3.76	4.90	6.53